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On the Theory of Vector Valued Hyperfunctions

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Contents

§ 0. Introduction

Recently the theory of vector valued hyperfunctions has been developed by P. D. F. Ion and T. Kawai, 1975, [14], and by Y. Ito, 1977, [15]. The former extended directly Sato's theory [28] [12] [21] [27], while the latter extended the method of A. Martineau and P. Schapira [25], [29] and then established Sato's theory indirectly and rather elementarily. They consider vector valued hyperfunctions on n-dimensional real Euclidean space. But recently when the author was studying the general theory of analytic linear mappings, he was forced to study vector valued hyperfunctions on an n-dimensional real analytic manifold M which is countable at infinity. Namely we have need of the theory of vector valued hyperfunctions in proving Martineau-Harvey's theorem in the case of vector valued functions. This theorem characterizes analytic linear mappings with a certain compact carrier.

The author has established that a vector valued hyperfunction is some class of analytic linear mappings and a vector valued hyperfunction with compact support in M is nothing else but a real analytic linear mapping with compact support. The vector valued hyperfunctions, by localization, form a flabby sheaf over M and their section modules are realized as relative cohomology groups with coefficients in a sheaf of vector valued holomorphic functions, as in Sato-Ion-Kawai's theory.

§1. Holomorphic functions and analytic functions

Let M be an n-dimensional real analytic manifold which is countable at infinity and X a complexification of M. We denote by ${}^E\mathcal{O}={}^E\mathcal{O}_X$ the sheaf of germs of holomorphic functions valued in a Fréchet space E over X, and by ${}^E\mathfrak{a}={}^E\mathfrak{a}_M={}^E\mathcal{O}_X|M$ the sheaf of germs of real analytic functions valued in E over M. We put $\mathcal{O}={}^{\mathbf{C}}\mathcal{O}$ and $\mathfrak{a}={}^{\mathbf{C}}\mathfrak{a}$, where $\mathbf{C}$ denotes the complex number field.

In the following we assume that E always denotes a Fréchet space.

If Ω is an open set in X, we set

$$\mathcal{O}(\Omega)=\Gamma(\Omega, \mathcal{O}),$$

the section module on Ω. This space has an FS-space topology for semi-norms

$$p_K(f)=\sup_K |f|,$$

where K runs over the family of compact subsets of Ω. It is known that $\mathcal{O}(\Omega)$ is a Fréchet nuclear space. Let K be a compact subset of X. We put

$$\mathcal{O}(K)=\varinjlim_{\Omega\supset K} \mathcal{O}(\Omega).$$

$\mathcal{O}(K)$ is the space of holomorphic functions in a neighborhood of K endowed with the locally convex topology of the inductive limit of $\mathcal{O}(\Omega)$ where Ω runs the family of open neighborhoods of K. It is a nuclear DFS-space (in particular, it is Hausdorff) and its dual $\mathcal{O}'(K)$ is a nuclear FS-space.

Further, any bounded subset of $\mathcal{O}(K)$ is contained and bounded in a space $\mathcal{O}(\Omega)$ (cf. A. Martineau [26] or H. Komatsu [21]).

If K is a compact subset of M, we have an isomorphism

$$\mathfrak{a}(K)=\mathcal{O}(K),$$

where $\mathfrak{a}(K)$ denotes the space of real analytic functions in a neighborhood of K in M. $\mathfrak{a}(K)$ is endowed with the topology of $\mathcal{O}(K)$. Then $\mathcal{O}(X)$ is dense in $\mathfrak{a}(K)$ by virtue of the embedding theorem (cf. H. Grauert [6]).

If Ω is an open set in M, let $\mathfrak{a}(\Omega)$ be the space of real analytic functions on Ω equipped with the topology

$$\mathfrak{a}(\Omega)=\varprojlim_{K\subset\Omega} \mathfrak{a}(K).$$

Then $\mathfrak{a}(\Omega)$ is a complete barreled nuclear space whose dual is a complete nuclear space.

Now we have

Proposition. *Let M_i be an n_i-dimensional real analytic manifold which is* *countable at infinity and X_i a complexification of M_i $(i=1, 2)$. Then we have the following canonical isomorphisms*:

(i) $\mathcal{O}(\Omega_1)\hat{\otimes}\mathcal{O}(\Omega_2)\simeq\mathcal{O}(\Omega_1\times\Omega_2)$,

$(\Omega_i\subset X_i$ *open set* $(i=1, 2))$;

(ii) $\mathcal{O}(K_1)\hat{\otimes}\mathcal{O}(K_2)\simeq\mathcal{O}(K_1\times K_2)$,

$(K_i\subset X_i$ *compact set* $(i=1, 2))$;

(iii) $\mathfrak{a}(K_1)\hat{\otimes}\mathfrak{a}(K_2)\simeq\mathfrak{a}(K_1\times K_2)$,

$(K_i\subset M_i$ *compact set* $(i=1, 2))$;

(iv) $\mathfrak{a}(\Omega_1)\hat{\otimes}\mathfrak{a}(\Omega_2)\simeq\mathfrak{a}(\Omega_1\times\Omega_2)$,

$(\Omega_i\subset M_i$ *open set* $(i=1, 2))$.

Proof. (i) See F. Treves [35], Theorem 51.6, p. 530 or A. Grothendieck [10], Chap. 2, p. 81.

(ii), (iii) They follow from the following

Lemma A. *Let E (resp. F) be a locally convex Hausdorff topological vector space which is an inductive limit of a family $\{E_i\}$ (resp. $\{F_j\}$) of locally convex Hausdorff topological vector spaces. Then, on $E\otimes F$, the projective topology and the inductive limit topology of projective tensor products $E_i\otimes_\pi F_j$ with respect to the natural linear mapping of each of these spaces into $E\otimes F$ coincide. On the subspace of $E\hat{\otimes}_\pi F$ generated by the images of the spaces $E_i\hat{\otimes}_\pi F_j$, the topology induced by $E\hat{\otimes}_\pi F$ coincides with the inductive limite topology of $E_i\hat{\otimes}_\pi F_j$.*

Proof of the lemma A. By duality, the necessary and sufficient condition that two locally convex topologies of one Hausdorff topological vector space coincide is that they give the same family of equicontinuous sets of linear functionals. But, since we have $(E\otimes_\pi F)'=(E\hat{\otimes}_\pi F)'=B(E, F)$, where $B(E, F)$ is the space of continuous bilinear forms on $E\times F$, the assertion of the lemma is equivalent to the assertion that the necessary and sufficient condition that the set of bilinear forms on $E\times F$ be equicontinuous is that its restriction to each $E_i\times F_j$ is equicontinuous. It is evident from the topologization of $E\times F=\varinjlim E_i\times F_j$. This completes the proof of the lemma A.

(iv) follows from the following

Lemma B. *Let E (resp. F) be a locally convex Hausdorff topological vector space which is a projective limit of a family $\{E_i\}$ (resp. $\{F_j\}$) of complete locally convex Hausdorff topological vector spaces. Assume that the natural continuous linear mappings u_i: $E\to E_i$ (resp. v_j: $F\to F_j$) are of dense range. Then we have $E\hat{\otimes}_\pi F=\varprojlim E_i\hat{\otimes}_\pi F_j$.*

Proof of the lemma B. We have first a natural inclusion $E\hat{\otimes}_\pi F\subset\varprojlim E_i\hat{\otimes}_\pi F_j$. Next we note that $E\otimes_\pi F=\varprojlim E_i\otimes_\pi F_j$, (cf. L. Schwartz [32], Exposé n°7, Proposition 5). Hence we have $E\hat{\otimes}_\pi F=\varprojlim [u_i\hat{\otimes}v_j(E\hat{\otimes}_\pi F)]$, where $[u_i\hat{\otimes}v_j(E\hat{\otimes}_\pi F)]$ denotes the closure of $u_i\hat{\otimes}v_j(E\hat{\otimes}_\pi F)$ in the space $E_i\hat{\otimes}_\pi F_j$. Since $u_i(E)$ (resp. $v_j(F)$) is dense in E_i (resp. F_j), $u_i(E)\otimes_\pi v_j(F)$ is dense in $E_i\otimes_\pi F_j$. Hence $u_i(E)\otimes_\pi v_j(F)$ is dense in $E_i\hat{\otimes}_\pi F_j$ and $u_i\hat{\otimes}v_j(E\hat{\otimes}_\pi F)$ is dense in $E_i\hat{\otimes}_\pi F_j$. Hence we have $[u_i\hat{\otimes}v_j(E\hat{\otimes}_\pi F)]=E_i\hat{\otimes}_\pi F_j$. Hence we have $E\hat{\otimes}_\pi F=\varprojlim E_i\hat{\otimes}_\pi F_j$. Q.E.D.

We have the following two corollaries to Lemmas A and B.

Corollary 1. *Let $E=\varinjlim E_i$ and $F=\varinjlim F_j$ be DFS*-spaces such as injective *limit spaces of compact injective sequences $\{E_i\}$ and $\{F_j\}$ of F- or DF-spaces. Then $E\otimes_\pi F=\varinjlim E_i\otimes_\pi F_j$ and $E\hat{\otimes}_\pi F=\varinjlim E_i\hat{\otimes}_\pi F_j$ are also DFS-spaces.*

Corollary 2. *Let $E=\varprojlim E_i$ and $F=\varprojlim F_j$ be FS-spaces such as projective limit spaces of compact projective sequences $\{E_i\}$ and $\{F_j\}$ of F- or DF-spaces. Then $E\otimes_\pi F=\varprojlim E_i\otimes_\pi F_j$ and $E\hat{\otimes}_\pi F=\varprojlim E_i\hat{\otimes}_\pi F_j$ are also FS-spaces.*

Proof of Corollaries 1 and 2 follows from the fact that a continuous linear mapping u of a locally convex Hausdorff topological vector space E into a locally convex Hausdorff topological vector space F is injective if and only if its transpose ${}^tu$ is of dense range, and from A. Grothendieck [10], Chap. I, § 1, n°3, Lemma 4.

Q.E.D.

§ 2. Analytic linear mappings

Definition 2.1. *Let E be a Fréchet space and Ω be an open subset of X. Elements of $L(\mathcal{O}(\Omega); E)(\equiv L_b(\mathcal{O}(\Omega); E))$ are called local analytic linear mappings on Ω valued in E or simply analytic linear mappings on Ω. We say that $u\in L(\mathcal{O}(\Omega); E)$ is carried by a compact set K in Ω if u can be extended to $\mathcal{O}(K)$. We then call K the carrier of u. We denote by $\mathcal{O}'(\Omega; E)$ the space $L(\mathcal{O}(\Omega); E)$. $\mathcal{O}'(K; E)=L(\mathcal{O}(K); E)$, $\mathfrak{a}'(K; E)=L(\mathfrak{a}(K); E)$ and $\mathfrak{a}'(\Omega; E)=L(\mathfrak{a}(\Omega); E)$ are defined in the same way. We also call their elements analytic linear mappings.*

Proposition 2.1. *Let E be a Fréchet space. Then we have:*

(i) $\mathcal{O}'(\Omega; E)=L(\mathcal{O}(\Omega); E)\simeq\mathcal{O}'(\Omega)\hat{\otimes}E$,

(Ω: *an open set in X*).

(ii) $\mathcal{O}'(K; E)=L(\mathcal{O}(K); E)\simeq\mathcal{O}'(K)\hat{\otimes}E$,

(K: *a compact set in X*).

(iii) $\mathfrak{a}'(K; E)=L(\mathfrak{a}(K); E)\simeq\mathfrak{a}'(K)\hat{\otimes}E$,

(K: *a compact set in M*).

(iv) $\mathfrak{a}'(\Omega; E) = L(\mathfrak{a}(\Omega); E) \simeq \mathfrak{a}'(\Omega) \hat{\otimes} E$,

(Ω: *an open set in* M).

Proof. See F. Treves [35], Proposition 50.5, p. 522. Q.E.D.

Proposition 2.2. *Let* E *be a Fréchet space, and* K *a compact subset of* X *with the Runge property in the sense of A. Martineau* [26] *and Y. Ito* [16], *and* $u \in L(\mathcal{O}(X); E)$. *Then* u *is carriable by* K *if and only if it is carriable by all open neighborhood of* K.

Proof. See Y. Ito [16], Proposition 2.14. Q.E.D.

The elements of $L(\mathfrak{a}(M); E)$ are called real analytic linear mappings. They are analytic linear mappings on X which are carried by real compact set in M.

Theorem 2.1. *Let* $u \in L(\mathfrak{a}(M); E)$, $u \neq 0$. *There exists the smallest real compact set which carries* u. *We call it the support of* u *and denote it by* $\operatorname{supp}(u)$.

Proof. See Y. Ito [16], Corollary 1 to Theorem 5.1. Q.E.D.

We remark that

$$\operatorname{Supp}(u_1 + u_2) \subset \operatorname{supp}(u_1) \cup \operatorname{supp}(u_2),$$

$$\operatorname{supp}(\lambda u) \subset \operatorname{supp}(u), \ \lambda \in \mathbf{C}.$$

Proposition 2.3. *Let* $K = \bigcup_{i=1}^{p} K_i$ *be the union of real compact sets. Let* $u \in L(\mathfrak{a}(M); E)$ *such that* $\operatorname{supp}(u) \subset K$. *Then there exists* $u_i \in L(\mathfrak{a}(M); E)$ ($i = 1, \ldots, p$) *such that*

$$u = \sum_{i=1}^{p} u_i, \quad \operatorname{supp}(u_i) \subset K_i.$$

Proof. We note first that the mapping

$$\mathfrak{a}(K) \longrightarrow \sum_{i=1}^{p} \mathfrak{a}(K_i),$$

$$f \longrightarrow (f | K_i)_{1 \leqq i \leqq p}$$

is injective and of closed range. By this mapping we can identify $\mathfrak{a}(K)$ with the nuclear closed subspace of $\sum_{i=1}^{p} \mathfrak{a}(K_i)$. Hence by virtue of A. Grothendieck [10], Chap. 2, Proposition 10, p. 69, we have the surjection

$$\prod_{i=1}^{p} L(\mathfrak{a}(K_i); E) \longrightarrow L(\mathfrak{a}(K); E),$$

$$(u_i)_{1\leqq i\leqq p} \longrightarrow \sum_{i=1}^{p} u_i. \qquad \text{Q.E.D.}$$

We now remark that the distributions with compact support valued in E are analytic linear mappings, for, by virtue of Stone-Weierstrass' theorem and Grauert's embedding theorem, the continuous injection

$$\mathfrak{a}(M) \longrightarrow C^\infty(M)$$

is of dense range. Analogously $\mathfrak{a}(\Omega)$ is dense in $C^\infty(\Omega)$.

Proposition 2.4. *Let $u \in \mathscr{E}'(M; E) \equiv L_b(\mathscr{E}(M); E)$. We denote temporarily by* $\mathrm{supp}_{\mathscr{D}'}(u)$ *its support considering it as a distribution and by* $\mathrm{supp}(u)$ *its support considering it as an analytic linear mapping. We then have*

$$\mathrm{supp}(u) = \mathrm{supp}_{\mathscr{D}'}(u).$$

Proof. Let $K = \mathrm{supp}_{\mathscr{D}'}(u)$. For any open set $\Omega \supset K$, u can be extended to $C^\infty(\Omega)$, hence to $\mathfrak{a}(\Omega)$. Consequently

$$\mathrm{supp}(u) \subset K.$$

Conversely, let $u \in \mathscr{E}'(M; E)$ such that u can be extended to $\mathfrak{a}(K)$ and let $\phi \in \mathscr{D}(M)$ such that

$$\mathrm{supp}(\phi) \cap K = \phi.$$

We must show that

$$u(\phi) = 0.$$

By virtue of the partition of unity we may assume that $\mathrm{supp}(\phi)$ is contained in a coordinate neighborhood. Hence we may consider ϕ is defined on $\mathbf{R}^n$ and may show this in the Euclidean case. This is shown as in the proof of Proposition 2.4 in Y. Ito [15]. Q.E.D.

Let now Ω be an open subset of M and K a compact subset of Ω. We call "envelope of K" (in Ω) and denote by $\tilde{K}$, the union of K and the relatively compact connected components (in Ω) of $\Omega - K$. It is again a compact set.

Proposition 2.5. *Let Ω be a relatively compact open subset of M and K a compact subset of Ω such that $K = \tilde{K}$. Then $\mathfrak{a}'(\partial\Omega; E)$ is dense in $\mathfrak{a}'(\overline{\Omega - K}; E)$.*

Proof. It is sufficient to see that the mapping of $\mathfrak{a}(\overline{\Omega - K})$ into $\mathfrak{a}(\partial\Omega)$ is injective. By the identity theorem it is sufficient to see that $\overline{\Omega - K}$ has no connected component which is disjoint from $\partial\Omega$. Let ω be such a component. Then ω is open in $\overline{\Omega - K}$. Hence

$$\omega = \Omega' \cap \overline{\Omega - K}$$

where Ω' is open in M. By definition of the closure, we have $\Omega' \cap (\Omega - K) \neq \phi$. Hence $\Omega' \cap (\Omega - K)$ is a nonempty open set and since ω is closed in $\overline{\Omega - K}$, $\omega \cap (\Omega - K)$ is closed in $\Omega - K$. If $\omega \cap \partial\Omega = \phi$, we have thus obtained a relatively compact connected component of $\Omega - K$. This is a contradiction. Q.E.D.

§3. Operations on analytic linear mappings

In this section we now define several operations on analytic linear mappings.

a) Multiplication by a holomorphic or an analytic function

Let Ω be an open set in X. For $f \in \mathcal{O}(\Omega)$ and $u \in \mathcal{O}'(\Omega; E)$, we define

$$fu \in \mathcal{O}'(\Omega; E)$$

by the formula

$$(fu)(g) = u(fg) \qquad \text{for all} \quad g \in \mathcal{O}(\Omega).$$

By this definition $\mathcal{O}'(\Omega; E)$ is an $\mathcal{O}(\Omega)$-module.

For a compact subset K of X (or M) and an open subset Ω of M, we can define an $\mathcal{O}(K)$- (resp. $\mathfrak{a}(K)$-, resp. $\mathfrak{a}(\Omega)$-) module structure of $\mathcal{O}'(K; E)$ (resp. $\mathfrak{a}'(K; E)$, resp. $\mathfrak{a}'(\Omega; E)$) in the same way.

For a real analytic linear mapping u and a real analytic function f, we have

$$\operatorname{supp}(fu) \subset \operatorname{supp}(u).$$

b) Tensor products of analytic linear mappings

We recall first the tensor product of analytic functionals.

Proposition 3.1. *Let M_i be an n_i-dimensional real analytic manifold which is countable at infinity and X_i its complexification $(i=1, 2)$. Then we have the following canonical isomorphisms*:

(i) $\mathcal{O}'(\Omega_1) \hat{\otimes} \mathcal{O}'(\Omega_2) \simeq L(\mathcal{O}(\Omega_1); \mathcal{O}'(\Omega_2)) \simeq \mathcal{O}'(\Omega_1 \times \Omega_2)$,

$(\Omega_i \subset X_i$ *open set* $(i=1, 2))$.

(ii) $\mathcal{O}'(K_1) \hat{\otimes} \mathcal{O}'(K_2) \simeq L(\mathcal{O}(K_1); \mathcal{O}'(K_2)) \simeq \mathcal{O}'(K_1 \times K_2)$,

$(K_i \subset X_i$ *compact set* $(i=1, 2))$.

(iii) $\mathfrak{a}'(K_1) \hat{\otimes} \mathfrak{a}'(K_2) \simeq L(\mathfrak{a}(K_1); \mathfrak{a}'(K_2)) \simeq \mathfrak{a}'(K_1 \times K_2)$,

$(K_i \subset M_i$ *compact set* $(i=1, 2))$.

(iv) $\mathfrak{a}'(\Omega_1)\hat{\otimes}\mathfrak{a}'(\Omega_2)\simeq L(\mathfrak{a}(\Omega_1); \mathfrak{a}'(\Omega_2))\simeq \mathfrak{a}'(\Omega_1\times\Omega_2)$,

$(\Omega_i\subset M_i$ *open set* $(i=1, 2))$.

Proof. See F. Treves [35], Proposition 50.5 and 50.7, and A. Grothendieck [10], Chap. 2, Theorem 12, and Lemmas of Proposition of § 1. Q. E. D.

Next we consider tensor products of analytic linear mappings. In the following, we assume that E_1 and E_2 be two Fréchet spaces. ω stands for ε- or π-topology in the sense of F. Treves [35].

Proposition 3.2. *Let* M_i *and* X_i *be as in Proposition* 3.1 $(i=1, 2)$. *Then we have the following canonical isomorphisms*:

(i) $\mathcal{O}'(\Omega_1; E_1)\hat{\otimes}_\omega\mathcal{O}'(\Omega_2; E_2)\simeq\mathcal{O}'(\Omega_1\times\Omega_2; E_1\hat{\otimes}_\omega E_2)$,

$(\Omega_i\subset X_i$ *open set* $(i=1, 2))$.

(ii) $\mathcal{O}'(K_1; E_1)\hat{\otimes}_\omega\mathcal{O}'(K_2; E_2)\simeq\mathcal{O}'(K_1\times K_2; E_1\hat{\otimes}_\omega E_2)$,

$(K_i\subset X_i$ *compact set* $(i=1, 2))$.

(iii) $\mathfrak{a}'(K_1; E_1)\hat{\otimes}_\omega\mathfrak{a}'(K_2; E_2)\simeq\mathfrak{a}'(K_1\times K_2; E_1\hat{\otimes}_\omega E_2)$,

$(K_i\subset M_i$ *compact set* $(i=1, 2))$.

(iv) $\mathfrak{a}'(\Omega_1; E_1)\hat{\otimes}_\omega\mathfrak{a}'(\Omega_2; E_2)\simeq\mathfrak{a}'(\Omega_1\times\Omega_2; E_1\hat{\otimes}_\omega E_2)$,

$(\Omega_i\subset M_i$ *open set* $(i=1, 2))$.

Proof. Since the tensor products of locally convex Hausdorff spaces are commutative and associative, it is sufficient to apply Propositions 2.1 and 3.1. Q. E. D.

Thus we have the following definitions of the tensor products of analytic linear mappings.

Definition 3.1. *We use the notations of Proposition* 3.2.
Let $u_i=\phi_i\otimes\mathbf{e}_i\in\mathcal{O}'(\Omega_i; E_i)$, $\phi_i\in\mathcal{O}'(\Omega_i)$, $\mathbf{e}_i\in E_i$ $(i=1, 2)$. *Then we define*

$$u_1\otimes_\omega u_2=(\phi_1\otimes\phi_2)\otimes(\mathbf{e}_1\otimes_\omega\mathbf{e}_2),$$

that is,

$$(u_1\otimes_\omega u_2)(f_1\otimes f_2)=\phi_1(f_1)\phi_2(f_2)(\mathbf{e}_1\otimes_\omega\mathbf{e}_2)$$
$$\text{for}\quad f_i\in\mathcal{O}(\Omega_i),\qquad (i=1, 2).$$

In all other cases we define the tensor products of analytic linear mappings analogously.

In all the real cases, we have

$$\mathrm{supp}(u_1 \otimes_\omega u_2) \subset \mathrm{supp}(u_1) \times \mathrm{supp}(u_2).$$

§4. Hyperfunctions valued in a Fréchet space E

We consider first hyperfunctions on a relatively compact open set in M valued in a Fréchet space E.

Let Ω be a relatively compact open subset of M. We put

$$\mathscr{B}(\Omega; E) = \mathfrak{a}'(\bar{\Omega}; E)/\mathfrak{a}'(\partial\Omega; E).$$

Definition 4.1. *The elements of $\mathscr{B}(\Omega; E)$ are called the hyperfunctions on Ω valued in a Fréchet space E or the E-valued hyperfunctions on Ω.*

Let K be a compact set containing Ω. Then we have

$$K = (K - \Omega) \cup \bar{\Omega}.$$

By virtue of Proposition 2.3, every element $u \in \mathfrak{a}'(K; E)$ can be written as follows:

$$u = u_1 + u_2,\ u_1 \in \mathfrak{a}'(K - \Omega; E) \quad \text{and } u_2 \in \mathfrak{a}'(\bar{\Omega}; E).$$

This shows that the canonical mapping:

$$\mathfrak{a}'(\bar{\Omega}; E)/\mathfrak{a}'(\partial\Omega; E) \longrightarrow \mathfrak{a}'(K; E)/\mathfrak{a}'(K - \Omega; E),$$

which is evidently injective, is also surjective. Hence, we have

$$\mathscr{B}(\Omega; E) \simeq \mathfrak{a}'(K; E)/\mathfrak{a}'(K - \Omega; E),\ K \supset \Omega.$$

Let now ω be an open set contained in Ω.

The mapping

$$\mathfrak{a}'(\bar{\Omega}; E) \longrightarrow \mathfrak{a}'(\bar{\Omega}; E)/\mathfrak{a}'(\bar{\Omega} - \omega; E)$$

defines a mapping

$$\mathscr{B}(\Omega; E) \longrightarrow \mathscr{B}(\omega; E)$$

called the restriction.

If $T \in \mathscr{B}(\Omega; E)$, we denote by $T|\omega$ its image in $\mathscr{B}(\omega; E)$. It is clear that if $\Omega_3 \subset \Omega_2 \subset \Omega_1$, and $T \in \mathscr{B}(\Omega_1; E)$, we have

$$(T|\Omega_2)|\Omega_3 = T|\Omega_3,$$

hence that the collection of $\mathscr{B}(\omega; E)$ defines a presheaf (of vector spaces) over Ω which we temporarily denote by ${}^E B|\Omega$.

Proposition 4.1. *Let Ω be a relatively compact open subset of M.*

1) *The presheaf* ${}^{E}B|\Omega$ *is a sheaf.*
2) *This sheaf is flabby.*
3) *If* K *is a compact subset of* Ω,

$$\Gamma_K(\Omega, {}^{E}B|\Omega) = \mathfrak{a}'(K; E).$$

4) *If* $F = \bigcup_{i=1}^{p} F_i$ *is a union of closed subsets of* Ω *and* $T \in \Gamma_F(\Omega, {}^{E}B|\Omega)$, *there exist* $T_i \in \Gamma_{F_i}(\Omega, {}^{E}B|\Omega)$ *such that*

$$T = \sum_{i=1}^{p} T_i.$$

5) *If* ω *is an open subset of* Ω,

$$({}^{E}B|\Omega)|\omega = {}^{E}B|\omega.$$

Proof. 1) (i) Let $\Omega = \bigcup_{i \in I} \Omega_i$ and $T \in \mathscr{B}(\Omega; E)$ such that $T|\Omega_i = 0$ for all $i \in I$. This is equivalent to say that, if $u_T \in \mathfrak{a}'(\bar{\Omega}; E)$ is a representative of T, the image of u_T in $\mathfrak{a}'(\bar{\Omega}; E)/\mathfrak{a}'(\bar{\Omega} - \Omega_i; E)$ is zero for all i. From here we have

$$\operatorname{supp}(u_T) \cap \Omega_i = \phi \quad \text{for all } i,$$

hence,

$$\operatorname{supp}(u_T) \subset \partial\Omega,$$

that is, $T=0$.

(ii) Let $\Omega = \Omega_1 \cup \Omega_2$ and $T_i \in \mathscr{B}(\Omega_i; E)$ $(i=1, 2)$ with

$$T_1|\Omega_1 \cap \Omega_2 = T_2|\Omega_1 \cap \Omega_2 = T.$$

Let $u_T \in \mathfrak{a}'(\overline{\Omega_1 \cap \Omega_2}; E)$ and $u_{T_i} \in \mathfrak{a}'(\bar{\Omega}_i; E)$ be representatives of T and T_i $(i=1, 2)$, respectively. Since

$$\operatorname{supp}(u_{T_i} - u_T) \subset \bar{\Omega}_i - \Omega_1 \cap \Omega_2$$

and since

$$\bar{\Omega}_i - \Omega_1 \cap \Omega_2 = (\overline{\Omega_i - \Omega_1 \cap \Omega_2}) \cup (\bar{\Omega}_i - \Omega_i),$$

we can, by replacing u_{T_i} with a equivalent u'_{T_i}, suppose that

$$u_{T_i} = u_T + v_i, \ \operatorname{supp}(v_i) \subset (\overline{\Omega_i - \Omega_1 \cap \Omega_2}).$$

We put

$$u'_T = u_T + v_1 + v_2 \in \mathfrak{a}'(\overline{\Omega_1 \cup \Omega_2}; E).$$

Let T' be the image of u'_T in $\mathscr{B}(\Omega_1 \cup \Omega_2; E)$. We have $T'|\Omega_i = T_i$, for

$\operatorname{supp}(u'_T - u_{T_i}) \cap \Omega_i = \operatorname{supp}(v_j) \cap \Omega_i$ (with $j \neq i$) and this set is contained in

$$(\overline{\Omega_j - \Omega_1 \cap \Omega_2}) \cap \Omega_i = \phi.$$

(iii) Let now $\Omega = \bigcup_{i \in I} \Omega_i$ and $T_i \in \mathscr{B}(\Omega_i; E)$, with

$$T_i | \Omega_i \cap \Omega_j = T_j | \Omega_i \cap \Omega_j.$$

We can suppose the covering is countable and by virtue of (ii) increasing. Thus we can suppose $\Omega_j \Subset \Omega_{j+1}$ and, since the envelope (in Ω) of a compact subset of Ω is a compact subset of Ω, we can suppose by (ii) that

$$\Omega = \bigcup_{j=0}^{\infty} \Omega_j,$$

$$\Omega_j \Subset \Omega_{j+1},$$

$$\tilde{\bar{\Omega}}_j = \bar{\Omega}_j \quad \text{(where } \tilde{\bar{\Omega}}_j \text{ is the envelope of } \bar{\Omega}_j \text{ in } \Omega\text{)},$$

$$T_j \in \mathscr{B}(\Omega_j; E), \quad T_{j+k} | \Omega_j = T_j.$$

Let $u_{T_j} \in \mathfrak{a}'(\bar{\Omega}_j; E)$ be a representative of T_j. Let d_j be a metric defining the topology of $\mathfrak{a}'(\bar{\Omega} - \Omega_j; E)$ and $v_j \in \mathfrak{a}'(\partial\Omega; E)$ such that

$$d_i(u_{T_{j+1}} - v_{j+1} - (u_{T_j} - v_j)) \leqq 2^{-j}, \quad \text{for all} \quad i \leqq j.$$

We construct v_j's by recurrence in virtue of Proposition 2.5. The sequence $u_{T_j} - v_j$ converges to an element $u_T \in \mathfrak{a}'(\bar{\Omega}; E)$. We have

$$\begin{aligned} u_T &= u_T - (u_{T_j} - v_j) + (u_{T_j} - v_j) \\ &= (u_{T_j} - v_j) + \lim_k \{u_{T_k} - v_k - (u_{T_j} - v_j)\}. \end{aligned}$$

Since the sequence

$$\{u_{T_k} - v_k - (u_{T_j} - v_j)\}_k$$

converges in $\mathfrak{a}'(\bar{\Omega} - \Omega_j; E)$,

$$u_T = u_{T_j} - v_j + w_j, \; w_j \in \mathfrak{a}'(\bar{\Omega} - \Omega_j; E).$$

Hence, we have

$$T | \Omega_j = T_j,$$

where T is the image of u_T in $\mathscr{B}(\Omega; E)$.

2) The sheaf ${}^E B | \Omega$ is flabby, for, if $\omega \subset \Omega$, $T \in \mathscr{B}(\omega; E)$, there exists $u_T \in \mathfrak{a}'(\bar{\omega}; E)$ and the image of u_T in $\mathscr{B}(\Omega; E)$ is an extension of T.

3) We have an injection if $K \subset \Omega$:

$$\mathfrak{a}'(K;\, E) \longrightarrow \mathfrak{a}'(\bar{\Omega};\, E)/\mathfrak{a}'(\partial\Omega;\, E).$$

The image of $\mathfrak{a}'(K;\, E)$ is the set of $T \in \mathscr{B}(\Omega;\, E)$ which are zero on $\Omega - K$, hence, is

$$\Gamma_K(\Omega,\, {}^E B|\Omega).$$

4) Let $\bar{F}$ and $\bar{F}_i$ be the closures of F and F_i in $\bar{\Omega}$, respectively, and u_T a representative of T in $\mathfrak{a}'(\bar{\Omega};\, E)$ so that

$$\operatorname{supp}(u_T) \subset \partial\Omega \cup \bar{F}.$$

Hence, by applying Proposition 2.3, we can suppose

$$\operatorname{supp}(u_T) \subset \bar{F}.$$

Let $u_{T_i} \in \mathfrak{a}'(\bar{F}_i;\, E)$

$$u_T = \sum_{i=1}^{p} u_{T_i}.$$

If T_i is the image of u_{T_i} in $\mathscr{B}(\Omega;\, E)$, we have

$$T = \sum_{i=1}^{p} T_i.$$

5) If $\omega' \subset \omega \subset \Omega$ are open sets, we have

$$\Gamma(\omega',\, {}^E B|\Omega) = \mathscr{B}(\omega';\, E) = \Gamma(\omega',\, {}^E B|\omega). \qquad \text{Q.E.D.}$$

Next we consider hyperfunctions on M valued in a Fréchet space E.

Let ${}^E B_1$ be the presheaf over M defined as follows:

If Ω is not relatively compact, $\mathscr{B}_1(\Omega;\, E) = \{0\}$.

If Ω is relatively compact, $\mathscr{B}_1(\Omega;\, E) = \mathscr{B}(\Omega;\, E)$.

The restrictions are defined by

$$\begin{array}{lll} \mathscr{B}_1(\Omega;\, E) \longrightarrow \mathscr{B}_1(\omega;\, E) & & \\ \qquad 0 \longrightarrow 0 & & \text{if } \Omega \text{ is not relatively compact,} \\ \qquad T \longrightarrow T|\omega & & \text{if } \Omega \text{ is relatively compact.} \end{array}$$

This presheaf satisfies the axiom (S1) of sheaves but not (S2) (cf. G. E. Bredon [2], p. 5, or R. Godement [5], p. 109).

We denote by ${}^E\mathscr{B}$ the sheaf associated to this presheaf ${}^E B_1$. It is a sheaf of vector space over $\mathbf{C}$.

Definition 4.2. *The sheaf ${}^E\mathscr{B}$ is called the sheaf of E-valued hyperfunctions over M.*

If $T \in \Gamma(\Omega, {}^E\mathscr{B}) = \mathscr{B}(\Omega; E)$, T is an E-valued hyperfunction on Ω. Hence an E-valued hyperfunction on Ω is defined by the following:

a covering $\Omega = \bigcup_{i \in I} \Omega_i$, where Ω_i's are relatively compact open sets,

$T_i \in \mathscr{B}(\Omega_i; E)$ satisfying $T_i|\Omega_i \cap \Omega_j = T_j|\Omega_i \cap \Omega_j$:

Two such couples $(\Omega_i, T_i)_{i \in I}$ and $(\Omega_{i'}, T_{i'})_{i' \in I'}$ define the same E-valued hyperfunction if

$$T_i|\Omega_i \cap \Omega_{i'} = T_{i'}|\Omega_i \cap \Omega_{i'}, \qquad \text{for all} \quad i \in I, \quad \text{and all} \quad i' \in I'.$$

Theorem 4.1. 1) *For all relatively compact open sets Ω in M, we have*

$${}^E\mathscr{B}|\Omega = {}^E B|\Omega.$$

2) *The sheaf ${}^E\mathscr{B}$ is flabby.*

3) *If K is a compact subset of M, we have*

$$\Gamma_K(M, {}^E\mathscr{B}) = \mathfrak{a}'(K; E).$$

4) *If $F = \bigcup_{i=1}^{p} F_i$ is a union of closed subsets of an open set Ω in M and if $T \in \Gamma_F(\Omega, {}^E\mathscr{B})$, there exist $T_i \in \Gamma_{F_i}(\Omega, {}^E\mathscr{B})$ with*

$$T = \sum_{i=1}^{p} T_i.$$

We write $\mathscr{B}_F(\Omega; E)$ for $\Gamma_F(\Omega, {}^E\mathscr{B})$. We write supp$(T)$ *for the support of an E-valued hyperfunction T.*

Proof. 1) is evident.

2) Let $T_0 \in \mathscr{B}(\Omega_0; E)$ and Ω_0 an open set in M. Let $\mathscr{F}$ be the family of couples (Ω, T) with

$$\Omega_0 \subset \Omega, \ T|\Omega_0 = T_0.$$

$\mathscr{F}$ is ordered and inductive for the relation

$$(\Omega, T) \prec (\Omega', T') \qquad \text{if} \quad \Omega \subset \Omega', \quad T'|\Omega = T.$$

Let (Ω, T) be a maximal element and we suppose that there exists $x_0 \notin \Omega$. Let ω be a relatively compact open set containing x_0. The E-valued hyperfunction $T|\Omega \cap \omega$ can be extended to $T_\omega \in \mathscr{B}(\omega; E)$ by virtue of Proposition 4.1. Hence there exists $S \in \mathscr{B}(\Omega \cup \omega; E)$ with

$$S|\omega = T_\omega, \quad S|\Omega = T,$$

which is a contradiction.

3) follows from 1) and Proposition 4.1.

4) For simplification of notations we suppose that $\Omega = M$ and $F = F_1 \cup F_2$. Let $\mathscr{F}$ be the family of triplets (Ω, T_1, T_2) such that

$$T_i \in \mathscr{B}_{F_i}(\Omega; E) \qquad (i=1, 2),$$

$$T_1 + T_2 = T|\Omega.$$

$\mathscr{F}$ is ordered and inductive for the relation of order of inclusion and extension.

Let (Ω, T_1, T_2) be a maximal element and suppose that there exists $x_0 \notin \Omega$. Let ω be a relatively compact open set containing x_0.

Let $T_i|\Omega \cap \omega \in \mathscr{B}_{F_i}(\Omega \cap \omega; E)$ can be extended to $T'_i \in \mathscr{B}_{\overline{F_i \cap \Omega}}(\omega; E)$ and

$$T|\omega - T'_1 - T'_2 \in \mathscr{B}_{\{F_1 \cup F_2 - (F_1 \cup F_2) \cap \Omega\}}(\omega; E).$$

Hence, by virtue of Proposition 4.1, there exist $S_i \in \mathscr{B}_{F_i - F_i \cap \Omega}(\omega; E)$ such that

$$T|\omega = T'_1 + T'_2 + S_1 + S_2.$$

Since $(T'_i + S_i)|\Omega \cap \omega = T_i|\Omega \cap \omega$, there exist $T''_i \in \mathscr{B}(\Omega \cup \omega; E)$ such that

$$T''_i|\Omega = T_i, \ T''_i|\omega = T'_i + S_i.$$

Hence we have

$$T''_i \in \mathscr{B}_{F_i}(\Omega \cup \omega; E) \text{ and}$$

$$T|\Omega \cup \omega = T''_1 + T''_2,$$

which is a contradiction.

Theorem 4.2. *The sheaf* ${}^E\mathscr{D}'$ *of E-valued distributions over M is a subsheaf of* ${}^E\mathscr{B}$.

Proof. Let Ω be an open set in M. We define the mapping

$$\mathscr{D}'(\Omega; E) \longrightarrow \mathscr{B}(\Omega; E)$$

in the following way where we put $\mathscr{D}'(\Omega; E) \equiv L_b(\mathscr{D}(\Omega); E)$. Let Ω_j be a sequence of open sets with

$$\Omega_j \Subset \Omega_{j+1}, \quad \bigcup_{j=0}^{\infty} \Omega_j = \Omega.$$

Let $\phi_j \in \mathscr{D}(\Omega_{j+1})$ and $\phi_j = 1$ in a neighborhood of $\bar{\Omega}_j$. Let $T \in \mathscr{D}'(\Omega; E)$ and put $T_j = \phi_j T$. Then $T_j \in \mathscr{E}'(\Omega; E)$, hence $T_j \in \mathfrak{a}'(\Omega; E)$ and $T_j|\Omega_j \in \mathscr{B}(\Omega_j; E)$, where we denote by $T_j|\Omega_j$ the image of $T_j \in \mathfrak{a}'(\bar{\Omega}_j; E)$ in $\mathscr{B}(\Omega_j; E)$. If $k > j$,

$$T_k - T_j \in \mathscr{E}'(\Omega_{k+1} - \bar{\Omega}_j).$$

Hence $\operatorname{supp}(T_k - T_j) \cap \Omega_j = \phi$ and

$$T_k|\Omega_j = T_j|\Omega_j \qquad \text{in} \quad \mathscr{B}(\Omega_j; E).$$

The sequence $T_j|\Omega_j$ defines an E-valued hyperfunction $T' \in \mathscr{B}(\Omega; E)$. It is easy to verify that T' is independent of choices of $\{(\Omega_j, \phi_j)\}$ and that we have thus constructed a linear mapping of $\mathscr{D}'(\Omega; E)$ into $\mathscr{B}(\Omega; E)$ which commutes with restrictions.

If $T\in \mathscr{D}'(\Omega; E)$ is of image zero, it is equivalent to say that for all n

$$\operatorname{supp}(\phi_j T) \cap \Omega_j = \phi.$$

Hence, by virtue of Proposition 2.4, the restriction of T to $\mathscr{D}'(\Omega_j; E)$ is zero. Hence $T=0$. Q.E.D.

§5. Operations on hyperfunctions valued in a Fréchet space E

In this section we define several operations on E-valued hyperfunctions on M.

a) Multiplication by a real analytic function

Let Ω be an open set in M. If $f\in \mathfrak{a}(\Omega)$ and $T\in \mathscr{B}(\Omega; E)$ and $\{\Omega_j\}_{j=0}^{\infty}$ be an open covering of Ω with $\Omega_j \Subset \Omega_{j+1}$, we shall define fT as follows. Let $u_{T_j}\in \mathfrak{a}'(\bar{\Omega}_j; E)$ such that

$$u_{T_j}|\Omega_j = T|\Omega_j = T_j,$$

where $u_{T_j}|\Omega_j$ denotes the image of u_{T_j} in $\mathscr{B}(\Omega_j; E)$. Since $\mathfrak{a}'(\bar{\Omega}_j; E)$ is an $\mathfrak{a}(\bar{\Omega}_j)$-module, we have

$$f u_{T_{j+k}}|\Omega_j = f u_{T_j}|\Omega_j, \qquad \text{for} \quad k\geqq 0.$$

Hence, $f u_{T_j}|\Omega_j$'s define an E-valued hyperfunction which depends only on f and on T and which we denote by fT.

We have verified that we thus define on $\mathscr{B}(\Omega; E)$ a structure of $\mathfrak{a}(\Omega)$-module and at the same time that the sheaf ${}^{E}\mathscr{B}$ is an $\mathfrak{a}$-module.

b) Tensor product of E-valued hyperfunctions

Let now E_1 and E_2 be two Fréchet spaces, and ω stands for ε- or π-topology. Let then Ω_i be an open set in an n_i-dimensional real analytic manifold M_i which is countable at infinity ($i=1, 2$). Let $T_1\in \mathscr{B}(\Omega_1; E_1)$ and $T_2\in \mathscr{B}(\Omega_2; E_2)$. Let

$$\Omega_1 = \bigcup_{j=1}^{\infty} \Omega_{1j} \quad \text{and} \quad \Omega_2 = \bigcup_{j=1}^{\infty} \Omega_{2j}$$

with

$$\Omega_{1j} \Subset \Omega_{1j+1} \quad \text{and} \quad \Omega_{2j} \Subset \Omega_{2j+1},$$

$$u_{T_{1j}} \in \mathfrak{a}'(\bar{\Omega}_{1j}; E_1), \quad u_{T_{1j}}|\Omega_{1j} = T|\Omega_{1j},$$

$$u_{T_{2j}} \in \mathfrak{a}'(\bar{\Omega}_{2j}; E_2), \quad u_{T_{2j}}|\Omega_{2j} = T|\Omega_{2j}.$$

We have

$$(u_{T_{1j+k}} \otimes_\omega u_{T_{2j+k}}) | \Omega_{1j} \times \Omega_{2j}$$
$$= (u_{T_{1j}} \otimes_\omega u_{T_{2j}}) | \Omega_{1j} \times \Omega_{2j}$$

and the sequence $(u_{T_{1j}} \otimes_\omega u_{T_{2j}}) | \Omega_{1j} \times \Omega_{2j}$ defines a hyperfunction on $\Omega_1 \times \Omega_2$ which is $T_1 \otimes_\omega T_2$.

We can verify that this product has properties of tensor products of vector valued distributions and extends them. In particular we have

$$\operatorname{supp}(T_1 \otimes_\omega T_2) \subset \operatorname{supp}(T_1) \times \operatorname{supp}(T_2).$$

c) Image of an E-valued hyperfunction by an analytic isomorphism

Let Ω_1 and Ω_2 be open sets in M and y an analytic diffeomorphism of Ω_1 onto Ω_2;

$$y: \Omega_1 \longrightarrow \Omega_2.$$

If $u \in \mathfrak{a}'(\Omega_2; E)$, we define

$$u \circ y \in \mathfrak{a}'(\Omega_1; E)$$

by the formula

$$(u \circ y)(f) = u((f \circ y^{-1}) |J|), \qquad \text{for} \quad f \in \mathfrak{a}(\Omega_1),$$

where $|J|$ is the Jacobian of the mapping y^{-1}. The mapping thus defined

$$y^*: \mathfrak{a}'(\Omega_2; E) \longrightarrow \mathfrak{a}'(\Omega_1; E)$$

is linear and verifies

$$\operatorname{supp}(y^* u) = y^{-1}(\operatorname{supp}(u)) \qquad \text{for} \quad u \in \mathfrak{a}'(\Omega_2; E).$$

Hence y^* can be prolonged to a morphism of sheaves

$$y^*: {}^E\mathscr{B} | \Omega_2 \longrightarrow {}^E\mathscr{B} | \Omega_1.$$

§6. Boundary values of holomorphic functions valued in a Fréchet space E

1. The resolution of the sheaf of holomorphic functions valued in a Fréchet space E

Let M be an n-dimensional real analytic manifold which is countable at infinity, and X its complexification which is in turn considered as a $2n$-dimensional real analytic manifold.

Let Ω be an open set in X. Let ${}^E\mathscr{F}$ be one of the sheaves ${}^E\mathfrak{a}_X$, ${}^E\mathscr{E}_X$, ${}^E\mathscr{D}'_X$, and ${}^{E}\mathscr{B}_X$ over X.

A differential form f with coefficients in $\mathscr{F}(\Omega; E)$ is called of type (p, q) if, in every coordinate neighborhood U, we can write it as follows:

$$f=\sum_{|I|=p}{}' \sum_{|J|=q}{}' f_{I,J}\omega^I \wedge \bar{\omega}^J,$$

where, $\omega^1,\dots, \omega^n$ being n forms of type (1, 0) with C^∞ coefficients in U such that at every points in U

$$<\omega^j, \omega^k> = \delta_{jk}, \quad j, k=1,\dots, n,$$

we put, for $I=(i_1,\dots, i_p)$ and $J=(j_1,\dots, j_q)$,

$$\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}, \quad \text{and}$$

$$\bar{\omega}^J = \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q},$$

$f_{I,J} \in \mathscr{F}(U; E)$ is assumed to be antisymmetric both in I and in J and $\sum'$ means that summation is extended only over increasing multi-indices. (cf. L. Hörmander [13], p. 112.) We then define the sheaf ${}^{E}\mathscr{F}^{p,q}$ of differential forms of type (p, q) with coefficients in ${}^{E}\mathscr{F}$ and ∂ and $\bar{\partial}$ are the morphisms of sheaves:

$$\partial\colon {}^{E}\mathscr{F}^{p,q} \longrightarrow {}^{E}\mathscr{F}^{p+1,q},$$

$$\bar{\partial}\colon {}^{E}\mathscr{F}^{p,q} \longrightarrow {}^{E}\mathscr{F}^{p,q+1}$$

defined in U by, for the above f,

$$\partial f=\sum_{i=1}^{n} \sum_{|I|=p}{}' \sum_{|J|=q}{}' \partial f_{I,J}/\partial\omega^i \omega^i \wedge \omega^I \wedge \bar{\omega}^J + \cdots,$$

$$\bar{\partial} f=\sum_{i=1}^{n} \sum_{|I|=p}{}' \sum_{|J|=q}{}' \partial f_{I,J}/\partial\bar{\omega}^i \bar{\omega}^i \wedge \omega^I \wedge \bar{\omega}^J + \cdots$$

where the dots indicate terms in which no $f_{I,J}$ is differentiated.

We define in the same way the sheaf ${}^{E}\mathscr{O}^p = {}^{E}\mathscr{O}^p_X$ of differential forms of type $(p, 0)$ with coefficients in ${}^{E}\mathscr{O}_X$. We then have a complex of sheaves:

$$0 \longrightarrow {}^{E}\mathscr{O}^p \longrightarrow {}^{E}\mathscr{F}^{p,0} \xrightarrow{\bar{\partial}} {}^{E}\mathscr{F}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} {}^{E}\mathscr{F}^{p,n} \longrightarrow 0,$$

for $\bar{\partial}\circ\bar{\partial}=0$ and, if $f \in {}^{E}\mathscr{F}^{p,0}(\Omega)$ has holomorphic coefficients, we have $\bar{\partial} f=0$.

If ${}^{E}\mathscr{F}$ is one of the sheaves ${}^{E}\mathscr{E}$ or ${}^{E}\mathscr{D}'$, it is well known that the complex is an exact sequences of sheaves, hence a resolution of ${}^{E}\mathscr{O}^p$ (cf. L. Hörmander [13], and P. D. F. Ion and T. Kawai [14]). If ${}^{E}\mathscr{F}={}^{E}\mathscr{E}$, it is the resolution of E-Dolbeault-Grothendieck.

If ${}^{E}\mathscr{F}={}^{E}\mathfrak{a}$, the complex is again a resolution of ${}^{E}\mathscr{O}^p$. In order to see this, we have only to take into account the resolution of $\mathscr{O}^p$ by the sheaves $\mathfrak{a}^{p,q}$ and the argument of P. D. F. Ion and T. Kawai [14].

Theorem 6.1. *The sequence*

$$0 \longrightarrow {}^{E}\mathcal{O}^{p} \longrightarrow {}^{E}\mathcal{B}^{p,0} \xrightarrow{\bar{\partial}} {}^{E}\mathcal{B}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} {}^{E}\mathcal{B}^{p,n} \longrightarrow 0$$

is an exact sequence of sheaves over X.

Proof. The argument is local. So that we have only to show this in a coordinate neighborhood of every point of X. So we have only to show this in the Euclidean space. This has been done in Y. Ito [15], Theorem 6.2. Q.E.D.

2. Sato's theory

a) Cohomology groups with coefficients in the sheaf ${}^{E}\mathfrak{a}$

Let ${}^{E}\mathfrak{a}$ be the sheaf of E-valued real analytic functions over M and ${}^{E}\mathcal{O}$ the sheaf of E-valued holomorphic functions over X, the complexification of M. If $x \in M$, we have an isomorphism

$${}^{E}\mathfrak{a}_{x} \simeq {}^{E}\mathcal{O}_{x}.$$

Hence, for all open subset Ω of M, we have

$${}^{E}\mathfrak{a}|\Omega = {}^{E}\mathcal{O}|\Omega.$$

Since every open set in X is paracompact, it follows from Theorem B42 of P. Schapira [29], p. 38, that

$$\mathfrak{a}(\Omega;\mathrm{E}) = \varinjlim_{\tilde{\Omega}\cap M=\Omega} \mathcal{O}(\tilde{\Omega}; E),$$

where $\tilde{\Omega}$ is an open neighborhood in X of an open set Ω in M such that $\tilde{\Omega} \cap M = \Omega$ and $\mathfrak{a}(\Omega; E)$ is the section module of ${}^{E}\mathfrak{a}$ on Ω and $\mathcal{O}(\tilde{\Omega}; E)$ is the section module of ${}^{E}\mathcal{O}$ on $\tilde{\Omega}$.

In the following of this section, $\tilde{\Omega}$ always denotes an open set in X and Ω denotes an open set in M as far as the contrary is not explicitely mentioned.

Theorem 6.2. *Let Ω be an arbitrary open set in M. Then we have*

$$H^{p}(\Omega, {}^{E}\mathfrak{a}) = 0$$

for every positive integer p.

Proof. We know, by virtue of Grauert's Theorem (cf. H. Grauert [6] or H. Komatsu [21], Theorem V. 2.5, p. 194), that Ω has a fundamental system of Stein neighborhoods. Then, it follows, from Oka-Cartan Theorem B (cf. P. D. F. Ion and T. Kawai [14], E. Bishop [1], and L. Bungart [3]) and Theorem B42 of P. Schapira [29], p. 38, that, for $p>0$, we have

$$H^p(\Omega, {}^E\alpha) = \varinjlim_{\tilde{\Omega} \cap M = \Omega} H^p(\tilde{\Omega}, {}^E\mathcal{O}) = 0. \qquad \text{Q.E.D.}$$

b) Malgrange's Theorem

Theorem 6.3 (Malgrange's Theorem). *Let $\tilde{\Omega}$ be an open set in X and F a closed subset of $\tilde{\Omega}$. Then we have*

(i) $H^p_F(\tilde{\Omega}, {}^E\mathcal{O}) = 0$, *for* $p > n$.

(ii) $H^p(\tilde{\Omega}, {}^E\mathcal{O}) = 0$, *for* $p \geqq n$.

Proof. (i) By virtue of Theorem 6.1, and Theorem B32 of P. Schapira [29], p. 27, the cohomology group $H^p_F(\tilde{\Omega}, {}^E\mathcal{O})$ is isomorphic to the p-th cohomology group of the complex:

$$0 \longrightarrow \mathscr{B}^{0,0}_F(\tilde{\Omega}; E) \xrightarrow{\bar{\partial}} \mathscr{B}^{0,1}_F(\tilde{\Omega}; E) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{B}^{0,n}_F(\tilde{\Omega}; E) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots.$$

Hence, for $p > n$, this cohomology group is zero.

(ii) We apply this result to a Stein neighborhood V of $\tilde{\Omega}$ with $F = V - \tilde{\Omega}$ and have

$$H^p_{V-\tilde{\Omega}}(V, {}^E\mathcal{O}) = 0, \qquad \text{for} \quad p > n.$$

We write the exact sequence of cohomology groups with support in $V - \tilde{\Omega}$ (cf. P. Schapira [29], Corollary 1 of Theorem B35, p. 32):

$$\cdots \longrightarrow H^p(V, {}^E\mathcal{O}) \longrightarrow H^p(\tilde{\Omega}, {}^E\mathcal{O}) \longrightarrow H^p_{V-\tilde{\Omega}}(V, {}^E\mathcal{O}) \longrightarrow H^{p+1}(V, {}^E\mathcal{O}) \longrightarrow \cdots.$$

The theorem then follows from the fact that

$$H^p(V, {}^E\mathcal{O}) = 0, \qquad \text{for} \quad p > 0. \qquad \text{Q.E.D.}$$

c) Cohomology groups with support in a compact subset of X. (Martineau-Harvey's Theorem)

Theorem 6.4. *Let K be a compact subset of X such that it admits a Stein neighborhood V and satisfies*

$$H^p(K, \mathcal{O}) = 0, \qquad \text{for} \quad p > 0.$$

Then we have

$$H^p_K(V, {}^E\mathcal{O}) = 0, \qquad \text{for} \quad p \neq n,$$

and isomorphisms

$$H^n_K(V, {}^E\mathcal{O}) \simeq H^{n-1}(V-K, {}^E\mathcal{O}) \simeq \mathcal{O}'(K; E).$$

Further, if $K_1 \subset K_2$ *satisfy the hypotheses of this theorem, the diagram*

$$\begin{array}{ccc} H^n_{K_1}(V, {}^E\mathcal{O}) & \longrightarrow & H^n_{K_2}(V, {}^E\mathcal{O}) \\ \downarrow & & \downarrow \\ \mathcal{O}'(K_1; E) & \longrightarrow & \mathcal{O}'(K_2; E) \end{array}$$

is commutative.

Proof. See Y. Ito [16], Theorem 7.1. Q. E. D.

d) The relative cohomology groups with support in M. (Sato's Theorem)

Theorem 6.5. *Let Ω be an open subset of M and V a Stein neighborhood in X of M.*

(i) *The relative cohomology groups $H^p_\Omega(V, {}^E\mathcal{O})$ are zero for $p \neq n$.*

(ii) *The presheaf over M*

$$\Omega \longrightarrow H^n_\Omega(V, {}^E\mathcal{O})$$

is a sheaf.

(iii) *This sheaf is isomorphic to the sheaf ${}^E\mathcal{B}$ of E-valued hyperfunctions over M.*

Proof. (i), (ii): Let Ω be a relatively compact open subset of M. We have the exact sequence

$$\cdots \longrightarrow H^p_{\partial\Omega}(V, {}^E\mathcal{O}) \longrightarrow H^p_{\bar{\Omega}}(V, {}^E\mathcal{O}) \longrightarrow H^p_\Omega(V, {}^E\mathcal{O})$$
$$\longrightarrow H^{p+1}_{\partial\Omega}(V, {}^E\mathcal{O}) \longrightarrow \cdots$$

(cf. H. Komatsu [21], Theorem II. 3.2, p. 77, or P. Schapira [29], Theorem B.35, p. 31). Since $\bar{\Omega}$ and $\partial\Omega$ are real compact sets which consequently satisfy the hypotheses of Theorem 6.4, we have

$$H^p_\Omega(V, {}^E\mathcal{O}) = 0, \qquad p < n-1,$$

and we have the exact sequence

$$0 \longrightarrow H^{n-1}_\Omega(V, {}^E\mathcal{O}) \longrightarrow H^n_{\partial\Omega}(V, {}^E\mathcal{O})$$
$$\longrightarrow H^n_{\bar{\Omega}}(V, {}^E\mathcal{O}) \longrightarrow H^n_\Omega(V, {}^E\mathcal{O}) \longrightarrow 0.$$

Since the morphism

$$H^n_{\partial\Omega}(V, {}^E\mathcal{O}) \longrightarrow H^n_{\bar{\Omega}}(V, {}^E\mathcal{O})$$

is isomorphic to the morphism

$$\mathfrak{a}'(\partial\Omega; E) \longrightarrow \mathfrak{a}'(\bar{\Omega}; E)$$

which is injective, we have

$$H^{n-1}_{\Omega}(V, {}^E\mathcal{O})=0.$$

Then we consider the sheaves over M:

$$\mathcal{H}^p_M({}^E\mathcal{O})$$

associated with the presheaves

$$\Omega \longrightarrow H^p_{\Omega}(V, {}^E\mathcal{O}).$$

These sheaves are zero for $p<n$ and since $H^p_{\Omega}(V, {}^E\mathcal{O})=0$, if $p>n$, by virtue of Theorem 6.3. The parts (i) and (ii) of the theorem follow from Theorem II.3.18 of H. Komatsu [21], p. 89, or Theorem B 36 of P. Schapira [29], p. 34.

(iii) Let Ω be an open subset of M. From the exact sequence

$$0 \longrightarrow H^n_{\partial\Omega}(V, {}^E\mathcal{O}) \longrightarrow H^n_{\bar{\Omega}}(V, {}^E\mathcal{O}) \longrightarrow H^n_{\Omega}(V, {}^E\mathcal{O}) \longrightarrow 0,$$

we deduce that the sheaf

$$\mathcal{H}^n_M({}^E\mathcal{O})$$

is flabby.

If K is a real compact set, the relative cohomology group $H^n_K(V, {}^E\mathcal{O})$ is isomorphic to $\mathfrak{a}'(K; E)$ by virtue of Theorem 6.4. Hence for all relatively compact open set Ω, the relative cohomology groups

$$H^n_{\Omega}(V, {}^E\mathcal{O})=H^n_{\bar{\Omega}}(V, {}^E\mathcal{O})/H^n_{\partial\Omega}(V, {}^E\mathcal{O})$$

and

$$\mathcal{B}(\Omega; E)=\mathfrak{a}'(\bar{\Omega}; E)/\mathfrak{a}'(\partial\Omega; E)$$

are isomorphic. Consequently the sheaves

$$\mathcal{H}^n_M({}^E\mathcal{O}) \quad \text{and} \quad {}^E\mathcal{B}$$

are isomorphic. Q.E.D.

Proposition. *Let Ω be an open subset of M and V a Stein neighborhood in X of M. Then we have*

$$H^n_{\Omega}(V, {}^E\mathcal{O})\simeq H^{n-1}(V-\Omega, {}^E\mathcal{O}).$$

Proof. This is an immediate consequence of Oka-Cartan Theorem B and the canonical long exact sequence of relative cohomology groups,

$$0=H^{n-1}(V, {}^E\mathcal{O}) \longrightarrow H^{n-1}(V-\Omega, {}^E\mathcal{O})$$
$$\longrightarrow H^n_\Omega(V, {}^E\mathcal{O}) \longrightarrow H^n(V, {}^E\mathcal{O})=0. \qquad \text{Q.E.D.}$$

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