J. Math. Tokushima Univ.
Vol. 11 (1977), 1–10

A Note on One-parameter Semi-groups of $*$-endomorphisms of C^*-algebras

By

Yoshifumi ITO

(Received May 15, 1977)

§**0.** So far many authors have studied one-parameter semi-groups of linear operators on Banach spaces and locally convex spaces, some of which are naturally considered as one-parameter semi-groups of $*$-endomorphisms of C^*-algebras. So in this note we study one-parameter semi-groups of $*$-endomorphisms of C^*-algebras.

A $*$-endomorphism of a C^*-algebra is in a sense a linear operator of the C^*-algebra as a Banach space. So the structure theorem of a one-parameter semi-group of linear operators on a Banach space can be translated word for word. The only difference is the fact that the infinitesimal generator of a one-parameter semi-group of $*$-endomorphisms of a C^*-algebra is on one hand an unbounded linear operator and on the other hand moreover an unbounded $*$-derivation of the C^*-algebra. At this point, we borrow many things from the theory of linear operators on Banach spaces, especially from K. Yosida's method of the theory of one-parameter semi-groups of linear operators on Banach spaces [4].

In this note we give an answer to the question of which derivation becomes an infinitesimal generator of a one-parameter semi-group of $*$-endomorphisms of a C^*-algebra. That is Theorem 6 in §2.

§**1.** Let R be a C^*-algebra. Let $\{\tau_t; 0 \leqq t < \infty\}$ be a one-parameter semi-group of $*$-endomorphisms of the C^*-algebra R satisfying the following conditions:

$$\tau_t \tau_s = \tau_{t+s},\ \tau_0 = I\ (=\text{the identity endomorphism}), \tag{1}$$

$$\lim_{t \to t_0} \tau_t(x) = \tau_{t_0}(x),\ (\lim = \text{the strong limit}), \tag{2}$$

$$0 \leqq t_0 < \infty, \qquad x \in R.$$

Further we assume that if R contains the unit e then $\tau_t(e) = e$.

In general a $*$-endomorphism τ of a C^*-algebra R has the property

$$\|\tau(x)\| \leqq \|x\| \qquad \text{for} \quad x \in R, \tag{3}$$

[cf. Sakai, S. [3], p. 5]. Hence the one-parameter semi-group of $*$-endomorphisms of the C^*-algebra R has the property

$$\sup_t \|\tau_t(x)\| \leqq \|x\|, \qquad \text{for} \quad x \in R. \tag{4}$$

We now deduce some properties of the one-parameter semi-group $\{\tau_t\}$ of $*$-endomorphisms of the C^*-algebra R such as strong differentiability of $\{\tau_t\}$, some properties of the infinitesimal generator of $\{\tau_t\}$ and the representation formula of $\{\tau_t\}$.

We may define the integral

$$C_\phi(x) = \int_0^\infty \phi(s)\tau_s(x)ds, \qquad (x \in R), \tag{5}$$

for complex-valued continuous function $\phi(s)$ such that $\int_0^\infty |\phi(s)|ds < \infty$ following after K. Yosida and other authors.

PROPOSITION 1. *Let S be the set of all $C_\phi(x)$ for all $x \in R$ and for all complex-valued continuously differentiable functions ϕ such that*

1) $\int_0^\infty |\phi(s)|ds < \infty$

and

2) $$\lim_{h \to 0} \int_h^\infty \left| \frac{\phi(s-h) - \phi(s)}{h} + \phi'(s) \right| ds = 0 \tag{6}$$

hold. Then S is dense in R.

PROOF. Put

$$\phi_\eta(s) = \eta \exp(-\eta s), \qquad \eta > 0. \tag{7}$$

Then for any $\eta > 0$ $\phi_\eta(s)$ satisfies 1) and 2) and moreover, for any $x \in R$,

$$\lim_{\eta \to \infty} C_\phi(x) = x. \tag{8}$$

For $C_\phi(x) \in S$, we have by (1)

$$\frac{1}{h}(\tau_h - I)\ (C_\phi(x)) = \int_h^\infty \frac{\phi(s-h) - \phi(s)}{h}\tau_s(x)ds - \frac{1}{h}\int_0^h \phi(s)\tau_s(x)ds.$$

Thus, by (6) and (1), $\lim_{h \downarrow 0} \frac{1}{h}(\tau_h - I)(C_\phi(x))$ exists and

$$\lim_{h \downarrow 0} \frac{1}{h}(\tau_h - I)\ (C_\phi(x)) = C_{-\phi'}(x) - \phi(0)x \tag{9}$$

holds.

If we denote by D the totality of $x \in R$ for which

$$w\text{-}\lim_{h \downarrow 0} h^{-1}(\tau_h - I)(x) = \delta(x) \tag{10}$$

exists, where w-lim means the weak limit. Then we have

THEOREM 1. *δ is a densely defined closed $*$-derivation from D to R with the properties*:

$$\lim_{h \to 0} h^{-1}(\tau_{t+h} - \tau_t)(x) = \delta(\tau_t(x)) = \tau_t(\delta(x)), \tag{11}$$

for any $x \in D$.

PROOF. By (9), $S \subseteqq D$. Hence, by Proposition 1, D is dense in R.

We have, by (10), for $x \in D$,

$$w\text{-}\lim_{h \downarrow 0} \frac{1}{h}(\tau_h - I)(\tau_t(x)) = w\text{-}\lim_{h \downarrow 0} \frac{1}{h}(\tau_{t+h} - \tau_t)(x) = \tau_t\left(w\text{-}\lim_{h \downarrow 0} \frac{1}{h}(\tau_h - I)(x)\right).$$

Hence $\tau_t(D) \subseteqq D$ and $\delta(\tau_t(x)) = \tau_t(\delta(x))$ for any $x \in D$, that is, δ is commutative with every τ_t and the right weak derivative $D^+\tau_t(x)$ exists and

$$D^+\tau_t(x) = \delta(\tau_t(x)) = \tau_t(\delta(x)) \qquad \text{for any} \quad x \in D.$$

Hence by the continuity (2) we have, for any $f \in R'$ (=the conjugate space of R as a Banach space),

$$f(\tau_t(x)) - f(x) = \int_0^t D^+ f(\tau_s(x))ds = \int_0^t f(\tau_s(\delta(x)))ds$$
$$= f\left(\int_0^t \tau_s(\delta(x))ds\right)$$

and therefore

$$\tau_t(x) - x = \int_0^t \tau_s(\delta(x))ds, \qquad \text{for } x \in D. \tag{12}$$

Thus we have the strong differentiability.

We now prove the closedness of δ. Let $x_n \in D$ $(n = 1, 2, \ldots)$ and let $\lim_{n \to \infty} x_n = x$, $\lim_{n \to \infty} \delta(x_n) = z$. Then, by (12),

$$\tau_t(x) - x = \int_0^t \tau_t(z)ds. \tag{13}$$

Hence $x \in D$ and $z = \delta(x)$.

Now let x, y be in D, then xy, $x^* \in D$ and

$$\begin{cases} \delta(xy) = \delta(x)y + x\delta(y), \\ \delta(x^*) = (\delta(x))^*. \end{cases} \tag{14}$$

That is, δ is a $*$-derivation of the C^*-algebra R. The theorem is proved.

Further, if the unit $e \in R$, then $e \in D$ and $\delta(e)=0$.

We now deduce the representation formula of $\{\tau_t\}$. Put

$$I_n = C_{\phi_n} \qquad (n=1, 2, \dots). \tag{15}$$

Then, by (7) and (9), we have

PROPOSITION 2.

(i) *The range* $\mathrm{Ran}(I_n)=\{I_n(x);\ x \in R\} \subseteqq D$,

$$\delta I_n = n(I_n - I). \tag{16}$$

(ii) $\|I_n\| \leqq 1, \quad \lim_{n\to\infty} I_n(x) = x, \qquad (x \in R).$ (17)

PROOF. We have by (4) and (7)

$$\|I_n(x)\| \leqq \left(\int_0^\infty n \exp(-ns)ds\right)\|x\| = \|x\|. \tag{18}$$

The proposition is proved.

Since by the above proposition

$$\delta I_n = n(I_n - I), \tag{19}$$

we have

$$\exp(t\delta I_n) = \sum_{m=0}^{\infty} \frac{(t\delta I_n)^m}{m!} = \exp(tn(I_n - I)), \qquad 0 \leqq t < \infty. \tag{20}$$

Thus

$$\begin{aligned} \|\exp(t\delta I_n)\| &= \|\exp(tnI_n)\exp(-tnI)\| \\ &\leqq \exp(tn)\exp(-tn) = 1. \end{aligned} \tag{21}$$

Since δI_n is commutative with each τ_t, we have, for $x \in D$,

$$\begin{aligned} \|\tau_t(x) - \exp(t\delta I_n)x\| &= \left\|\int_0^t \frac{d}{ds}((\exp(t-s)\delta I_n)\tau_s(x))ds\right\| \\ &= \left\|\int_0^t (\exp(t-s)\delta I_n)\tau_s(\delta - \delta I_n)x\, ds\right\| \\ &\leqq \int_0^t \|(\delta - \delta I_n)x\|ds, \qquad \text{by (4) and (21)} \\ &= t\|(\delta - \delta I_n)x\|. \end{aligned} \tag{22}$$

We have

$$\delta I_n x = I_n \delta(x), \qquad \text{for } x \in D, \tag{23}$$

for δ is a closed derivation commutative with each τ_t. Thus, by (22),

$$\|\tau_t(x) - \exp(t\delta I_n)x\| \leqq t\|(I - I_n)\delta x\|, \qquad \text{for} \quad x \in D. \tag{24}$$

By (4) and (21), we have

THEOREM 2. *For* $x \in R$,

$$\tau_t(x) = \lim_{n\to\infty} \exp(t\delta I_n)x = \lim_{n\to\infty} \sum_{m=0}^{\infty} (m!)^{-1}(t\delta I_n)^m x$$

uniformly for t in any finite interval.

PROOF. We have only to note that D is dense in R. The theorem is proved.

We now deduce some properties of the infinitesimal generator δ of $\{\tau_t\}$.

THEOREM 3. $\|(\delta - nI)x\| \geqq n\|x\| \qquad (n = 1, 2,\dots)$ (25)
for $x \in D$.

PROOF. Assume the contrary and let $\|(\delta - nI)x\| = a < n$ for a certain $x \in D$ with $\|x\| = 1$. Let $f \in R'$ be such that $f(x) = 1$, $\|f\| = 1$. Then, by

$$\frac{d}{dt}\tau_t(x) = \tau_t(\delta x) = n\tau_t(x) + \tau_t((\delta - nI)x),$$

we obtain

$$\begin{cases} \dfrac{d}{dt}\phi(t) = n\phi(t) + \psi(t), \quad \text{where} \\ \phi(t) = f(\tau_t(x)),\ \psi(t) = f(\tau_t((\delta - nI)x)). \end{cases}$$

Since $\phi(0) = 1$, we have

$$\phi(t) = \exp(nt)\left(\int_0^t \exp(-nt)\psi(t)dt + 1\right).$$

And hence, by

$$|\psi(t)| \leqq \|f\|\ \|\tau_t((\delta - nI)x)\| \leqq \|(\delta - nI)x\| = a,$$

we have

$$|\phi(t)| \geqq \exp(nt)(1 - an^{-1}(1 - \exp(-nt))).$$

Thus $\phi(t)$ is unbounded in t when $t \to \infty$, contrary to $|\phi(t)| \leqq \|f\|\ \|\tau_t(x)\| \leqq \|x\| = 1$. The theorem is proved.

THEOREM 4. $\operatorname{Ran}(\delta - nI) = R \qquad (n = 1, 2,\dots)$.

PROOF. We first show that $\mathrm{Ran}(\delta-nI)$ is dense in R. If otherwise, there exists $f\in R'$, $f\neq 0$, such that $f(\delta x-nx)=0$ on D. Thus, by $\tau_t(D)\subseteqq D$, we have $f(\delta\tau_t(x))=nf(\tau_t(x))$. And hence

$$\frac{d}{dt}f(\tau_t(x))=nf(\tau_t(x)).$$

Therefore, by $f(\tau_0(x))=f(x)$, we obtain $f(\tau_t(x))=f(x)\exp(nt)$. This is a contradiction. In fact, by $f\neq 0$ and by the fact that D is dense in R, there exists $x\in D$ such that $f(x)\neq 0$. Then $f(x)\exp(nt)$ is unbounded in t when $t\to\infty$, contrary to $|f(\tau_t(x))|\leqq\|f\|\,\|\tau_t(x)\|\leqq\|f\|\,\|x\|$. Thus $\mathrm{Ran}(\delta-nI)$ is dense in R. Therefore, for any $y\in R$, there exists a sequence $\{x_h\}\subseteqq D$ such that $\lim_{h\to\infty}(\delta-nI)x_h=y$. Because of $\|(\delta-nI)(x_h-x_k)\|\geqq n\|x_h-x_k\|$ by Theorem 3, $\{x_h\}$ is a Cauchy sequence. Let $\lim_{h\to\infty}x_h=x$, then by $\lim_{h\to\infty}(\delta-nI)x_h=y$ and by the closedness of δ we have $y=(\delta-nI)x$. Thus the theorem is proved.

THEOREM 5. *Let y_n be the unique solution of $(\delta-nI)y_n=y$ $(n=1, 2,\dots)$ by Theorems 3 & 4, then*

$$\lim_{n\to\infty}\delta(-ny_n)=\delta(y)\qquad\text{for}\quad y\in D. \tag{26}$$

PROOF. We have, by (16), $\delta I_n y-nI_n y=-ny$. And hence

$$-ny_n=I_n y.$$

Thus (17) and (23) imply (26). Thus the theorem is proved.

§**2.** Main theorem. We now answer the question of which derivation of R generates a one-parameter semi-group of $*$-endomorphisms of R.

THEOREM 6. *Let conversely δ be a densely defined closed $*$-derivation with the domain D such that*

(i) $\|(\delta-nI)x\|\geqq n\|x\|$ $\quad(n=1, 2,\dots)$ *for* $x\in D$,

(ii) $\mathrm{Ran}(\delta-nI)=R$ $\quad(n=1, 2,\dots)$.

Then there exists a unique one-parameter semi-group τ_t of $*$-endomorphisms of R which satisfies

1) $\tau_t\tau_s=\tau_{t+s}$, $\tau_0=I$ (27)

2) $\|\tau_t(x)\|\leqq\|x\|$, *for* $0\leqq t<\infty$, (28)

3) $\lim_{t\to t_0}\tau_t(x)=\tau_{t_0}(x)$, $\quad 0\leqq t_0<\infty$, $x\in R$, (29)

4) $\lim_{h\to 0} h^{-1}(\tau_{t+h}-\tau_t)(x)=\delta(\tau_t(x))=\tau_t(\delta(x))$ (30)

for any $x\in D$.

Proof. By (i) and (ii), the operator J_n defined by

$$J_n y=-ny_n \qquad (n=1, 2,\dots), \tag{31}$$

where y_n is the unique solution of $(\delta-nI)y_n=y$ $(n=1, 2,\dots)$, satisfies

$$\|J_n\|\leqq 1 \tag{32}$$

$$J_n=-n(\delta-nI)^{-1}, \tag{33}$$

$$J_nJ_m=J_mJ_n, \tag{34}$$

$$\delta J_n=J_n\delta, \tag{35}$$

$$\lim_{n\to\infty} J_nx=x, \qquad x\in R, \tag{36}$$

$$\lim_{n\to\infty}\delta J_nx=\delta x, \qquad x\in D. \tag{37}$$

Since

$$\delta J_ny=\delta(-ny_n)=-n(y+ny_n)=n(J_n-I)y, \tag{38}$$

we have, by (32),

$$\|\exp(t\delta J_n)y\|=\|\exp(ntJ_n)\exp(-ntI)y\|$$
$$\leqq\exp(nt)\exp(-nt)\,\|y\|=\|y\|. \tag{39}$$

Hence the linear operator defined by

$$\tau_t^{(n)}=\exp(t\delta J_n) \tag{40}$$

satisfies

$$\|\tau_t^{(n)}x\|\leqq\|x\|, \tag{41}$$

$$\tau_t^{(n)}x-x=\int_0^t\tau_s^{(n)}\delta J_nxds, \tag{42}$$

$$\lim_{h\to 0}h^{-1}(\tau_{t+h}^{(n)}-\tau_t^{(n)})x=\tau_t^{(n)}\delta J_nx. \tag{43}$$

By (34) and (35), δJ_n is commutative with $\tau_t^{(m)}$ and hence

$$\|(\tau_t^{(m)}-\tau_t^{(n)})x\|=\left\|\int_0^t\frac{d}{ds}((\exp\ (t-s)\delta J_n)\tau_s^{(m)}x)ds\right\|$$

$$=\left\|\int_0^t(\exp(t-s)\delta J_n)\tau_s^{(m)}(\delta J_m-\delta J_n)xds\right\|$$

$$\leqq \int_0^t (\delta J_m - \delta J_n)x ds, \quad \text{by (41) and (43),}$$

$$= t\|(\delta J_m - \delta J_n)x\| .$$

Therefore, by (37),

$$\tau_t(y) = \lim_{n\to\infty} \tau_t^{(n)} y \qquad (y \in D) \tag{44}$$

exists uniformly for t in any finite interval. Since D is dense in R and since we have (41), we see that the limit $\tau_t(y)$ exists for all $y \in R$ and that τ_t satisfies (27)–(29). Hence, by letting $n \to \infty$ in (42), we obtain

$$\tau_t(y) - y = \int_0^t \tau_s(\delta(y))ds, \qquad y \in D . \tag{45}$$

Thus, $\{\tau_t;\ 0 \leqq t < \infty\}$ is a unique strongly continuous one-parameter semi-group of linear operators of R whose infinitesimal generator is δ.

Lastly, we now show that τ_t is a $*$-endomorphism of R. Put

$$\phi(t) = \tau_t(xy) - \tau_t(x)\tau_t(y) \qquad \text{for} \quad x,\ y \in R . \tag{46}$$

Since δ is a derivation of R, we have

$$\begin{cases} \dfrac{d}{dt}\phi(t) = \delta\phi(t) \\ \phi(0) = 0 \end{cases} \tag{47}$$

for $x,\ y \in D$. This Cauchy problem is correct in the meaning of S. G. Krein [1] (p. 47, Theorem 2.8). Thus $\phi(t) \equiv 0$ for $x,\ y \in D$. Since D is dense in R, we have by (28)

$$\tau_t(xy) = \tau_t(x)\tau_t(y) \qquad \text{for} \quad x,\ y \in R . \tag{48}$$

Next we put

$$\phi(t) = \tau_t(x^*) - \tau_t(x)^*, \qquad x \in R . \tag{49}$$

Then, since δ is a $*$-derivation of R, we have

$$\begin{cases} \dfrac{d}{dt}\phi(t) = \delta\phi(t) \\ \phi(0) = 0 \end{cases} \tag{50}$$

for $x \in D$. This Cauchy problem is also correct. Hence $\phi(t) \equiv 0$. Thus we have

$$\tau_t(x^*) = \tau_t(x)^*, \qquad \text{for} \quad x \in D . \tag{51}$$

Since D is dense in R, we have by (28)

$$\tau_t(x^*)=\tau_t(x)^*, \qquad x \in R. \tag{52}$$

§3. Examples. We have some examples.

EXAMPLE 1. $R=C([0, \infty))$=the C^*-algebra of all complex-valued bounded continuous functions on the interval $[0, \infty)$. The product is defined by pointwise multiplication. We define $\|x\|=\sup_s |x(s)|$ and $x^*(s)=\overline{x(s)}$ for $x \in R$. Then if we define

$$\tau_t(x)(s)=x(s+t) \qquad \text{for} \quad 0 \leqq t < \infty, \tag{53}$$

$\{\tau_t; 0 \leqq t < \infty\}$ is a one-parameter semi-group of $*$-endomorphisms of R. The infinitesimal generator δ of τ_t is defined by

$$\delta(x)(s)=\frac{dx}{ds}(s), \tag{54}$$

where D is the set of functions in $C([0, \infty))$ whose first derivatives are also in $C([0, \infty))$.

EXAMPLE 2. Let H be a Hilbert space and $L(H)$ be the C^*-algebra of all continuous operators on H. We define $R=C([0, \infty); L(H))$=the C^*-algebra of all $L(H)$-valued bounded continuous functions on $[0, \infty)$. The product of R is defined by

$$(ST)(s)=S(s)T(s). \tag{55}$$

We define $|||T|||=\sup_s \|T(s)\|$ and $T^*(s)=T(s)^*$=the adjoint operator of $T(s)$. Then if we define

$$\tau_t(T)(s)=T(s+t), \qquad 0 \leqq t < \infty, \tag{56}$$

$\{\tau_t; 0 \leqq t < \infty\}$ is a one-parameter semi-group of $*$-endomorphisms of R. The infinitesimal generator δ of τ_t is defined by

$$\delta(T)(s)=\frac{dT}{ds}(s), \tag{57}$$

where D is the set of $L(H)$-valued functions in $C([0, \infty); L(H))$ whose first derivatives are also in $C([0, \infty); L(H))$.

EXAMPLE 3. Let A be a C^*-algebra. We define $R=C([0, \infty); A)$=the C^*-algebra of all A-valued bounded continuous functions on $[0, \infty)$. The product of R is defined by

$$(xy)(s)=x(s)y(s). \tag{58}$$

We define $|||x||| = \sup_s \|x(s)\|$ and $x^*(s) = x(s)^*$ where $*$ in the right-hand side is the involution in A. Then, if we define

$$\tau_t(x)(s) = x(s+t), \qquad 0 \leqq t < \infty, \tag{59}$$

$\{\tau_t; 0 \leqq t < \infty\}$ is a one-parameter semi-group of $*$-endomorphisms of R. The infinitesimal generator δ of τ_t is defined by

$$\delta(x)(s) = \frac{dx}{ds}(s), \tag{60}$$

where D is the set of A-valued functions in $C([0, \infty); A)$ whose first derivatives are also in $C([0, \infty); A)$.

Department of Mathematics
College of General Education
Tokushima University

References

[1] Krein, S. G.: Linear Differential Equations in Banach Space. American Mathematical Society, Providence, Rhode Island, 1971. (English translation of Lineĭnye different͡sial'nye uravnenii͡a v banakhovom prostranstve.)

[2] Robinson, W.: Unbounded derivation of C^*-algebras. International Symposium on Mathematical Problems in Theoretical Physics, Ed. by H. Araki, Springer-Verlag, Berlin, Heidelberg, New York, 1975. pp. 303–311.

[3] Sakai, S.: C^*-algebras and W^*-algebras. Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[4] Yosida, K.: On the differentiability and the representation of one-parameter semi-group of linear operators. J. Math. Soc. Japan. **1** (1948) 15–21.

[5] ————: Topological Analysis I. Iwanami, Tokyo, 1951. (In Japanese.)

[6] ————: Functional Analysis. Springer-Verlag, Berlin, Heidelberg, New York; Kinokuniya, Tokyo, 1971.