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Theory of Analytic Linear Mappings, I. General Theory

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Contents

§ 0. Introduction

In this paper we study the general theory of analytic linear mappings defined on a complex analytic manifold valued in a locally convex Hausdorff topological vector space. Analytic linear mappings are by definition continuous linear mappings of the space of holomorphic functions into a locally convex Hausdorff topological vector space. That is, analytic linear mappings are so to speak "vector valued analytic functionals".

Up to now, the theory of analytic linear mappings has been studied by many authors in the special situations. See, for example, J. S. Silva [49], G. Köthe [28], A. Grothendieck [10], A. Martineau [35], P. Lelong [30], M. Morimoto [37], Y. Ito [18] and so on. Especially, the theory of analytic functionals has been developed extensively.

Meanwhile, the concept of distributions has been introduced by L. Schwartz [48] as the dual object of the function. That is, the space of distributions on $\boldsymbol{R}^n$ is nothing else but the dual space of the space of indefinitely differentiable functions with compact support. This gives the generalization of the concept of functions. This has been generalized into many directions. The dual object of the function space has produced many things newly. Among them, there is the analytic functional, a special form of analytic linear mappings, as a dual object of holomorphic functions.

In another direction of the generalization of the concept of functions, M. Sato [40] has introduced the concept of hyperfunctions as boundary values of holomorphic functions. Many authors follow him, for example, A. Martineau [33], F. Harvey [15], H. Komatsu [25], P. Schapira [42], T. Kawai [21], M. Morimoto [37] and so on. Among them, A. Martineau reformulated the theory of hyperfunctions starting from the concept of analytic functionals. He showed that a hyperfunction is a class of analytic functionals, and especially, the hyperfunction with compact support is nothing else but an analytic functional with compact support.

In one another direction of generalizations of L. Schwartz's theory, there is a theory of vector valued distributions which was initiated by L. Schwartz [45], [46]. The space of vector valued distributions is the space of continuous linear mappings of the space of indefinitely differentiable functions with compact support into a locally convex Hausdorff topological vector space. Here again, the space of continuous linear mappings of a function space into a locally convex Hausdorff topological vector space is seen to be meaningful. The author has in this point studied the theory of analytic linear mappings on the Euclidean space and has used it to reformulate the theory of vector valued hyperfunctions initiated by P. D. F. Ion and T. Kawai [17] as A. Martineau did in the scalar case, see [18], [54]. So it is seen to be worthwhile to study the theory of analytic linear mappings in the general situation. So we will do this in this paper.

§ 1. Preliminaries: Spaces of holomorphic functions

In this section, we mention some preliminary facts about spaces of holomorphic functions on a complex analytic manifold, following A. Martineau [35].

Let V be a complex analytic manifold, not necessarily connected. $H(V)$ denotes always in the following the vector space of holomorphic functions on V. If A is a subset of V, $H_A(V)$ denotes the inductive limit of vector spaces $H(W)$, W running over the family $\mathscr{V}(A)$ of open sets containing A, $W_1 \leqq W_2$ if $W_1 \subset W_2$, the mappings $i_{W_2,W_1}: H(W_2) \to H(W_1)$ being the restriction mappings. An element of $H_A(V)$ is called a holomorphic function on V defined in a neighborhood of A. It is convenient to write:

$$H_A(V) = \bigcup_{W \in \mathscr{V}(A)} H(W).$$

Remark. Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on V supposed to be countable at infinity, $\mathcal{O} \mid A$ the restriction of $\mathcal{O}$ to A. By virtue of Theorem 3.3.1 of R. Godement [8], and Lemma II.2.13 of H. Komatsu [25], all section of $\mathcal{O} \mid A$ can be prolonged to a section of $\mathcal{O}$ on a neighborhood of A, hence we have

$$H_A(V) \cong \Gamma(A, \mathcal{O} \mid A).$$

If B is another subset of V, $B \subset A$, there exists a homomorphism of $H_A(V)$ into $H_B(V)$ which we denote by $i_{A,B}$, obtained as follows: if $W_1 \in \mathscr{V}(A)$, $W_2 \in \mathscr{V}(B)$, then $W_3 = W_1 \cap W_2 \in \mathscr{V}(B)$, hence there exists $i_{W_1,W_3}: H(W_1) \to H(W_3)$. W_3's form a family cofinal in $\mathscr{V}(B)$, hence by passing to the inductive limit we obtain $i_{A,B}$ which we call homomorphism of restriction.

Let ϕ be a family of subsets of A such that

1) The X's of ϕ cover A,

2) If $B \in \phi$, $C \in \phi$, then $B \cup C \in \phi$.

Then with the homomorphisms of restriction $i_{B,C}$ if $B \supset C$, the $H_B(V)$'s form a projective system. If ϕ contains all the convergent sequences of A, we can easily verify that the projective limit of $H_B(V)$, $\varprojlim_{B \in \phi} H_B(V)$, is isomorphic to $H_A(V)$ by the homomorphism which we obtain by passing to the projective limit in B over the homomorphisms $i_{A,B}$. We then identify $\varprojlim_{B \in \phi} H_B(V)$ with $H_A(V)$.

This can be shown as follows. For all x, $x \in A$, there exists a neighborhood $\omega(x)$ of x such that $f_x \in H_A(V)$ can be prolonged analytically in $\omega(x)$ to f^x, $\sup_{z \in \omega(x)} |f^x(z)| \leqq M_x$, and such that, for all $y \in \omega(x) \cap A$ we have $f_y = (f^x)_y$. In fact, suppose that it is not so and thus implies the existence of a sequence of points x_n tending to x, such that $f_{x_n} \neq (f^x)_{x_n}$. We consider $L = \{x\} \cup (\bigcup_n \{x_n\})$. It is a compact subset of A. Hence, since $H_L(V) = \varinjlim_{W \in \mathscr{V}(L)} H(W)$, f is prolongable to a neighborhood ω_L of L, which is in particular a neighborhood of the point x. Hence it contains a simply connected neighborhood Ω_x of this point. For sufficiently large n, $x_n \in \Omega_x$ from which $f_{x_n} = (f^x)_{x_n}$, which is the contradiction.

We can now recollect f^y's.

We now assume that the manifold V is countable at infinity. Hence we can equip it with a metric d. For all x, $x \in V$, there exist a neighborhood of x, $\pi(x)$, a space $\boldsymbol{C}^{N(x)}$, an open set $\omega_1(x)$ of $\boldsymbol{C}^{N(x)}$ and an isomorphism $i(x)$ of $\pi(x)$ onto $\omega_1(x)$.

We can find a subfamily X of V such that the $\pi(x)$'s, $x \in X$, form a locally finite covering of V. A set C included in $\pi(x)$ will be said to be convex in the chart if $i(x)(C)$ is convex in $\omega_1(x)$. We denote by $B(r, y)$ the set of points z of V such that $d(z, y)\leqq r$. For all y, $y\in V$, there exists a $\rho_1(y)$ such that $B(\rho_1(y), y)$ is included in each $\pi(x)$ containing y and convex in each of these charts. If $z\in A$, we will take $\rho(z)$ such that $\rho(z)\leqq\rho_1(z)$ and such that $B(\rho(z), z)\subset\pi(z)$. We set $\pi'(z)=B((1/2)\rho(z), z)$. Then let $\pi'=\bigcup_{x\in A}\pi'(x)$. The function f is prolongable to π'. In fact, if f^y and f^z denote the prolongations of germs f_y and f_z to $\pi(y)$ and $\pi(z)$, and if $u\in\pi(y)\cap\pi'(z)$, being placed in a chart containing y, hence containing z, the value at u of f^y can be obtained by following the "segment" $i(x)^{-1}([i(x)(y), i(x)(z)])$ then the "segment" $i(x)^{-1}([i(x)(z), i(x)(u)])$. The paths (y, z), (z, u) are in $B(\rho(y), y)$ and the path (z, u) is in $\pi'(z)$. Hence we have $f^y(u)=f^z(u)$, which we wish to show.

If ψ is a family of subsets of A whose finite unions form a family ϕ it is convenient to write

$$H_A(V)=\bigcap_{B\in\psi} H_B(V).$$

In particular we have the formula

$$H_A(V)=\bigcap_{x\in A} H_{\{x\}}(V).$$

If ω_i is a covering of a manifold V by open subsets, we have thus

$$H(V)=\bigcap_{i\in I} H(\omega_i).$$

Equipping each $H(\omega_i)$ with the topology of the uniform convergence on all compact subset, we can easily verify that the topology of $H(V)$ is the topology of the projective limit of $H(\omega_i)$. Further in the case where V is countable at infinity this follows from Theorem 1 of A. Martineau [35], p. 5.

Proposition. *Let K be a compact subset of V. $H_K(V)$ admits a topology of space* (DFS).

Proof. See A. Grothendieck [11] and A. Martineau [35], p. 8. Q. E. D.

This topology is that of the inductive limit of $H(W)$, W running over the family of relatively compact open neighborhoods of K. We will write $H_K(V) = \varinjlim_{W\in\mathscr{V}(K)} H(W)$. Let now A be a certain subset of V. We have

$$H_A(V)=\bigcup_{W\in\mathscr{V}(A)} H(W)=\bigcap_{K\in\psi} H_K(V),$$

where ψ denotes a family of compact subsets containing all the convergent sequences of A. We can hence equip A with diverse topologies (being able to be all identical).

We denote by $H_{P,\psi,A}(V)$ the vector space $H_A(V)$ equipped with the topology induced from the product topology:

$\prod_{K \in \psi} H_K(V)$ onto its closed subspace $\bigcap_{K \in \psi} H_K(V)$.

This space is always a complete Schwartz space. We can hence apply Theorem 2 of A. Martineau [35], p. 6 to it. We denote by $H_{P,A}(V)$ the space obtained by taking for family ψ the family of all the compact subsets of A.

We denote then by $H_{I,A}(V)$ the space $\varinjlim_{W \in \mathscr{V}(A)} H(W)$. The latter space is ultra-bornologic, and its topology is finer than that of $H_{P,A}(V)$. We possibly have $H_{P,A} \neq H_{I,A}(V)$ even when A is closed in V.

§ 2. General notions on the analytic linear mappings

1. Definitions

In the following of this paper, E is always assumed to be an arbitrary locally convex Hausdorff topological vector space over the complex number field as far as the contrary is not mentioned.

Definition 2.1. *Let V be a complex analytic manifold which is countable at infinity. We call all the elements of the space $H'(V; E) \equiv L_b(H(V); E)$ of continuous linear mappings of $H(V)$ into E equipped with the topology of bounded convergence "analytic linear mappings" defined on V valued in a locally convex Hausdorff topological vector space E or shortly (E-valued) analytic linear mappings on V.*

Analytic linear mappings defined on V valued in a complex one dimensional Euclidean space $\boldsymbol{C}$ are nothing else but analytic functionals defined on V. Thus, E-valued analytic linear mappings on V are so to speak "E-valued analytic functionals" on V.

We define multiplication of an analytic linear mapping T on V and a holomorphic function f on V by the following formula:

$$(fT)(g) = T(fg) \qquad \text{for all} \quad g \in H(V).$$

Then the space $H'(V; E)$ of all E-valued analytic linear mappings on V becomes an $H(V)$-module.

Proposition 2.1. *Let V be as in Definition 2.1, and E a complete locally convex Hausdorff topological vector space. Then we have the isomorphism*

$$H'(V; E) \cong H'(V) \hat{\otimes} E.$$

Proof. Since the space of analytic functionals on V, $H'(V)$, is a (DFS) nuclear space, it follows from F. Trèves [52], Proposition 50.5, p. 522. Q. E. D.

Let μ be an E-valued measure with compact support in V, that is, an element of the space of continuous linear mappings of $C(V)$ into E, $L_b(C(V); E) \equiv C'(V; E)$, where $C(V)$ is the space of all continuous functions on V equipped with the topology of compact convergence. The vector space $C'(V; E)$ is equipped with the topology of bounded convergence. Then the mapping

$$f \longrightarrow \int_V f d\mu, \quad f \in H(V)$$

defines an E-valued analytic linear mapping which we denote by $T(\mu)$.

Definition 2.2. *We say that the E-valued measure μ with compact support defined on V represents an analytic linear mapping T, or that T is representable by μ, if, for all $f \in H(V)$, we have*

$$T(f) = \int_V f d\mu.$$

We then write μ_T, hence $T(\mu_T) = T$; μ_T is not, in any way, unique. But all the E-valued analytic linear mappings are representable by E-valued measures with compact support. Namely, we have

Proposition 2.2. *Let E and V be as in Definition* 2.1. *Further assume that E is quasi-complete. Then all E-valued analytic linear mapping defined on V is representable by an E-valued measure with compact support.*

Proof. Since $H(V)$ is a nuclear subspace of $C(V)$, it follows immediately from Proposition 10 of Chapter 2, § 3, n°1 of A. Grothendieck [13]. Q. E. D.

Replacing $C(V)$ by $\mathscr{E}(V)$, the space, defined by L. Schwartz [48], of indefinitely differentiable functions on V, $H(V)$ is again a closed nuclear subspace of $\mathscr{E}(V)$. Thus, $H'(V; E)$ is the quotient space of $\mathscr{E}'(V; E) \equiv L_b(\mathscr{E}(V); E)$, the space of E-valued distributions with compact support on V.

2. Extension of definitions

Let V be a complex analytic manifold and A a subset, for a moment anyone, of V. We consider the vector space of locally holomorphic functions defined in a neighborhood of A, $H_A(V)$.

Definition 2.3. *We call all element of $H'_{P,A}(V; E) \equiv L_b(H_{P,A}(V); E)$ an E-valued local analytic linear mapping on A.*

Definition 2.4. *An element of $H'_{I,A}(V; E) \equiv L_b(H_{I,A}(V); E)$ is called an E-valued local analytic linear mapping defined almost on A.*

The injection of $H_{I,A}(V)$ into $H_{P,A}(V)$ being continuous, it follows that all local analytic linear mapping defined on A is a local analytic linear mapping defined almost on A.

If A is a compact subset K of V, we have, by definition, $H_{P,K}(V)=H_{I,K}(V)$, which is denoted by $H_K(V)$. If A is an open subset Ω of V, we have, by definition, $H_{I,\Omega}(V)=H(\Omega)$.

Proposition 2.3. *Let V be a complex analytic manifold which is countable at infinity. We denote $H'(K;E)\equiv L_b(H_K(V);E)$ for a compact subset K of V. Then we have*

$$H'(V;E)=\varinjlim_{K\subset V} H'(K;E).$$

Proof. It follows easily from the definitions of the topologies of $H(V)=\varprojlim_{K\subset V} H_K(V)$ and of $L_b(H(V);E)$ and $\varinjlim_{K\subset V} L_b(H_K(V);E)$. Q.E.D.

Thus we have, for a complex analytic manifold V,

Proposition 2.4. *If A is a compact subset, or an open subset of V, which is a union of a countable family of its compact subsets, the notions of local analytic linear mappings defined on A and of local analytic linear mappings defined almost on A coincide.*

We encounter the more complicated situations where we have again the same circumstances.

If A and B are two subset of V with $A\subset B$, ${}^t i_{A,B}$ denote the natural mapping of $H_B(V)$ into $H_A(V)$. ${}^t i_{A,B}$ is evidently continuous from $H_{I,B}(V)$ into $H_{I,A}(V)$. Let $i_{A,B}$ be the transposed mapping of this mapping, which maps $H'_{I,A}(V;E)$ into $H'_{I,B}(V;E)$.

Definition 2.5. *We say that $T\in H'_{I,B}(V;E)$ is representable by $U_T\in H'_{I,A}(V;E)$ if $T=i_{A,B}(U_T)$.*

It is reduced to the same thing to say that T defined on the subspace ${}^t i_{A,B}(H_{I,B}(V))$ of $H_A(V)$ is prolongable to $H_{I,A}(V)$. Thus we can then say that T is prolongable to $H_{I,A}(V)$.

Proposition 2.5. *Let A be a subset of a complex analytic manifold V. Then we have*

$$H'_{P,A}(V;E)=\varinjlim_{K\subset A} H'(K;E),$$

where K runs over all compact subsets of A.

Proof. It can be proved in the same way as in Proposition 2.3. Q.E.D.

Proposition 2.6. *Let K be a compact subset of V which is a finite union of* *compact subsets K_h of V and E a complete locally convex Hausdorff topological vector space. Then we have the surjective homomorphism of*

$$\prod_h H'(K_h;\, E) \text{ onto } H'(K;\, E).$$

That is, for every $T \in H'(K;\, E)$, there exist $T_h \in H'(K_h;\, E)$ such that

$$T = \sum_h i_{K_h,K}(T_h).$$

Proof. Since $H_K(V)$ and $H_{K_h}(V)$ are nuclear (DFS) spaces, and since we have

$$H'(K;\, E) \cong H'_K(V) \hat{\otimes} E, \quad H'(K_h;\, E) \cong H'_{K_h}(V) \hat{\otimes} E,$$

it suffices to prove that the mapping

$$H_K(V) \longrightarrow \sum_h H_{K_h}(V),$$

$$f \longrightarrow ({}^t i_{K_h,K}(f))_h$$

is injective and of closed range, which is easy to verify. Q. E. D.

Proposition 2.7. *Let A be as in Proposition 2.5. Then all E-valued local analytic linear mapping defined on A is representable by an E-valued local analytic linear mapping defined on a compact subset of A.*

Proof. This is the same thing as Proposition 2.5. Q. E. D.

3. The notion of carrier

Let T be an E-valued analytic linear mapping defined on V.

Definition 2.6. a) *We say that T is quasi-carriable by a subset A of V if T is representable by an E-valued local analytic linear mapping defined almost on A. A is a quasi-carrier of T.* b) *We say that T is carriable by A if it is representable by an E-valued local analytic linear mapping defined on a compact subset of A. A is a carrier of T.*

Proposition 2.8 (Transitivity of the notion of carrier). *If A is a quasi-carrier of T and if B contains A, B is a quasi-carrier of T. If A is a carrier of T and if B contains A, B is a carrier of T.*

Proof. Let $\bar{T}$ be a prolongation of T to $H_{I,A}(V)$ and ${}^t i_{A,B}$ the natural restriction mapping of $H_{I,B}(V)$ into $H_{I,A}(V)$. Then $\bar{T} \circ {}^t i_{A,B}$ is an analytic linear mapping quasi-carriable by B whose restriction to $H(V)$ is equal to T.

The second assertion follows immediately from the first. Q. E. D.

Proposition 2.9. *If A is an open subset of V, which is a union of a countable* *family of its compact subsets, then T is carriable by A if and only if it is carriable by a compact subset of A.*

Proof. It follows immediately from Proposition 2.3. Q. E. D.

Proposition 2.10. *Let E be quasi-complete and A an open subset of a complex analytic manifold. Then an E-valued analytic linear mapping T is carriable by A if and only if it is representable by an E-valued measure with compact support in A.*

Proof. The sufficience is clear. We note that the necessity follows from the application of Proposition 2.2 to $H'_{I,A}(V; E)=H'(A; E)$. Q. E. D.

The notions which we will introduce are employed at the paragraph 3.3. When we speak of submanifold of V we suppose them closed.

If W is a submanifold of V and A a subset of W, we can then consider the spaces $H_{I,A}(W)$ and $H_{P,A}(W)$, and the definitions given with respect to V are valid also for W. But further, if B is a subset of V with $B \supset A$, we have an (algebraic) homomorphism of restriction to a neighborhood of A in W, ${}^t\rho_{(A,W),(B,V)}$, of $H_B(V)$ into $H_A(W)$. We can easily verify that this mapping is continuous from $H_{I,B}(V)$ into $H_{I,A}(W)$.

Definition 2.7. *A local analytic linear mapping T defined in V almost on B will be said to be strictly quasi-carriable by A in W if it provide a local analytic linear mapping defined in W almost on A, that is, if*

$$T=\rho_{(A,W),(B,V)}(U) \quad \textit{where} \quad U \in H'_{I,A}(W; E).$$

In these conditions we say that W is a strict carrier of T.

Proposition 2.11. *Let Ω be an open subset of an analytic submanifold W of V. A necessary and sufficient condition that T defined on V be strictly carriable by Ω is that it is strictly carriable by a compact subset in Ω.*

Proposition 2.12. *Let E be quasi-complete and Ω an open subset of an analytic submanifold W of V. A necessary and sufficient condition that an E-valued analytic linear mapping T defined on V be strictly carriable by Ω is that there exists an E-valued measure μ with compact support included in Ω which represents T.*

Let Φ be a family of subsets of V.

Definition 2.8. *$A \in \Phi$ will be said to be a Φ-quasi-support of $T \in H'(V; E)$ (respectively a Φ-support of T) if A is minimal for the order relation of the inclusion among the elements of Φ which are quasi-carriers of T (respectively carriers of T).*

Let V and W be two complex analytic manifold, $A \subset V$ and $B \subset W$. We denote by an analytic mapping p of (A, V) into (B, W) a mapping defined in an open neighborhood of A into an open neighborhood of B and analytic in these neighborhoods. We often say that p is a projection of (A, V) into (B, W). Then we deduce from it the mapping ${}^t p$ of $H_B(W)$ into $H_A(V)$ by : $\phi \to \phi \circ p$. This mapping is evidently continuous from $H_{I,B}(W)$ into $H_{I,A}(V)$, its transpose, denoted newly by p, maps $H'_{I,A}(V; E)$ into $H'_{I,B}(W; E)$.

Definition 2.9. *If $T \in H'_{I,A}(V; E)$, $p(T)$ will be said to be the image of T by p. We very often say that $p(T)$ is the projection of T by p.*

This notion generalizes in fact all the formerly introduced notions where p is injective. These properties are resumed in the

Proposition 2.13. a) *If $C \subset A$ is a quasi-carrier (respectively a carrier) of T in V, $p(C)$ is a quasi-carrier (respectively a carrier) of $p(T)$ in W.*

b) *If the image of a neighborhood of A in V is contained in a submanifold X of W, $p(T)$ is strictly carriable by X in W for all $T \in H'_{I,A}(V; E)$.*

c) *If q is a projection of (B, W) into (D, Y), $q \circ p$ is a projection of (A, V) into (D, Y) and we have*

$$(q \circ p)(T) = q(p(T)) \qquad \text{for all} \quad T \in H'_{I,A}(V; E).$$

4. Identifications

We give the definition of Runge property of the subset of V following A. Martineau [35].

Definition 2.10. *We say that a subset A of V has the Runge property (of order zero) if all bounded subset of $H_{I,A}(V)$ is in the closure of a bounded subset, in $H_{I,A}(V)$, of elements of $H(V)$.*

In these conditions, $i_{A,V}$ is injective from $H'_{I,A}(V; E)$ into $H'(V; E)$ (without taking topologies into consideration) and we consent to identify local analytic linear mappings defined almost on A with the subspace $i_{A,V}(H'_{I,A}(V; E))$ of $H'(V; E)$.

In the more general manner, if $B \subset A$, we say that B has the Runge property with respect to A if $H_{I,A}(V)$ is strictly dense in $H_{I,B}(V)$, that is, if all bounded subset of $H_{I,B}(V)$ is in the closure of a bounded subset of $H_{I,B}(V)$ included in ${}^t i_{B,A}(H_{I,A}(V))$.

We say classically that an open subset ω of V has the Runge property with respect to $H(V)$ if $H(V)$ is dense in $H(\omega)$. This definition is equivalent to Definition 2.10.

$\mathscr{R}$ denotes in the following the family of subsets of V having the Runge property with respect to V. If $A \in \mathscr{R}$, we can then identify all element of $H'_{I,A}(V; E)$ with an analytic linear mapping defined on V. In this case we have no more to say about local analytic linear mappings defined almost on A, but only about analytic linear mappings on V which are quasi-carriable by A.

Proposition 2.14. *$T \in H'(V; E)$ is quasi-carriable by $A \in \mathscr{R}$ if (and only if) it is carriable by all open neighborhood ω of A.*

Proof. It is clear from the definitions. Q.E.D.

For a quasi-complete locally convex Hausdorff topological vector space E, we have

Proposition 2.15. *$T \in H'(V; E)$ is quasi-carriable by $A \in \mathscr{R}$ if (and only if), whatever is an open neighborhood ω of A, there exists an E-valued measure μ_ω with compact support in ω such that*

$$T(f) = \int f d\mu_\omega \qquad \text{for all} \quad f \in H(V).$$

Proof. We denote by $\bar{T}$ the prolongation of T to $H_{I,A}(V)$. The "restriction" of $\bar{T}$ to every space $H(\omega)$ defines an analytic linear mapping T_ω of "restriction" T to $H(V)$. Hence there exists well, for all open neighborhood ω of A, an E-valued measure μ_ω with compact support representing T_ω in ω, in particular, for all $f \in H(V)$

$$T(f) = \int_\omega f d\mu_\omega = \int_V f d\mu_\omega.$$

Reciprocally to say that, whatever is an open neighborhood ω of A, there exists an E-valued measure μ_ω with compact support in ω such that $T(f) = \int f d\mu_\omega$ for all $f \in H(V)$ implies that we can extend T to $H_{I,A}(V)$.

In fact, if $f \in H_{I,A}(V)$, there exist an open neighborhood ω_1 of A and a sequence of functions f_i of $H(V)$ such that $f = \lim_{i\to\infty} f_i$ in $H(\omega_1)$.

If we put $\bar{T}(f) = \int_{\omega_1} f d\mu_{\omega_1}$, where μ_{ω_1} is an E-valued measure which represents T in ω_1, we have

$$\bar{T}(f) = \lim_{i\to\infty} T(f_i).$$

This definition is independent of the choice of the open subset ω_1. In fact if ω_2 is an another open subset in which the approximation of f by a sequence $g_1, g_2, \dots, g_n, \dots$ of elements of $H(V)$ is possible, by the same reasoning as the preceding, $\lim_{i\to\infty} T(g_i)$ exists.

But f_i's tend to f in $H(\omega_1 \cap \omega_2)$ and g_i's tend to f in $H(\omega_1 \cap \omega_2)$.

If $\mu_{\omega_1 \cap \omega_2}$ represents T in $\omega_1 \cap \omega_2$

$$\int_{\omega_1 \cap \omega_2} f d\mu_{\omega_1 \cap \omega_2} = \lim_{i\to\infty} T(f_i) = \lim_{i\to\infty} T(g_i).$$

Hence let $\bar{T}$ be the linear prolongation thus constructed. The restriction of $\bar{T}$

to every space $H(\omega)$ is continuous. In fact if $f_i \to f_0$ in $H(\omega)$, $\{f_0\} \cup \bigcup_{i=1}^{\infty} \{f_i\}$ is a bounded subset of $H(\omega)$, hence there exists an open subset $\omega_3 \subset \omega$ such that $\{f_0\} \cup (\bigcup_{i=1}^{\infty} \{f_i\})$ belongs to the closure in $H(\omega_3)$ of $H(V)$. If μ_{ω_3} represents T in ω_3, since $f_i \to f_0$,

$$\bar{T}(f_0) = \int f_0 d\mu_{\omega_3} = \lim_{i\to\infty} \int f_i d\mu_{\omega_3} = \lim_{i\to\infty} T(f_i) .$$

By the definition of the topology of $H_{I,A}(V)$, $\bar{T}$ is hence continuous. Q. E. D.

5. The case of open or compact sets

In all the following of this paper, we suppose now, without explicit mention of the contrary, that V is a Stein manifold (of complex dimension n) (cf. L. Hörmander [16]). If K is a compact subset of V, we recall that we mean by envelope of K, say $\hat{K}$, the set of points y of V such that $|f(y)| \leqq \sup_{x\in K} |f(x)|$ for all $f \in H(V)$. The definition of Stein manifolds (cf. L. Hörmander [16]) assures that $\hat{K}$ is a compact subset of V. We consent that a compact subset equal to its envelope is $H(V)$-convex. The definition of Stein manifolds also assures that functions of $H(V)$ separate points of V and that all point of V possesses a local coordinate system formed by functions of $H(V)$. Then Lemma 5.3.7 of L. Hörmander [16], p. 126, assures that $\hat{K}$ admits a fundamental system of open neighborhoods ω_i each belonging to $\mathscr{R}$. A bounded subset of $H_{\hat{K}}(V)$ is formed by functions which are holomorphic and bounded in one of neighborhoods ω_{i_0}, hence is the closure in $H(\omega_{i_0})$ and a fortiori in $H_{\hat{K}}(V)$ of a bounded subset formed by functions of $H(V)$. The compact subset $\hat{K}$ hence belongs to the class $\mathscr{R}$.

If $T \in H'(V; E)$ is carriable by all neighborhood of $\hat{K}$, then, by virtue of Proposition 2.14, it is carriable by $\hat{K}$.

If ω is an open subset of V, we mean by $\hat{\omega}$ the set $\bigcup_{K\subset\omega} \hat{K}$. It is the envelope of ω.

We propose to prove:

Theorem 2.1. *If an analytic linear mapping $T \in H'(V; E)$ is carriable by an open subset ω of V, it is carriable by $\hat{\omega}$.*

Conversely, for a quasi-complete locally convex Hausdorff topological vector space E, if an analytic linear mapping $T \in H'(V; E)$ is carriable by the envelope $\hat{\omega}$ of an open subset ω of V, it is carriable by ω.

Proof. The first assertion is trivial. We will now prove the second. Let $\bar{T}$ be the prolongation of T to $H(\hat{\omega})$ identified with a closed subspace of $H(\omega)$ (cf. Lemma 3 of Theorem 1.1 of A. Martineau [35], p. 21). By Proposition 10 of Chapter 2, § 3, n°1 of A. Grothendieck [13], there exists a prolongation $\bar{\bar{T}}$ of $\bar{T}$ to

$H(\omega)$. Hence T is carriable by ω. Q.E.D.

Definition 2.11. *An analytic linear mapping $T \in H'(V; E)$ is said to be weakly carriable by K if it is carriable by all neighborhood of K.*

Corollary of Theorem 2.1. *In the notation of Theorem 2.1, if $T \in H'(V; E)$ is weakly carriable by a compact subset K of V, it is carriable by $\hat{K}$.*

Conversely, for a quasi-complete E, if $T \in H'(V; E)$ is carriable by the envelope $\hat{K}$ of a compact subset K of V, it is weakly carriable by K.

Proof. If T is weakly carriable by K it is carriable by all neighborhood of K. Denoting by ω one of these neighborhoods, $\hat{\omega}$ is a neighborhood of $\hat{K}$ and $\hat{K}$ admitting a fundamental system of $H(V)$-convex neighborhoods we thus obtain evidently a fundamental system of neighborhoods of $\hat{K}$. Hence T is carriable by each $\hat{\omega}$ by virtue of Theorem 2.1, that is, by $\hat{K}$ since we have seen this in the above.

Conversely, for a quasi-complete E, an E-valued analytic linear mapping T being carriable by $\hat{K}$ is carriable by each $\hat{\omega}$ hence by each ω. Q.E.D.

Definition 2.12. *We say that K is a good compact subset, if all analytic linear mapping $T \in H'(V; E)$ is carriable by K if and only if it is carriable by $\hat{K}$.*

We introduce the following condition:

(γ) There exists an equicontinuous family Φ of mappings from $[0, 1]$ into $\hat{K}$ such that, for all $y \in \hat{K}$, there exists ϕ, $\phi \in \Phi$, satisfying $\phi(0) = y$, $\phi(1) \in K$.

All compact subset of V satisfies (γ) when V is of dimension 1. An analytic polyhedron of $\boldsymbol{C}^n$, a convex compact subset, the envelope of two polydiscs, and etc., satisfy (γ). We have

Theorem 2.2. *If K satisfies (γ) and E is quasi-complete, it is a good compact subset.*

Proof. It suffices to prove $i_{K,\hat{K}}: H'(K; E) \to H'(\hat{K}; E)$ is surjective. But this follows immediately from Proposition 10 of Chapter 2, § 3, n°1 of A. Grothendieck [13]. Q.E.D.

By summarizing we note:

Proposition 2.16. *If E is quasi-complete, for a compact subset K of V, the following assertions are equivalent:*

a) *K is a good compact subset;*
b) *$H_{\hat{K}}(V)$ is identified with a closed subspace of $H_K(V)$ (by the injective homomorphism of restriction);*
c) *$H_{\hat{K}}(V)$ is identified with the closure of $H(V)$ in $H_K(V)$;*
d) *For all open neighborhood ω of K, there exists an open neighborhood ω_1 of $\hat{K}$ such that all element of $H(\omega) \cap H_{\hat{K}}(V)$ can be prolonged to $H(\omega_1)$.*

Proof. It is trivial that b) implies a). Conversely, we will now show that a) implies b). In fact a) means that the mapping of representation of $H'(K; E)$ into $H'(\hat{K}; E)$ is surjective. Applying this for $E=\boldsymbol{C}$, we may conclude that the mapping of representation of $H'_K(V)$ into $H'_{\hat{K}}(V)$ is surjective. Hence its transpose, the homomorphism of restriction of $H_{\hat{K}}(V)$ into $H_K(V)$ has a closed range by virtue of Theorem 4 of preliminaries of A. Martineau [35], p. 6.

Other equivalences have been proved in the proof of Proposition 1.10 of Chapter 1 of A. Martineau [35], p. 25. Q.E.D.

Remark. J.-E. Björk [55] has shown that every compact set in $\boldsymbol{C}^n$ is a good compact set for analytic functionals. His work [55] is kindly informed the author by Prof. M. Morimoto. Proposition 2.16 shows that for a compact set in a Stein manifold to be a good compact set for analytic linear mappings is equivalent to the fact that it is a good compact set for analytic functionals. So every compact set in $\boldsymbol{C}^n$ is a good compact set for analytic linear mappings.

If K is a compact space and $\mathscr{A}$ a subalgebra of the algebra $C(K)$ of continuous functions on K and which separate points of K, there exists the smallest compact subset, called the Shilov boundary of K with respect to $\mathscr{A}$, such that all $f \in \mathscr{A}$ attains its maximum module on this compact subset (cf. L. Hörmander [16], Theorem 3.1.18, p. 67).

Definition 2.13. *When K is a compact subset of a Stein manifold V, we mean by the distinguished boundary of K, and denote by δK, the Shilov boundary of K with respect to the algebra of restrictions of $H(V)$ to K.*

A compact subset is said to be distinguished compact subset, in V, if it is equal to its distinguished boundary.

Proposition 2.17. *Let E be a quasi-complete locally convex Hausdorff topological vector space. If an E-valued analytic linear mapping T, defined on V, is weakly carriable by a compact subset K, it is weakly carriable by the distinguished boundary of K.*

Proof. T is carriable by $\hat{K}$ and, since $(\delta K)^\wedge=\hat{K}$ by virtue of Proposition 1.11 in Chapter I of A. Martineau [35], p. 27, T is also weakly carriable by δK by applying Corollary of Theorem 2.1. Q.E.D.

This proposition implies that if f is holomorphic in a neighborhood of $\hat{K}$, we can find in an arbitrary neighborhood $\omega(\delta K)$ of δK an E-valued measure with compact support μ such that $T(f)=\int_{\omega(\delta K)} f d\mu$.

Proposition 2.18. *Let E_m be a finite m-dimensional complex Euclidean space. A necessary and sufficient condition that an E_m-valued analytic linear mapping T*

carriable by a compact subset K can be represented by an E_m-valued measure with support included in the distinguished boundary δK of K is that, for all $f \in H(V)$, there exists a domination of the form

$$\|T(f)\| \leqq M \sup_{x \in K} |f(x)| .$$

Proof. Necessity. If $T(f) = \int f d\mu$ where μ is concentrated on δK,

$$\|T(f)\| \leqq \sup_{x \in \delta K} |f(x)| \cdot \|\mu\| \leqq \sup_{x \in K} |f(x)| \cdot \|\mu\| .$$

Sufficiency. If we have an inequality of the form $\|T(f)\| \leqq M \cdot \sup_{x \in K} |f(x)|$ for all $f \in H(V)$, $T(f)$ depends only on the function of $C(K)$ restriction of f to the compact subset K.

In fact, if f_1 and f_2 have the same restriction to K $\|T(f_1 - f_2)\| = 0$ from which $T(f_1) = T(f_2)$. We denote again by T this linear mapping defined on the subspace F of $C(K)$ formed by restrictions of functions of $H(V)$ to K.

Since δK is the distinguished boundary of K we have $\sup_{x \in \delta K} |f(x)| = \sup_{x \in K} |f(x)|$ and F can be identified with a closed subspace of the space $C(\delta K)$. By the Hahn-Banach type theorem there hence exists an E_m-valued measure μ defined on the space δK which prolongs T given on F. The image of this measure by the injection of δK into V is the desired measure. Q. E. D.

Corollary. *Let E_m be as in Proposition 2.18. All E_m-valued analytic linear mapping carriable by the interior of a compact subset K is representable by an E_m-valued measure with support included in the distinguished boundary of K.*

Proof. In fact, such an analytic linear mapping, say T, is representable by an E_m-valued measure μ with compact support L in K. Hence, if $f \in H(V)$, we have

$$\|T(f)\| \leqq \|\mu\| \cdot \sup_{y \in L} |f(y)| .$$

But we have

$$\sup_{y \in L} |f(y)| \leqq \sup_{y \in K} |f(x)| ,$$

from which we have the majorization

$$\|T(f)\| \leqq \|\mu\| \cdot \sup_{x \in K} |f(x)| .$$

This permits us to apply Proposition 2.18. Q. E. D.

Proposition 2.19. *Let E be a complete locally convex Hausdorff topological vector space. Let T be an E-valued local analytic linear mapping defined on an* *open subset ω of V. If $\omega=\bigcup_{i=1}^{p}\omega_i$ is a finite union of open subsets ω_i of V and if u_i denotes the natural mapping of $H'(\omega_i; E)$ into $H'(\omega; E)$, there exist E-valued local analytic linear mappings $T_1, \dots, T_p$ such that T_i is defined on ω_i and $T=\sum_{i=1}^{p} u_i(T_i)$.*

Proof. This can be proved by the same way as Proposition 2.6. Q. E. D.

Theorem 2.3. *Let E be a complete locally convex Hausdorff topological vector space. If an E-valued analytic linear mapping T defined on V is carriable by an open subset ω of V and if $\omega_1,\dots,\omega_p$ are p open subsets of V such that $(\bigcup_{i=1}^{p}\omega_i)^\wedge \supset \omega$, then there exist E-valued analytic linear mappings T_i such that T_i is carriable by ω_i and $T=\sum_{i=1}^{p} T_i$.*

Theorem 2.4. *Let E be complete. If an E-valued analytic linear mapping T defined on V is carriable by a compact subset K of V and if $K_1,\dots, K_p$ are p compact subsets of V such that $\bigcup_{i=1}^{p} K_i$ is a good compact subset of V and $(\bigcup_{i=1}^{p} K_i)^\wedge \supset K$, then there exist E-valued analytic linear mappings T_i such that T_i is carriable by K_i and $T=\sum_{i=1}^{p} T_i$.*

Proof of Theorem 2.3. If T is carriable by ω, it is a fortiori carriable by $(\bigcup_{i=1}^{p}\omega_i)^\wedge$. Hence, by virtue of Theorem 2.1, there exists an E-valued local analytic linear mapping Θ defined on $\bigcup_{i=1}^{p}\omega_i$ which represents T. Hence, by virtue of Proposition 2.19, there exist E-valued local analytic linear mappings Θ_i defined on ω_i such that

$$\Theta=\sum_{i=1}^{p} u_i(\Theta_i).$$

If T_i is the restriction of Θ_i to $H(V)$, we have well

$$T=\sum_{i=1}^{p} T_i,$$

and the problem is resolved. Q. E. D.

Proof of Theorem 2.4. The proof goes analogously noting Definition 2.12 and Proposition 2.6. Q. E. D.

§ 3. Carriers and supports

1. Existence and uniqueness of supports

Let $\mathscr{K}$ be a subset of the set of compact subsets of V such that, if A_α is a totally ordered subfamily of $\mathscr{K}$, $\bigcap_\alpha A_\alpha \in \mathscr{K}$. In conformity with Definition 2.8, a compact subset K of $\mathscr{K}$ is said to be $\mathscr{K}$-support of T if T is carriable by K and if K is minimal in the family of elements of $\mathscr{K}$ which are carriers of T. A compact subset K of $\mathscr{K}$ is said to be weak $\mathscr{K}$-support of T if T is weakly carriable by K and if K is minimal in the family of elements of $\mathscr{K}$ which are weak carriers of T.

Proposition 3.1 (Theorem of existence of supports). *If T is weakly carriable by an element of $\mathscr{K}$, then T admits at least one weak $\mathscr{K}$-support.*

Proof. Let K_α be a totally ordered by inclusion subfamily of compact subsets, elements of $\mathscr{K}$, which are weak carriers of T. Let $K=\bigcap_\alpha K_\alpha$. K belongs to $\mathscr{K}$. It is a weak carrier of T. In fact, let W be an open neighborhood of K. Then there exists α_0 such that $K_{\alpha_0}\subset W$. If otherwise, for all α, we have $L_\alpha=K_\alpha \cap CW\neq\phi$ and the family of L_α's forms a filter basis on one of L_α's, say L_{α_1}. Hence $\bigcap_\alpha L_\alpha\neq\phi$ which is a contradiction for $\bigcap_\alpha L_\alpha\subset K$. W is hence a carrier of T. This being true for all neighborhood W of K, K is a weak carrier of T. In these conditions, by virtue of Zorn's lemma, there exists at least one minimal element in $\mathscr{K}$ among weak carriers of T.

If the family $\mathscr{K}$ contains a fundamental system of compact subsets of V, all analytic linear mapping is carriable, by virtue of Proposition 2.9, by an element of $\mathscr{K}$, hence admits at least one weak $\mathscr{K}$-support. Q. E. D.

Examples.

1) $\mathscr{K}_0$ is the family of all compact subsets of V.

2) $\mathscr{K}_1$ is the family of all compact subsets of a subset of V. If V is a Stein neighborhood of a real analytic manifold R, and if $\mathscr{K}_1$ is the family of all compact subsets of R, a $\mathscr{K}_1$-support of T will be said to be a real support of T.

3) $\mathscr{K}_2$ is the family of all compact subsets of V belonging to the class $\mathscr{R}$ of subsets of V having the Runge property with respect to V.

We note that, as soon as the complex dimension of V is larger than one, we can find two compact subsets of $\mathscr{K}_2$ whose intersection does not belong to it.

4) $\mathscr{K}_3$ is the family of all compact subsets of V which are $H(V)$-convex.

A $\mathscr{K}_3$-support of T will be called to be an $H(V)$-convex support of T.

5) If V is a complex vector space, $\mathscr{K}_4$ is the family of all convex compact subsets of V.

A $\mathscr{K}_4$-support of T will be said to be a convex support of T.

$\mathscr{K}_\rho^0$, if ρ is a complex norm in V, denotes the family of all ρ-balls with center at zero, that is,

$$K \in \mathscr{K}_\rho^0 \Longleftrightarrow |\exists \ell \in \boldsymbol{R}^+,\ K=\{z \mid \rho(z) \leqq \ell\}|\,.$$

$\mathscr{K}_\rho^z$ is the family of all ρ-balls with center at z and $\mathscr{K}_\rho$ the family of all ρ-balls of V.

6) If $V=\boldsymbol{C}^n$.

$\mathscr{K}_5$ is the family of all compact polycylinders of V. We can also consider $\mathscr{K}_5 \cap \mathscr{K}_3$, $\mathscr{K}_5 \cap \mathscr{K}_4$.

$\mathscr{K}_6$ will be the family of all compact polydiscs of V. It is a particular case of $\mathscr{K}_\rho$. In all the families $\mathscr{K}_i$, $i \geqq 2$, the compact subsets are good compact subsets. We have the

Theorem 3.1 (Uniqueness). *Let E be a complete locally convex Hausdorff topological vector space. All $H(V)$-convex compact subset of V is the only one $H(V)$-convex support of an appropriate E-valued analytic linear mapping defined on V.*

Proof. Let K be an $H(V)$-convex compact subset of V and L an $H(V)$-convex compact subset of V which does not contain K. The sets L and K being $H(V)$-convex, the mappings $H'_K(V)$ and of $H'_L(V)$ into $H'(V)$ are injective. Thus the mappings of $H'_K(V; E) \cong H'_K(V) \hat{\otimes} E$ and on $H'_L(V; E) \cong H'_L(V) \hat{\otimes} E$ into $H'(V; E) \cong H'(V) \hat{\otimes} E$ are injective. We identify $H'_K(V; E)$ and $H'_L(V; E)$ with subspaces of $H'(V; E)$. The analytic linear mappings carriable by L and by K are the elements of $H'_K(V; E) \cap H'_L(V; E)$. We equip $H'_K(V; E) \cap H'_L(V; E)$ with the upper bound topology of topologies of $H'_K(V; E)$ and of $H'_L(V; E)$ which makes it evidently a Fréchet space.

The injection $H'_K(V; E) \cap H'_L(V; E) \to H'_K(V; E)$ being, by definition, continuous, the image of $H'_K(V; E) \cap H'_L(V; E)$ is a subspace of $H'_K(V; E)$, whether it is meager, or it is equal to $H'_K(V; E)$ by virtue of the celebrated theorem of Banach (cf. S. Banach [1], Theorem 3, p. 38).

Lemma 1 of Theorem 2.1 of Chapter 1 in A. Martineau [35], p. 33, assures that it is meager. In fact, the proof of Theorem 2.1 of Chapter 1 in A. Martineau [35], p. 33, shows that, if L does not contain K, the image of $H'_K(V) \cap H'_L(V)$ is meager in $H'_K(V)$. Thus the image of $H'_K(V; E) \cap H'_L(V; E)$ is meager in $H'_K(V; E)$. Lemma 2 of Theorem 2.1 of Chapter 1 in A. Martineau [35], p. 33, assures that the union of the spaces $H'_K(V; E) \cap H'_L(V; E)$ when L runs over the family of $H(V)$-convex compact subsets of V which do not contain K is included in the countably infinite union of some of these subspaces. Hence it is a meager subset of $H'_K(V; E)$ and its complement is nonempty.

Theorem 3.1 can be proved with the supplementary precision: In the Fréchet space of E-valued analytic linear mappings carriable by K, the set of those which admit K as only one $H(V)$-convex support is non-meager. Q. E. D.

Corollary. *Let E be as in Theorem 3.1. If $\mathscr{K}_0$ is the family of all compact subsets of V, a compact subset is a weak $\mathscr{K}_0$-support of a certain E-valued analytic linear mapping on V if and only if it is a distinguished compact subset.*

Proof. If an E-valued analytic linear mapping T on V is weakly carriable by a compact subset K, by virtue of Proposition 2.17, T is weakly carriable by the distinguished boundary δK of K. Hence, if K is a weak support, we have $K=\delta K$. On the other hand, if L is a distinguished compact subset in V, there exists by virtue of Theorem 3.1 an E-valued analytic linear mapping T whose only one $H(V)$-convex support is $\hat{L}$. Then, by virtue of Corollary of Theorem 2.1, T is weakly carriable by L, and since, by virtue of Corollary 1 to Proposition 1.11 of Chapter 1 in A. Martineau [35], p. 27, if L_1 and L_2 are two unconformable distinguished compact subsets, $\hat{L}_1 \neq \hat{L}_2$, T cannot be weakly carriable by a subset of L. Q. E. D.

Remark. In using, for example, Proposition 2.17, we see that all E-valued analytic linear mapping admits an infinity of weak $\mathscr{K}_0$-supports. Theorem 3.1 does not impede that a given E-valued analytic linear mapping could admit several $H(V)$-convex supports.

The support can not be stable for the topology. In fact, let K and L be two compact subsets of V, $L\subset K$, and let $i_{L,K}$ be the natural mapping of $H'_L(V; E)$ into $H'_K(V; E)$. If X is a topological space, Y a subset of X, we mean in abuse of language by "connected component" of Y in X the union of the connected components of the points of Y. We have

Proposition 3.2. *Let E be a complete locally convex Hausdorff topological vector space. A necessary and sufficient condition that $i_{L,K}(H'_L(V; E))$ is dense in $H'_K(V; E)$ is that the connected component of L in K is equal to K.*

Proof. In fact, to say that $i_{L,K}(H'_L(V; E))$ is dense in $H'_K(V; E)$ is reduced to say that ${}^t i_{L,K}$ is injective from $H_K(V)$ into $H_L(V)$. It is to say, by the principle of analytic continuation, that the connected component of L in K is equal to K.

Q. E. D.

Corollary. *Let x_0 be a point of V. All analytic linear mapping defined on V carriable by the connected component of x_0 in V is the limit, in $H'(V; E)$, of analytic linear mappings carriable by x_0.*

Proof. We apply the preceding proposition as follows: If N is a compact subset and $T\in H'_N(V; E)$, N included in the connected component of x_0, we can find a connected compact subset K containing $N \cup \{x_0\}$. We then apply the proposition to K and to $L=\{x_0\}$. We can hence find that a filter T_α of elements of $H'_{x_0}(V; E)$ converges to T in $H'_K(V; E)$, hence a fortiori in $H'(V; E)$.

We note further that we can always suppose that T_α is in a countable basis.

Q. E. D.

2. The intersections of carriers

We consider in this section only the systems of carriers which are all compact or all open.

Let $K_0, K_1,\dots, K_l$ be $(l+1)$ compact (or open) subsets of V. They form a covering $\mathfrak{U}$ of $K=\bigcup_{i=0}^{l} K_i$. We denote by $K_{i_0\cdots i_k}$ the set $K_{i_0}\cap K_{i_1}\cap\cdots\cap K_{i_k}$. The topological vector space of alternate k-cochains of $\mathfrak{U}$ with values in the sheaf $\mathcal{O}$ of germs of holomorphic functions on V is the topological vector space

$$C^k(\mathfrak{U};\mathcal{O})=\prod_{\sigma=(i_0<i_1<\cdots<i_k)} H_{K_{i_0\cdots i_k}}(V).$$

The coboundary operator ∂, which maps $C^k(\mathfrak{U};\mathcal{O})$ into $C^{k+1}(\mathfrak{U};\mathcal{O})$, is defined by:

$$((\partial f)_{i_0\cdots i_{k+1}})=\Big(\sum_{h=0}^{k+1}(-1)^h f_{i_0\cdots \hat{i}_h\cdots i_{k+1}}\Big),$$

the sum, $i_0,\dots, i_{k+1}$ being given, having a sense in $H_{K_{i_0\cdots i_{k+1}}}(V)$, in abuse of language.

Definition 3.1. *We call entire k-cochains of* $\mathfrak{U}$ *the elements* $(f_{i_0\cdots i_k})$ *of* $C^k(\mathfrak{U};\mathcal{O})$, *such that, for all* $\sigma=(i_0,\dots, i_k)$, $f_{i_0\cdot i_k}$ *is in the image of* $H(V)$ *in* $H_{K_{i_0\cdots i_k}}(V)$.

$Z^k(\mathfrak{U};\mathcal{O})$ *denotes the kernel of*

$$\partial\colon C^k(\mathfrak{U};\mathcal{O})\longrightarrow C^{k+1}(\mathfrak{U};\mathcal{O}).$$

It is a closed subspace of $C^k(\mathfrak{U};\mathcal{O})$. *We say that an element of* $Z^k(\mathfrak{U};\mathcal{O})$ *is an alternate k-cocycle of covering.*

$Z^k_e(\mathfrak{U};\mathcal{O})$ *denotes the subspace of entire k-cocycles of covering.*

We introduce the temporary definition.

Definition 3.2. *The compact (respectively open) subset K is said to have the Runge property of order k with respect to the covering of* $K=\bigcup_{i=0}^{l} K_i$ *by* K_i*'s, if* $Z^k_e(\mathfrak{U};\mathcal{O})$ *is dense in* $Z^k(\mathfrak{U};\mathcal{O})$.

We have:

Proposition 3.3. *Let* $K_0, K_1,\dots, K_l$ *be* $(l+1)$ $H(V)$*-convex compact (or open) subsets of V, such that the connected component of* $K_{0\cdots l}$ *in V contains* $\bigcup_{i=0}^{l} K_{0\cdots\hat{i}\cdots l}$. *Let E be quasi-complete.*

If K has not the Runge property of order $l-1$ *with respect to* $\mathfrak{U}$, *there exists an E-valued analytic linear mapping T defined on V, carriable by each* $K_{0\cdots\hat{i}\cdots l}$ *and is not carriable by* $K_{0\cdots l}$.

(T is inevitably different from zero).

Proof. We denote by $T_{0\cdots\hat{i}\cdots l}$ the prolongation of T to $K_{0\cdots\hat{i}\cdots l}$. We use

Lemma 1. *If $K_{0\cdots l}$ is a carrier of T, for all $(l-1)$ cocycle $f_{0\cdots\hat{i}\cdots l}$ of covering we have*

$$\sum_{i=0}^{l}(-1)^i T_{0\cdots\hat{i}\cdots l}(f_{0\cdots\hat{i}\cdots l})=0.$$

Proof of Lemma 1. We have noted that, if each K_i is an $H(V)$-convex compact subset, $K_{i_0\cdots i_k}$ is $H(V)$-convex for all system $(i_0,\dots, i_k)$ of indices. It is the same if each K_i is an $H(V)$-convex open subset.

We denote by $T_{0\cdots l}$ the prolongation of T to $H_{K_{0\cdots l}}(V)$. Taking consideration of the uniqueness of prolongation, we have

$$T_{0\cdots\hat{i}\cdots l}(f_{0\cdots\hat{i}\cdots l})=T_{0\cdots l}(f_{0\cdots\hat{i}\cdots l}) \qquad \text{for all} \quad i=0, 1,\dots, l.$$

Consequently,

$$\sum_{i=0}^{l}(-1)^i T_{0\cdots\hat{i}\cdots l}(f_{0\cdots\hat{i}\cdots l})=\sum_{i=0}^{l}(-1)^i T_{0\cdots l}(f_{0\cdots\hat{i}\cdots l})$$

$$=T_{0\cdots l}\Big(\sum_{i=0}^{l}(-1)^i f_{0\cdots\hat{i}\cdots l}\Big)=T_{0\cdots l}(0)=0.$$

Lemma 2 *(K_i's are not more supposed to be $H(V)$-convex). Let*

$$(u_{0\cdots\hat{i}\cdots l})\in\prod_{i=0}^{l}H'_{K_{0\cdots\hat{i}\cdots l}}(V;\, E)=L(C^{l-1}(\mathfrak{U};\, \mathcal{O});\, E).$$

A necessary and sufficient condition that each of local analytic linear mappings $u_{0\cdots\hat{i}\cdots l}$ provides an analytic linear mapping u defined on V, is that the continuous linear mapping $(u_{0\cdots\hat{i}\cdots l})$ is zero on the subspace $Z_e^{l-1}(\mathfrak{U};\, \mathcal{O})$ of $C^{l-1}(\mathfrak{U};\, \mathcal{O})$.

Proof of Lemma 2. Necessity. Let $(f_{0\cdots\hat{i}\cdots l})\in Z_e^{l-1}(\mathfrak{U};\, \mathcal{O})$. The topological hypothesis of Proposition 3.3 implies that

$$\sum_{i=0}^{l}(-1)^i f_{0\cdots\hat{i}\cdots l}=0$$

on the connected component of $\bigcup_i K_{0\cdots\hat{i}\cdots l}$ in V.

If f_1 and f_2, two elements of $H(V)$, coincide on this component, we have surely, for all i:

$$u(f_1)=u_{0\cdots\hat{i}\cdots l}(f_1)=u_{0\cdots\hat{i}\cdots l}(f_2)=u(f_2).$$

Consequently:

$$\sum_{i=0}^{l}(-1)^{i}u_{0\cdots\hat{i}\cdots l}(f_{0\cdots\hat{i}\cdots l})$$

$$=u\Big(\sum_{i=0}^{l}(-1)^{i}f_{0\cdots\hat{i}\cdots l}\Big)=u(0)=0.$$

Sufficiency. Let u be the restriction of $u_{\hat{0}\cdots l}$ to $H(V)$. u is an analytic linear mapping defined on V, prolongable to $H_{K_{\hat{0}\cdots l}}(V)$.

Let $f\in H(V)$ and $f_{0\cdots\hat{i}\cdots l}$ its image in $H_{K_{0\cdots\hat{i}\cdots l}}(V)$. For given i_0, we put

$$g_{0\cdots\hat{i}_0\cdots l}=(-1)^{i_0+1}f_{0\cdots\hat{i}_0\cdots l}$$

$$g_{\hat{0}\cdots l}=f_{\hat{0}\cdots l}$$

$$g_{0\cdots\hat{h}\cdots l}=0 \qquad \text{if} \quad h\neq 0,\ i_0.$$

Then $\sum_{i=0}^{l}(-1)^{i}g_{0\cdots\hat{i}\cdots l}=f_{\hat{0}\cdots l}-f_{0\cdots\hat{i}_0\cdots l}=0$, $g_{0\cdots\hat{i}\cdots l}$ is an entire $(l-1)$ cocycle.

Hence, by hypothesis,

$$u_{\hat{0}\cdots l}(f_{\hat{0}\cdots l})+(-1)^{i_0+1}(-1)^{i_0}u_{0\cdots\hat{i}\cdots l}(f_{0\cdots\hat{i}_0\cdots l})=0,$$

that is,

$$u(f)=u_{0\cdots\hat{i}_0\cdots l}(f).$$

$u_{0\cdots\hat{i}_0\cdots l}$ realizes well the prolongation of u to $K_{0\cdots\hat{i}_0\cdots l}$. This proves Lemma 2.

If $K=\bigcup_{i=0}^{l}K_i$ has not the Runge property of order $l-1$ with respect to $\mathfrak{U}$, there exists a continuous linear mapping of $Z^{l-1}(\mathfrak{U};\mathcal{O})$ into E, not identically zero, with zero restriction to $Z_e^{l-1}(\mathfrak{U};\mathcal{O})$. By Proposition 10 of Chapter 2, § 3, n°1 of A. Grothendieck [13], it can be prolonged to a continuous linear mapping $(T_{0\cdots\hat{i}\cdots l})$ not identically zero on $C^{l-1}(\mathfrak{U};\mathcal{O})$.

By virtue of Lemma 2, there exists an analytic linear mapping T defined on V carriable by each $K_{0\cdots\hat{i}\cdots l}$, $T_{0\cdots\hat{i}\cdots l}$ being a prolongation of T to $H_{K_{0\cdots\hat{i}\cdots l}}(V)$. By virtue of Lemma 1, since the restriction of $T_{0\cdots\hat{i}\cdots l}$ to $Z^{l-1}(\mathfrak{U};\mathcal{O})$ is not zero, T is not carriable by $K_{0\cdots l}$. Q.E.D.

Proposition 3.4. *Let $K_0, K_1,\dots, K_l$ be $(l+1)$ compact (or open) subsets of V (not necessarily $H(V)$-convex). If K has the Runge property of order $l-1$ with respect to $\mathfrak{U}$, a sufficient condition that all analytic linear mapping defined on V and carriable by each one of $(l+1)$-sets $K_{0\cdots\hat{i}\cdots l}$, is carriable by their intersection, is that $H^l(\mathfrak{U};\mathcal{O})=0$. If $K_{0\cdots l}$ and each one of $K_{0\cdots\hat{i}\cdots l}$'s are Runge compact (or open) subsets, this condition is necessary.*

Proof. We note first that $Z^l(\mathfrak{U};\mathcal{O})=C^l(\mathfrak{U};\mathcal{O})$. We consider the continuous linear mapping ∂ defined from $C^{l-1}(\mathfrak{U};\mathcal{O})$ into $C^l(\mathfrak{U};\mathcal{O})$.

When K has the Runge property of order l with respect to $\mathfrak{U}$, $\partial C^{l-1}(\mathfrak{U}; \mathcal{O})$ is dense in $C^l(\mathfrak{U}; \mathcal{O})$. In fact, let $f \in H(V)$ and $f_{\hat{0}\cdots l}$ the image of f in $H_{K\hat{0}\cdots l}(V)$, $f_{0\cdots\hat{i}\cdots l}=0$ if $i \neq 0$, $f_{0\cdots l}$ the image of f in $H_{K_{0\cdots l}}(V)$; $\partial f_{\hat{0}\cdots l}=f_{0\cdots l}$. Hence, since, by hypothesis, $H(V)$ is of dense image in $H_{K_{0\cdots l}}(V)$, $\partial C^{l-1}(\mathfrak{U}; \mathcal{O})$ is a fortiori of dense image in $C^l(\mathfrak{U}; \mathcal{O})$. If K_i's are compact subsets, $C^{l-1}(\mathfrak{U}; \mathcal{O})$ and $C^l(\mathfrak{U}; \mathcal{O})$ are (DFS) spaces; if they are open subsets, they are (FS) spaces. Consequently (Theorem 4 of Preliminaries in A. Martineau [35], p. 6) the following properties are equivalent:

a) $\partial(C^{l-1}(\mathfrak{U}; \mathcal{O}))=C^l(\mathfrak{U}; \mathcal{O})$,

b) ∂ is a homomorphism,

c) ${}^t\partial$, which is injective, is such that ${}^t\partial((C^l(\mathfrak{U}; \mathcal{O}))')$ is closed in $(C^{l-1}(\mathfrak{U}; \mathcal{O}))'$.

The condition a) can be also expressed:

a′) $H^l(\mathfrak{U}; \mathcal{O})=0$.

The condition c) is equivalent to

c′) ${}^t\partial(L_b(C^l(\mathfrak{U}; \mathcal{O}); E))$ is closed in $L_b(C^{l-1}(\mathfrak{U}; \mathcal{O}); E)$ for any complete E.

Sufficiency of the condition. Let T be an analytic linear mapping defined on V and $T_{0\cdots\hat{i}\cdots l}$ a prolongation of T to $K_{0\cdots\hat{i}\cdots l}$. Suppose that ∂ is surjective. If $f_{0\cdots l} \in C^l(\mathfrak{U}; \mathcal{O})$, we put

$$T_{0\cdots l}(f_{0\cdots l})=\sum_{i=0}^{l}(-1)^i T_{0\cdots\hat{i}\cdots l}(f_{0\cdots\hat{i}\cdots l})$$

$$\text{for}\quad f_{0\cdots l}=\sum_{i=0}^{l}(-1)^i f_{0\cdots\hat{i}\cdots l}.$$

Two representations of $f_{0\cdots l}$ differ in an $(l-1)$-cocycle. But if $(f_{0\cdots\hat{i}\cdots l}) \in Z_e^{l-1}(\mathfrak{U}; \mathcal{O})$, by virtue of Lemma 2 to Proposition 3.3, we have

$$\sum_{i=0}^{l}(-1)^i T_{0\cdots\hat{i}\cdots l}(f_{0\cdots\hat{i}\cdots l})=0$$

and since $Z_e^{l-1}(\mathfrak{U}; \mathcal{O})$ is dense in $Z^{l-1}(\mathfrak{U}; \mathcal{O})$, for all $(l-1)$-cocycle $(\psi_{0\cdots\hat{i}\cdots l})$ of $\mathfrak{U}$ with values in $\mathcal{O}$ we have

$$\sum_{i=0}^{l}(-1)^i T_{0\cdots\hat{i}\cdots l}(\psi_{0\cdots\hat{i}\cdots l})=0.$$

Hence this definition does not depend on the chosen decomposition. ∂ being a homomorphism, the linear mapping $f_{0\cdots l} \to T_{0\cdots l}(f_{0\cdots l})$ is continuous.

Necessity. $H(V)$ is supposed to be dense in $H_{K_{0\cdots l}}(V)$. Suppose that $H^l(\mathfrak{U}; \mathcal{O}) \neq 0$. In these conditions ${}^t\partial(L_b(C^l(\mathfrak{U}; \mathcal{O}); E))$ is not closed by virtue of the condition c′). The set $\overline{{}^t\partial(L_b(C^l(\mathfrak{U}; \mathcal{O}); E))}$ is the set of elements which annihilate the kernel of ∂, that is, of $Z^{l-1}(\mathfrak{U}; \mathcal{O})$. If $(T_{0\cdots\hat{i}\cdots l})$ is an element of $\overline{{}^t\partial(L_b(C^l(\mathfrak{U}; \mathcal{O}); E))}$ not belonging to ${}^t\partial(L_b(C^l(\mathfrak{U}; \mathcal{O}); E))$, we have hence in particular for all $(f_{0\cdots\hat{i}\cdots l}) \in Z_e^{l-1}(\mathfrak{U}; \mathcal{O})$

$$\sum_{i=0}^{l}(-1)^i T_{0\cdots\hat{i}\cdots l}(f_{0\cdots\hat{i}\cdots l})=0.$$

Lemma 2 to Proposition 3.3 shows that there exists then T defined on V and carriable by each compact subset $K_{0\cdots\hat{i}\cdots l}$. The set $K_{0\cdots l}$ is not a carrier of T; in fact, if $K_{0\cdots l}$ is a carrier of T and $T_{0\cdots l}$ then denotes the unique prolongation of T to $H_{K_{0\cdots l}}(V)$, we will necessarily have

$$(T_{0\cdots\hat{i}\cdots l})\in {}^t\partial(T_{0\cdots l})$$

since each $K_{0\cdots\hat{i}\cdots l}$ has the Runge property. But this is not so.

Hence T is not carriable by $K_{0\cdots l}$ and the necessity of our condition is demonstrated. Q.E.D.

As corollaries we obtain the following theorems.

Theorem 3.2. *Let K_0 and K_1 be two compact (or open) subsets of V. If $K_0\cup K_1$ is $H(V)$-convex, all analytic linear mapping carriable by K_0 and by K_1 is carriable by $K_0\cap K_1$.*

Proof. The cohomology group of $K=K_0\cup K_1$ with values in $\mathcal{O}$ is equal to $H_K(V)$ in degree zero, and then we have

$$H^i(K, \mathcal{O})=0 \qquad \text{for} \quad i\geqq 1$$

by virtue of Oka-Cartan Theorem B (cf. L. Hörmander [16]). K being $H(V)$-convex, $H(V)$ has a dense image in $H_K(V)$. On the other hand, it is well known that the natural mapping of $H^1(\mathfrak{U}, \mathcal{O})$ into $H^1(K, \mathcal{O})$ is injective. Since $H^1(K, \mathcal{O})=0$, we have hence $H^1(\mathfrak{U}, \mathcal{O})=0$. We then apply Proposition 3.4. Q.E.D.

Theorem 3.3. *Let $K_0,\dots, K_l$ be $(l+1)$ $H(V)$-convex compact (or open) subsets of V. If $\bigcup_{i=0}^{l} K_i$ is $H(V)$-convex, all analytic linear mapping carriable by each $K_{0\cdots\hat{i}\cdots l}$ is carriable by $K_{0\cdots l}$.*

Proof. For $l=1$ it is a particular case of the preceding theorem. Suppose hence $l>1$. Since $K_0,\dots, K_l$ are $H(V)$-convex, $K_{i_0\cdots\hat{i}_k\cdots i_r}$ is then also $H(V)$-convex. Hence, by the Lemma to Theorem 2.3 in Chapter 1 of A. Martineau [35], p. 42, the entire r-coboundaries are dense in the subspace of $Z^r(\mathfrak{U}; \mathcal{O})$ formed by the r-coboundaries.

On the other hand, $H^j(K_{i_0\cdots i_h}, \mathcal{O})=0$ for all $j\geqq 1$, all system of indices $i_0,\dots,$ i_h, and all h. We can hence apply Leray's theorem on the acyclic covering, valuable if K_i's are all closed, or all open, which assures that the cohomology group of the covering is "equal" to that of the space (cf. R. Godement [8]). We have hence $H^{l-1}(\mathfrak{U}, \mathcal{O})=0=H^l(\mathfrak{U}, \mathcal{O})$, that is, all $(l-1)$-cocycle of the covering is an $(l-1)$-coboundary. Hence, since all entire $(l-1)$-coboundary is an entire $(l-1)$-cocycle,

K satisfies the Runge condition of order $(l-1)$ with respect to $\mathfrak{U}$.

We apply again Proposition 3.4. Q. E. D.

Theorem 3.4. *If V is of complex dimension n and if $K_0,\dots, K_l$ are $(l+1)$ $H(V)$-convex compact (or open) subsets of V, all analytic linear mapping carriable by the $(n+1)$ by $(n+1)$ intersections of these sets is carriable by $K_0 \cap \cdots \cap K_l$.*

Proof. We need the following lemma.

Lemma. *Let $K_0,\dots, K_l$ be $(l+1)$ $H(V)$-convex compact (or open) subsets of V. For $l \geqq n+1$ if an analytic linear mapping T is carriable by each set $K_{0\cdots\hat{i}\cdots l}$, T is carriable by $K_{0\cdots l}$.*

Proof of the Lemma. We know that $H^i(K, \mathcal{O})=0$ for $i \geqq (n+1)$.

By virtue of Malgrange's theorem, if ω is an open subset of a complex analytic manifold of dimension n without compact connected component, we have $H^n(\omega, \mathcal{O})=0$ (cf. B. Malgrange [32]). It is the same for a closed subset F of such a manifold after passing to the limit over the open neighborhoods of F in V (cf. R. Godement [8], Theorem 4.11.1, p. 193). Applying again Leray's theorem, we have hence, if $l \geqq n+1$, $H^{l-1}(\mathfrak{U}, \mathcal{O})=0$ and $H^l(\mathfrak{U}, \mathcal{O})=0$. By the reasoning of the preceding theorem $H^{l-1}(\mathfrak{U}, \mathcal{O})=0$ implies that K satisfies the Runge property of order $l-1$ with respect to $\mathfrak{U}$.

We then apply Proposition 3.4, which assures the lemma. Q. E. D.

We now turn to the proof of the theorem. Suppose that we have proved that T is carriable by all compact subset $K_{i_0\cdots i_{r-1}}$, $n+1 \leqq r < l$.

Let $i_0,\dots,i_r$ be a system of some indices of length $r+1$, T is carriable, by the hypothesis, by each set $K_{i_0\cdots\hat{i}_h\cdots i_r}$ $(h=0, 1,\dots, r)$. Hence, by virtue of the lemma, T is carriable by $K_{i_0\cdots i_r}$. We can conclude the theorem by induction. Q. E. D.

Let $\mathcal{O}^n$ be the sheaf of germs of holomorphic differential n-forms over V. If A is a subset of V, $H_*(A, \mathcal{O}^n)$ will denote the cohomology group with compact support of A with values in the sheaf $\mathcal{O}^n$.

We have

Proposition 3.5. *Let E be a complete locally convex Hausdorff topological vector space. Let $K_0, K_1,\dots, K_l$ be $(l+1)$ $H(V)$-convex nonempty open sets in a manifold V of dimension n. If $H_*^{n-l+1}(K, \mathcal{O}^n)=0$, all E-valued analytic linear mapping carriable by each $K_{0\cdots\hat{i}\cdots l}$ is carriable by $K_{0\cdots l}$.*

For $l>1$, if the connected component of $K_{0\cdots l}$ in V contains $\bigcup_{i=0}^{l} K_{0\cdots\hat{i}\cdots l}$ and if all E-valued analytic linear mapping carriable by each $K_{0\cdots\hat{i}\cdots l}$ is carriable by $K_{0\cdots l}$, we have $H_*^{n-l+1}(K, \mathcal{O}^n)=0$.

Proof. Noting Propositions 3.3 and 3.4, the reasoning of Proposition 2.5 in

Chapter 1 of A. Martineau [35], p. 43, is valid for this proposition. Q.E.D.

As a corollary we have

Theorem. 3.5. *Let V be a connected Stein manifold of dimension n. Let $K_0, \dots, K_n$ be $(n+1)$ $H(V)$-convex open sets in V and put $K = \bigcup_{i=0}^{n} K_i$. If CK has no compact connected component, all E-valued analytic linear mapping $T \neq 0$ carriable by each $K_{0 \cdots \hat{i} \cdots n}$ is carriable by $K_{0 \cdots n}$. Conversely, if all E-valued analytic linear mapping $T \neq 0$ carriable by each $K_{0 \cdots \hat{i} \cdots n}$ is carriable by $K_{0 \cdots n}$, CK has no compact connected component, where E is a complete locally convex Hausdorff topological vector space.*

Proof. We can go analogously to the proof of Theorem 2.5 in Chapter 1 of A. Martineau [35], p. 45. Q.E.D.

We now mention an application of Theorem 3.2.

Proposition 3.6. *Let T be an analytic linear mapping defined on V carriable by a countable compact set K. Then T admits only one $H(V)$-convex support, which is included in K.*

Proof. Noting Theorem 3.2, we can go analogously to the proof of the Corollary to Proposition 2.6 in Chapter 1 of A. Martineau [35], p. 47. Q.E.D.

3. The case of submanifold

We recall that, for the case of a submanifold, new definitions which we are going to use have been introduced at § 2.3, Definitions 2.7 and 2.8.

Proposition 3.7. *Let T be an analytic linear mapping defined on V. If K is an $H(V)$-convex compact carrier of T and if W is an analytic subset of V which is a quasi-carrier of T, $K \cap W$ is a carrier of T.*

Proof. Let ω be some neighborhood of $K \cap W$. We can find an open neighborhood of W, say ω_1, and an open neighborhood of K, say ω_2, such that $\omega_1 \cap \omega_2 \subset \omega$. There exist an $H(V)$-convex open neighborhood Ω_1 of W, included in ω_1, and an $H(V)$-convex open neighborhood Ω_2 of K, included in ω_2 (cf. Lemma 2 of no. 2 of § 2 in Chapter 1 of A. Martineau [35], p. 48). $\Omega_1 \cup \Omega_2$ is a neighborhood of $K \cup W$. Hence we can find an $H(V)$-convex open neighborhood Ω_3 of $K \cup W$, included in $\Omega_1 \cup \Omega_2$. The analytic linear mapping T is by hypothesis carriable by $\Omega_2 \cap \Omega_3$ and by $\Omega_1 \cap \Omega_3$ and, since the open set $(\Omega_2 \cap \Omega_3) \cup (\Omega_1 \cap \Omega_3) = \Omega_3$ is $H(V)$-convex, T is carriable by $\Omega_1 \cap \Omega_2 \cap \Omega_3$ by virtue of Theorem 3.2, hence a fortiori by ω. By virtue of Proposition 2.14, T is carriable by $K \cap W$. Q.E.D.

Corollary 1. *An analytic linear mapping T defined on V, quasi-carriable by* *a complex analytic subset W of V, is carriable by W.*

Proof. Let T be an analytic linear mapping quasi-carriable by W, and K an $H(V)$-convex carrier of T. By virtue of Proposition 3.7, T is carriable by $K \cap W$ which is a compact subset of W. Q.E.D.

Corollary 2. *If T is carriable by an $H(V)$-convex open subset Ω of V and by an analytic subset W of V, T is carriable by $W \cap \Omega$.*

Corollary 3. *If T is carriable by an analytic subset W of V, all $H(V)$-convex support of T is included in W.*

Corollary 4. *If an analytic linear mapping T defined on V is carriable by analytic subsets $W_0,\dots, W_k$, T is carriable by $W_0 \cap \cdots \cap W_k$.*

Theorem 3.6. *Let $W_0,\dots, W_k$ be $(k+1)$ submanifolds of V such that $W_0 \cap \cdots \cap W_k = W_{0\cdots k}$ is a manifold. If T is strictly carriable by each W_i and carriable by an $H(V)$-convex compact (or open) subset K of V, T is strictly carriable by $K \cap W_{0\cdots k}$ in $W_{0\cdots k}$.*

Proof. Suppose that Theorem 3.6 has been proved when K is compact. Let Ω be an $H(V)$-convex open carrier of T. T is carriable by an compact subset K of Ω which is $H(V)$-convex. Hence it is strictly carriable by $W_{0\cdots k} \cap K$ in $W_{0\cdots k}$, hence a fortiori by $\Omega \cap W_{0\cdots k}$ in $W_{0\cdots k}$. Hence we can suppose K to be compact.

T being strictly carriable by W_i is a fortiori carriable by W_i in V. Hence by virtue of Proposition 3.7 and its Corollary 4 T is carriable by $K \cap W_{0\cdots k}$. Let W be a submanifold of V and K an $H(V)$-convex compact subset of V. Let $L = W \cap K$. We denote by ${}^t\rho$ the restriction mapping ${}^t\rho_{(L,W),(K,V)}$ in Definition 2.7 continuous from $H_K(V)$ into $H_L(W)$. Then ${}^t\rho$ is a homomorphism of $H_K(V)$ onto $H_L(W)$.

Lemma 1. *An analytic linear mapping T defined on V is strictly carriable by a submanifold W of V if and only if, for all $f \in H(V)$ with zero restriction to W, we have $T(f)=0$.*

Lemma 2. *An analytic linear mapping T defined on V and carriable by an $H(V)$-convex compact set L included in an submanifold W of V is strictly carriable by L (included) in W, if, for all $f \in H_L(V)$ with zero restriction to W (${}^t\rho(f)=0$), we have $T(f)=0$ (cf. Definition 2.7).*

Proof of Lemmas 1 and 2. The restriction mapping ${}^t\rho$ is in each of these cases a homomorphism. This implies that $\rho(H'(W))$ (resp. $\rho(H'_L(W))$) is closed. Hence $\rho(H'(W; E))$ (resp. $\rho(H'_L(W; E))$) is closed. But we have expressed that T belongs to the annihilator of the kernel of ${}^t\rho$, hence to $\overline{\rho(H'(W; E))}$ (resp. $\overline{\rho(H'_L(W; E))}$), where the closure is taken with respect to the simple convergence topology. Consequently, $T \in \rho(H'(W; E))$ (resp. $T \in \rho(H'_L(W; E))$). Q.E.D.

Proof of Theorem 3.6.

Now let $L=K\cap W_{0\cdots k}$. At all point $x\in W_{0\cdots k}$, there exist a finite number of functions $f_{i,\alpha,x}\in H(V)$ which generate the ideal of the manifold W_i in a neighborhood of the point x, (and which vanish on W_i).

If $f\in H_L(V)$ vanishes on $W_{0\cdots k}$ in a neighborhood of x there hence exist a finite number of functions $g_{i,\alpha,x}\in H_{\{x\}}(V)$ such that $f=\sum_i\sum_\alpha f_{i,\alpha,x}g_{i,\alpha,x}$ in a neighborhood of the point x. Hence we can find a finite number of functions $h_{i,\beta}\in H_L(V)$ and $f_{i,\beta}$, $f_{i,\beta}$ vanishing on W_i for all β, such that $f=\sum_i(\sum_\beta h_{i,\beta}f_{i,\beta})$.

In $H_L(V)$, $h_{i,\beta}$ is the limit of a sequence $\phi_{i,\beta,n}$ of functions of $H(V)$. By virtue of Lemma 1, since T is strictly carriable by W_i, we have

$$T(\phi_{i,\beta,n}f_{i,\beta})=0.$$

Since T is carriable by L, we have

$$T(h_{i,\beta}f_{i,\beta})=\lim_{n\to\infty}T(\phi_{i,\beta,n}f_{i,\beta})=0$$

and finally

$$T(f)=0.$$

$W_{0\cdots k}$ being a manifold, by virtue of Lemma 2, T is strictly carriable by L in $W_{0\cdots k}$.
Q.E.D.

Corollary 1. *Let T be an analytic linear mapping defined on V and strictly carriable by a submanifold W of V. If K is an $H(W)$-convex strict support of T in W, it is an $H(V)$-convex support of T in V.*

Proof. If $K\subset W$ is $H(W)$-convex, it is $H(V)$-convex in V.

If $L\subset K$ is an $H(V)$-convex support of T, by virtue of Theorem 3.6 applied with $k=0$, T is strictly carriable by $L\cap W=L$. Hence we have $L=K$. Q.E.D.

Corollary 2. *Let T be strictly carriable by $(k+1)$ regular submanifolds $W_0,\dots,W_k$ of V. If $W_{0\cdots k}=\{x_0\}$, then T is of the form $\boldsymbol{a}\otimes\delta_{x_0}$, where $\boldsymbol{a}$ is some element of E and δ_{x_0} is the Dirac measure at the point x_0.*

Remark. Let $f\in H(V)$, we define fT by

$$(fT)(g)=T(fg).$$

Let $I(W)$ be the closed subspace of $H(V)$ formed by functions vanishing on W. We can state Lemma 1 of Theorem 3.6 in the form:

T is strictly carriable by W if and only if, for all $f\in J$, where J is a total subset of $I(W)$, we have $fT=0$.

§ 4. C^n and R^n, the case of convex sets

1. The systems of $(n+1)$ convex carriers

In the following of this section, C denotes a finite dimensional complex vector space, and R denotes a finite dimensional real vector space. When we need specify its (complex or real) dimension, say l (or m), we write C^l (or R^m).

If R is a real vector space, $C=R_c$ denotes its complexification. By tube of basis $K\subset R$, we mean the set K_T of C equal to

$$K+iR;$$

K is the basis of K_T.

C^* (resp. R^*) denotes the space of linear forms on C (resp. R).

Proposition 4.1. *Let T be an analytic linear mapping different from zero defined on C. If it is carriable by two convex compact (or open) subsets K_1, K_2, then $K_1\cap K_2\neq\phi$.*

Proof. It is clear that the case of open subsets is reduced to the case of compact subsets. So we now assume that K_1 and K_2 are compact.

Let K be the convex hull of $K_1\cup K_2$ supposed to be disjoint, and P a real hyperplane separating K_1 and K_2 (which exists). Let L be the union of real hyperplanes parallel to P and encountering $K_1\cup K_2$. The set CL is a union of hyperplanes. Hence $C(K\cap L)=CL\cup CK$ is a union of hyperplanes. The set $K\cap L$ is a compact subset of the form $L_1\cup L_2$, where $L_i\supset K_i$ $(i=1, 2)$, L_i is convex, $L_1\cap L_2=\phi$, which is polynomially convex by virtue of Lemma 3.1 of A. Martineau [35], Chapter 1, § 3, no. 1. The analytic linear mapping $T\neq 0$ is carriable by L_1 and L_2. Then, by virtue of Theorem 3.2, it is carriable by $L_1\cap L_2=\phi$. This is a contradiction. Q.E.D.

Let $B_0, B_1,\dots, B_n$ be convex open (or compact) subsets of C^n and T an analytic linear mapping carriable by each subset

$$B_{0\cdots\hat{i}\cdots n}=\bigcap_{k\neq i} B_k,$$

then T is carriable by $B_0\cap B_1\cap\cdots\cap B_n$. In order to prove this we apply Theorem 3.5.

Let $y\notin B_0\cup B_1\cup\cdots\cup B_n$.

There exists a closed half-space F_i whose boundary is a real hyperplane P_i, $y\in P_i$, which does not encounter B_i. Then $F_0\cap F_1\cap\cdots\cap F_n$ contains a half-line issuing from y. Hence all connected component of $C(B_0\cup\cdots\cup B_n)$ contains a half-line, hence is not relatively compact. Theorem 3.5 is applicable. Hence we have,

Theorem 4.1. *If $B_0, B_1, \dots, B_l$ are $(l+1)$ $(l \geqq n)$ convex all open (or all compact) subsets of C^n and T an analytic linear mapping carriable by each n by n intersection of these convex subsets, T is also carriable by $B_0 \cap B_1 \cap \cdots \cap B_l$.*

We underline some elementary aspects of this theorem.

Proposition 4.2. *Theorem 4.1 is a consequence of the following assertion: Let $P_0, P_1, \dots, P_n$ be $(n+1)$ open half-spaces. If an analytic linear mapping T defined on C^n is carriable by each $P_{0 \cdots \hat{i} \cdots n}$, then T is carriable by $P_{0 \cdots n}$.*

Proof. We prove this in the case of compact subsets. Let $\{P_\alpha\}$ be the set of open half-spaces such that each P_α contains at least one of B_i's. Let $P_{\alpha_0}, \dots, P_{\alpha_n}$ be a system of $(n+1)$ of these half-spaces. Then $P_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_n}$ contains by hypothesis one of subsets $B_{0 \cdots \hat{j} \cdots n}$. Hence it is a carrier of T. Consequently, by virtue of the hypothesis of Proposition 4.2, $P_{\alpha_0 \cdots \alpha_n}$ is a carrier of T. By virtue of Theorem 3.4, T is carriable by all finite intersections of P_α, hence finally by $B_0 \cap B_1 \cap \cdots \cap B_l$.

Q. E. D.

We note that we know how we can avoid the use of Theorem 3.5 in the following cases:

a) each B_i is a circled convex set in C^n;
b) each B_i is a convex tube in the complexification of R^n;
c) each B_i is a product of convex subsets of $\boldsymbol{C}$,

$$B_i = \prod_{j=1}^{n} B_i^j, \quad B_i^j \subset \boldsymbol{C}, \quad B_i \subset \boldsymbol{C}^n = \prod_{i=1}^{n} \boldsymbol{C}_i, \quad \boldsymbol{C}_i \cong \boldsymbol{C}.$$

We show first a).

Let A_i be the (circled) convex hull of $\bigcup_{j \neq i} B_{0 \cdots \hat{j} \cdots n}$. One can easily verify that $B_{0 \cdots \hat{i} \cdots n} = A_{0 \cdots \hat{i} \cdots n}$, that on the other hand $\bigcup_{j=0}^{n} A_j$ is convex. We then apply Theorem 3.3.

For b), let $G_0, \dots, G_n$ be the bases of the tubes $B_0, \dots, B_n$ of C^n.

Since T is carriable by each $B_{0 \cdots \hat{i} \cdots n}$, by virtue of Proposition 4.1, $B_{0 \cdots \hat{i} \cdots n} \cap B_{0 \cdots \hat{j} \cdots n} = B_{0 \cdots n} \neq \phi$ $(i \neq j)$. We carry out the same construction as before, and note that, under the hypothesis, $A_i \cap A_j \neq \phi$ (which has been verified), $\bigcup_{j=0}^{n} A_j$ is a convex tube.

We introduce a new weakened notion of carrier whose utility will appear essentially at Part II of this series of works.

Definition 4.1. *A closed convex set G is a semi-carrier of an E-valued analytic linear mapping T if T is carriable by all convex neighborhood of G.*

This notion is a priori weaker than that of quasi-carrier; T is said to be semi-carriable by G. If G is compact, as it admits a fundamental system of convex neighborhoods, if T is semi-carriable by G, then it is carriable by G. Theorem 4.1 then admits the corollary:

Corollary 1. *Let $B_0, B_1,\dots, B_n$ be $(n+1)$ closed convex subsets of C^n. Suppose that $B_0 \cap B_1 \cap \cdots \cap B_n$ be compact. Let T be an E-valued analytic linear mapping which is semi-carriable by each subset $B_{0\cdots\hat{i}\cdots n}$. Then T is carriable by $B_{0\cdots n}$.*

Proposition 4.3. *Let C_1^n and C_2 be two complex vector spaces of finite dimension, and T an E-valued analytic linear mapping defined on $C_1^n \times C_2$. Let $B_0,\dots, B_l$ be $(l+1)$ open (or compact) convex subsets of $C_1^n \times C_2$ $(l \geqq n)$, each B_i of which is of the form $K_i \times L$ where K_i is an open (or compact) convex subset of C_1^n $(i=0,\dots, l)$, and L is an open (or compact) subset of C_2. If T is carriable by each n by n intersection of B_i's, it is carriable by $B_0 \cap B_1 \cap \cdots \cap B_l$.*

Proof. The question is to verify that $K=\bigcup_i (K_i \times L)=(\bigcup_i K_i)\times L$ satisfies the Runge property of order $n-1$, and that $H^m(K; \mathcal{O})=0$ for $m \geqq n$ which permits us to apply Propositions of § 3.2. We reason this in the same way as in the proof of Proposition 3.3 of A. Martineau [35], Chapter 1, p. 56. Q.E.D.

We can show that, in Theorem 4.1 and its particular cases, hence also in Proposition 4.2 and in Proposition 4.3, $(n+1)$ is the best possible constant.

2. Applications

We consider in $\boldsymbol{C}^n$ a compact convex set of the form $K=\prod_{i=1}^{n} K_i$ where K_i is a (compact) convex subset of the complex plane.

We define in $\boldsymbol{C}^n$ a complex norm ρ_r, $r=(r_1, r_2,\dots, r_n)$, by the condition:

if $z=(z_1,\dots, z_n)$, $\rho_r(z) \leqq 1$ is equivalent to $|z_1| \leqq r_1$, $|z_2| \leqq r_2,\dots, |z_n| \leqq r_n$.

We have

Proposition 4.4. *Let r be fixed. If an E-valued analytic linear mapping T defined on $\boldsymbol{C}^n$ is carriable by all ρ_r-ball containing K, it is carriable by K.*

Proposition 4.4 ($\boldsymbol{k}$). *Let $C_1=\boldsymbol{C}^k \times C$ and T be an E-valued analytic linear mapping defined on C_1. Let Ω be an $H(C)$-convex open subset of C and $B_0,\dots, B_l$ the interior of $(l+1)$ ρ_r-balls of radius a of $\boldsymbol{C}^k$. If T is carriable by all subset $B \times \Omega$ where B is the interior of a ρ_r-ball of radius a containing $B_0 \cap \cdots \cap B_l$, it is carriable by $(B_0 \cap \cdots \cap B_l) \times \Omega$.*

Remark. If K is a product compact subset of $\boldsymbol{C}^k$ and if T is carriable by all set $B \times \Omega$ where $\operatorname{int}(B) \supset K$, B being a ρ_r-ball, it follows in general only that T is semi-carriable by $K \times \Omega$ (Ω convex).

Proofs. We can prove the above two Propositions in the same way as in the proof of Propositions 3.4 and 3.4(k) of A. Martineau [35], Chapter 1, p. 57.

Q. E. D.

Definition 4.2. *Let R be a real vector space, and let K_1 and K_2 be two closed convex subsets of R. We say that K_2 is K_1-ribboned if, for any hyperplane P of R which does not encounter K_1, there exist $x \in R$ and $\lambda \in \boldsymbol{R}^+$ such that $(x+\lambda K_2) \supset K_1$ and $(x+\lambda K_2) \cap P = \phi$.*

If the property takes place for all compact subset K_1, K_2 is said to be ribboned.

If K_1 is a ρ-ball and $K_2 = a \cdot K_1$ where $a > 1$, we can take $\lambda = 1$.

We can easily verify that a necessary and sufficient condition that K is ribboned is that it is compact, that its interior is nonempty, and that it admits a tangent hyperplane at each of its boundary points.

ρ_1 and ρ_2 being two real norms, if the unit ball of ρ_2 is B_1-ribboned where B_1 denotes the unit ball of ρ_1, we say that ρ_2 is ρ_1-ribboned.

If the unit ball of ρ is ribonned, we say that ρ is a ribboned norm.

We will prove in Part II the following proposition.

Proposition 4.5. *If ρ_1 and ρ_2 are two complex norms on a complex vector space C and if ρ_2 is ρ_1-ribboned, a necessary and sufficient condition that an E-valued analytic linear mapping T defined on C is carriable by the ρ_1-ball B of radius a is that T is carriable by all ρ_2-ball containing B.*

In the case $\rho_2 = \rho_1$, if T is carriable by all ρ_1-balls of radius $b > a$ which contains B, it is carriable by B.

And also we have,

Proposition 4.5 bis. *Let R be a real vector space, and C its complexification. Let ρ_1 and ρ_2 be two real norms on R, and assume that ρ_2 is ρ_1-ribboned.*

A necessary and sufficient condition that an E-valued analytic linear mapping T be semi-carriable by the tube whose basis is the ρ_1-ball B of radius a is that T be semi-carriable by all tube whose basis is a ρ_2-ball containing B. In the case $\rho_2 = \rho_1$, if T is semi-carriable by all tubes with a ρ_1-ball of radius $b > a$ which contains B as basis, T is semi-carriable by B_T.

From here we draw the following analogs of Proposition 4.4.

Theorem 4.2. *Let ρ be a ribboned complex norm, and K a compact convex subset of a complex vector space C. A necessary and sufficient condition that an E-valued analytic linear mapping T defined on C be carriable by K is that it be carriable by all ρ-ball containing K.*

Theorem 4.2 bis. *Let ρ be a ribboned real norm in R and K a convex compact subset of R. A necessary and sufficient condition that an E-valued analytic* *linear mapping T defined on the complexification C of R be semi-carriable by K_T is that it be semi-carriable by all tube whose basis is a ρ-ball containing K.*

Proof. We can prove these theorems in the same way as that of Theorems 3.2 and 3.2 bis in A. Martineau [35], Chapter 1, p. 60. Q. E. D.

3. Some special cases

Theorem 4.3. a) *Let R be a real vector space, and C its complexification. Let T be an E-valued analytic linear mapping defined on $C \times \boldsymbol{C}$. Suppose that it admits a convex compact carrier of the form $\Gamma \times B$ where Γ is a compact subset of R and B is a compact subset of $\boldsymbol{C}$. Then it admits the smallest carrier of this type.*

b) *Let T be an E-valued analytic linear mapping defined on C. Suppose that it admits a real compact carrier. Then it admits the smallest real compact carrier (which we will call real support of T).*

If T is defined on $\boldsymbol{C}^n$ and admits a real support, all polydisc centered in $\boldsymbol{R}^n$ which is a carrier of T contains the real support of T.

Proof. a) If T is carriable by $\Gamma_1 \times B_1$ and by $\Gamma_2 \times B_2$, $\Gamma_1 \times B_1$ being minimal for the inclusion among the carriers of this type, it is carriable a fortiori by

$$A=(\Gamma_1 \cup \Gamma_2)\times B_1 \quad \text{and by} \quad B=(\Gamma_1 \cup \Gamma_2)\times B_2.$$

The set $A \cup B$ is equal to

$$(\Gamma_1 \cup \Gamma_2)\times(B_1 \cup B_2).$$

Consequently it is polynomially convex. Hence T is carriable by $(\Gamma_1 \cup \Gamma_2)\times (B_1 \cap B_2)$ [cf. Theorem 3.2]. If Γ is the convex hull of $\Gamma_1 \cup \Gamma_2$, T is carriable by $\Gamma \times (B_1 \cap B_2)$ and $\Gamma_i \times B_i$ $(i=1, 2)$. $\{\Gamma \times (B_1 \cap B_2)\} \cup (\Gamma_i \times B_i)$ $(i=1, 2)$ is polynomially convex. Hence T is carriable by $\{\Gamma \times (B_1 \cap B_2)\} \cap (\Gamma_i \times B_i)$, that is, by $\Gamma_i \times (B_1 \cap B_2)$ $(i=1, 2)$. Since $(\Gamma_1 \cup \Gamma_2)\times(B_1 \cap B_2)$ is again polynomially convex, T is carriable by $(\Gamma_1 \cap \Gamma_2)\times(B_1 \cap B_2)$. Hence we have

$$(\Gamma_1 \cap \Gamma_2)\times(B_1 \cap B_2)=\Gamma_1 \times B_1.$$

b) If T is carriable by two real compact subsets K_1 and K_2 of R, K_1 supposed to be minimal among the real compact carriers of T, T is carriable by $K_1 \cap K_2$ since $K_1 \cup K_2$ is polynomially convex. Hence $K_1 = K_1 \cap K_2$.

If T is defined on $\boldsymbol{C}^n$ and admits a real support K, let L be a polydisc $L=\prod_{i=1}^{n} D_i$ where D_i is a disc centered on $\boldsymbol{R}$, which is a carrier of T. If we denote by M_i the smallest segment containing the projection $\mathrm{pr}_i K$ of K onto the i-th real component and the real diameter of D_i, $L \cup (\prod_{i=1}^{n} M_i)$ is polynomially convex. Hence T is carriable by $L \cap (\prod_{i=1}^{n} M_i)$, that is, by the trace of L on $\boldsymbol{R}^n$ which, by virtue of the uniqueness of real support, necessarily contains K. Q. E. D.

§ 5. The analytic linear mappings with real support

In the following we will consider only the real analytic manifolds W which are countable at infinity.

We denote by $H(W)$ the space $H_W(V)$ where V is a complexification of W. Since two complexifications coincide in a neighborhood of W [cf. H. Whitney et F. Bruhat [53], Proposition 1, p. 133], $H_W(V)$ does not depend on the choice of V, and the same does for $H_{I,W}(V)$ and $H_{P,W}(V)$ which we will temporarily denote by $H_I(W)$ and $H_P(W)$ (in fact we can prove that these spaces are equal).

We can suppose that W is a regular submanifold of a space $\boldsymbol{R}^N$ for a certain positive integer N, trace of one of its complexifications V, V regular submanifold of a suitable polynomially convex neighborhood Ω of $\boldsymbol{R}^N$. (cf. Grauert, H. [9] and Whitney, H. et F. Bruhat [53]). In these conditions an element of $H'_I(W; E)$ can be considered as an E-valued analytic linear mapping defined on Ω (or on $\boldsymbol{C}^N$) which is strictly carriable by V in Ω.

Lemma. *If an E-valued analytic linear mapping T defined on $\boldsymbol{C}^n$ is semi-carriable by $\boldsymbol{R}^n$, it is in fact carriable by $\boldsymbol{R}^n$.*

Proof. We can take as a fundamental system of convex neighborhoods of $\boldsymbol{R}^n$ the sets

$$B_m = \left\{ z = (x_1 + iy_1, \dots, x_n + iy_n);\ |y_1| < \frac{1}{m}, \dots, |y_n| < \frac{1}{m} \right\}.$$

Let T be an E-valued analytic linear mapping which is quasi-carriable by $\boldsymbol{R}^n$, hence carriable be B_1.

We consider the analytic mapping $w = (w_1, \dots, w_n)$

$$w_l = \operatorname{th}\left(\frac{\pi}{4} z_l\right) \qquad (l = 1, 2, \dots, n)$$

which is an analytic isomorphism of B_1 onto the polydisc P_1:

$$|w_1| < 1, \dots, |w_n| < 1, \ w_l = u_l + iv_l \quad (l = 1, 2, \dots, n).$$

The image of $\boldsymbol{R}^n$ is the product K of real open segments $|u_l| < 1$ $(l = 1, 2, \dots, n)$. Let $\Theta = w(T)$. Θ is an E-valued analytic linear mapping defined on the polynomially convex set P_1, which we can consider to be defined on the space $\boldsymbol{C}^n(w_1, \dots, w_n)$. It is carriable by each open subset $w(B_i)$, a fortiori by all neighborhood of $\overline{K}$, hence by $\overline{K}$. On the other hand, being carriable by P_1, it is carriable by a certain compact polydisc $Q(|w_1|\leqq r,\dots, |w_n|\leqq r)$, $0\leqq r<1$.

Then we apply Theorem 4.3 b). Hence Θ is carriable by $\bar{K}\cap Q$, that is, by the compact subset L of K defined by $|u_l|\leqq r$, $l=1, 2,\dots, n$. Consequently T is carriable by $w^{-1}(L)$ which is a compact subset of $\boldsymbol{R}^n$. Q.E.D.

Theorem 5.1. *Let W be a C-analytic subset of $\boldsymbol{R}^N$* (cf. H. Whitney et F. Bruhat [53]). *If an E-valued analytic linear mapping T defined on $\boldsymbol{C}^N$ is quasi-carriable by W, it is in fact carriable by W.*

Proof. If T is an E-valued analytic linear mapping defined on $\boldsymbol{C}^N$ and quasi-carriable by W, it is a fortiori semi-carriable by $\boldsymbol{R}^N$, hence by virtue of the Lemma, admits a real support K.

Let V be a complexification of W in a suitable polynomially convex neighborhood Ω of $\boldsymbol{R}^N$ in $\boldsymbol{C}^N$. Since K is $H(\boldsymbol{C}^N)$-convex, hence a fortiori $H(\Omega)$-convex, by virtue of Proposition 3.7, T is carriable by $K\cap V=K\cap W$. Q.E.D.

Complement. All E-valued local analytic linear mapping T defined almost on W is an E-valued analytic linear mapping defined on $\boldsymbol{C}^N$ which is carriable by W.

Proof. In Ω, by virtue of Lemma 2 of Proposition 2.7 in A. Martineau [35], Chapter 1, p. 48, V admits a fundamental system of $H(\Omega)$-convex neighborhoods, hence finally is polynomially convex in the sense of Definition 2.10. It is the same for $\boldsymbol{R}^N$ (cf. H. Cartan [6]). But we can easily verify that all neighborhood of W contains the intersection of a neighborhood of $\boldsymbol{R}^N$ and a neighborhood of V, hence polynomially convex open subset. Consequently, we can identify $H'_I(W; E)$ with a subspace of $H'(\boldsymbol{C}^N; E)$. It is then sufficient to apply Theorem 5.1. Q.E.D.

Corollary 1. *Let W be a real analytic manifold. For all E-valued analytic linear mapping $T\in H'_I(W; E)$ there exists the smallest compact subset K of W which is a carrier of T.*

Proof. The E-valued analytic linear mapping T which is carriable by $K\cap W$ according to Theorem 5.1 and strictly carriable by a complexification V of W, a regular submanifold of Ω, is strictly carriable in V by $K\cap W$ by virtue of Theorem 3.6. If K_1 is minimal, in V, among compact carriers of T, it is also that in $\boldsymbol{C}^N$ (Corollary 1 of Theorem 3.6). From here we have the existence and the uniqueness of support thanks to Theorem 4.3. Q.E.D.

Corollary 2. *Let W be a real analytic manifold, and W_1 a C-analytic subset of W. If an E-valued analytic linear mapping T defined on a sufficiently small Stein neighborhood V of W is quasi-carriable by W_1, it is carriable by W_1.*

Corollary 3. *If T is carriable by a compact subset K of W and (quasi-) carriable by a C-analytic subset of W, then T is carriable by $K\cap W$. If T is* *(quasi-)carriable by C-analytic subsets W_i of W, it is carriable by* $\bigcap_i W_i$.

§ 6. Algebraic operations on the analytic linear mappings

In this section, we mention first the properties of the tensor products of analytic linear mappings. In the following we assume that E be a complete locally convex Hausdorff topological vector space.

Proposition 6.1. *Let V_1 and V_2 be two complex analytic manifolds which are countable at infinity. Then we have the following canonical isomorphism*:

$$H(V_1)\hat{\otimes}H(V_2)\cong H(V_1\times V_2).$$

Proof. See A. Grothendieck [13], Chapter 2, p. 81 and F. Trèves [52], Theorem 51.6, p. 530. Q.E.D.

Proposition 6.2. *Let V_1 and V_2 be as in Proposition* 6.1. *We have the following canonical isomorphism*:

$$H'(V_1)\hat{\otimes}H'(V_2)\cong L_b(H(V_1);\ H'(V_2))\cong H'(V_1\times V_2).$$

Proof. See F. Trèves [52], Propositions 50.5 and 50.7, p. 522 and p. 524, and Proposition 6.1. Q.E.D.

Proposition 6.3. *Let V_1 and V_2 be as in Proposition* 6.1. *We have the following canonical isomorphism*:

$$H'(V_1;\ E_1)\hat{\otimes}_\omega H'(V_2;\ E_2)\cong H'(V_1\times V_2;\ E_1\hat{\otimes}_\omega E_2).$$

Here E_1 and E_2 denote complete locally convex Hausdorff topological vector spaces and ω stands for ε or π topology in the terminology of F. Trèves [52].

Proof. Since the tensor products of locally convex Hausdorff topological vector spaces are commutative and associative, it is sufficient to apply Propositions 2.1 and 6.2. Q.E.D.

Thus we have the following definition of the tensor product of analytic linear mappings.

Definition 6.1. *We use the notations of Proposition* 6.3. *Let* $T_i=u_i\otimes\mathbf{e}_i\in H'(V_i;\ E_i)$, $u_i\in H'(V_i)$, $\mathbf{e}_i\in E_i$ $(i=1, 2)$. *Then we define*

$$T_1\otimes_\omega T_2=(u_1\otimes u_2)\otimes(\mathbf{e}_1\otimes_\omega\mathbf{e}_2),$$

i.e.

$$(T_1\otimes_\omega T_2)(f_1\otimes f_2)=u_1(f_1)u_2(f_2)(\mathbf{e}_1\otimes_\omega\mathbf{e}_2)\qquad \textit{for}\ \ f_i\in H(V_i)\quad (i=1,2).$$

Proposition 6.4 *If the E_i-valued analytic linear mapping T_i is carriable by an open (or compact) subset L_i $(i=1, 2)$, $T_1 \otimes_\omega T_2$ is carriable by $L_1 \times L_2$.*

Next we mention the definition and properties of the convolutions of analytic linear mappings.

If V is a complex analytic manifold and τ is an analytic mapping of $V\times V$ into V, we can deduce from this a continuous linear mapping ${}^t\tau$ of $H(V)$ into $H(V)\hat{\otimes} H(V)$, which is isomorphic to $H(V\times V)$, as follows

$$f \longrightarrow {}^t\tau f = ((z_1, z_2) \longrightarrow f(\tau(z_1, z_2))), \quad z_i \in V_i, \quad i=1, 2.$$

The transpose of this mapping defines a continuous bilinear mapping of $H'(V; E_1) \times H'(V; E_2)$ into $H'(V; E_1 \hat{\otimes}_\omega E_2)$, which is denoted by

$$(T_1, T_2) \longrightarrow T_1 \tau_\omega T_2,$$

where E_1 and E_2 denote complete locally convex Hausdorff topological vector spaces, and ω stands for ε or π topology.

$T_1 \tau_\omega T_2$ is defined by the following equality

$$(T_1 \tau_\omega T_2)(f) = (T_1 \otimes_\omega T_2)(f(\tau(z_1, z_2))).$$

Definition 6.2. *We call the analytic linear mapping $T_1 \tau_\omega T_2$ the composition product associated with τ_ω of T_1 and T_2.*

In fact, in the following we interest only in the following case:

$V = C$ is a complex vector space, and τ is the addition of vectors of C. In this case we say that $T_1 \tau_\omega T_2$, which we denote by $T_1 *_\omega T_2$, is the convolution of analytic linear mappings T_1 and T_2.

If μ_1 is an E_1-valued measure which represents T_1, μ_2 an E_2-valued measure which represents T_2, then $\mu_1 \otimes_\omega \mu_2$ represents $T_1 \otimes_\omega T_2$ on $C\times C$. For if $\phi(x)\psi(y)$ is a decomposed holomorphic function on this space, we have, by the definition itself,

$$\begin{aligned}(T_1 \otimes_\omega T_2)(\phi \cdot \psi) &= T_1(\phi) \otimes_\omega T_2(\psi) = \mu_1(\phi) \otimes_\omega \mu_2(\psi)\\ &= (\mu_1 \otimes_\omega \mu_2)(\phi \cdot \psi).\end{aligned}$$

Consequently, $\mu_1 *_\omega \mu_2$ (in the sense of L. Schwartz [46]) represents $T_1 *_\omega T_2$.

Since, if K_1 is the support of an E_1-valued measure μ_1, and K_2 is that of an E_2-valued measure μ_2, $\mu_1 *_\omega \mu_2$ has its support in $K_1 + K_2$, we deduce by virtue of Corollary of Theorem 2.1,

Proposition 6.5. *If an E_i-valued analytic linear mapping T_i is (weakly) carriable by L_i, L_i being open (or compact) $(i=1, 2)$, then $T_1 *_\omega T_2$ is (weakly) carriable by $L_1 + L_2$.*

Corollary. *If T_1 and T_2, defined on a complexification C of a real vector space R, admit a real support, $T_1 *_\omega T_2$ admits a real support.*

§7. The analytic linear mappings as boundary values of vector valued holomorphic functions

1. The analytic linear mappings as cohomology classes. Martineau-Harvey's theorem

$\mathcal{O}$ denotes the sheaf of germs of holomorphic functions on a complex analytic manifold Y and ${}^E\mathcal{O}$ denotes the sheaf of germs of holomorphic functions on Y valued in a Fréchet space E.

Definition 7.1 (Martineau [33]). *Let Y be a complex analytic manifold. A compact subset K of Y is said to be almost convex if $H^i(K, \mathcal{O})=0$ for all $i>0$, and if it admits a Stein neighborhood.*

In particular if a compact subset K of Y admits a fundamental system of Stein neighborhoods, it is almost convex.

Theorem 7.1 (Martineau-Harvey). *Let Y be a complex manifold and K an almost convex compact subset of Y. Then, for a Stein neighborhood V of K, we have*

(i) $H_K^p(V, {}^E\mathcal{O})=0, \quad p\neq n,$

(ii) $H_K^n(V, {}^E\mathcal{O})\cong H^{n-1}(V-K, {}^E\mathcal{O})\cong L_b(\mathcal{O}(K); E).$

Here E denotes a Fréchet space.

Remark. In the above theorem, except for the first isomorphism in (ii), V has only to be an open neighborhood of K.

Proof. Let $\mathcal{F}^{p,q}$ denotes the sheaf of germs of differential forms of type (p, q) with coefficients in a sheaf $\mathcal{F}$ over Y. $\mathfrak{a}$ denotes the sheaf of germs of real analytic functions on Y and $\mathcal{B}$ the sheaf of germs of hyperfunctions on Y and ${}^E\mathcal{B}$ the sheaf of germs of E-valued hyperfunctions on Y. Then we have the resolutions of $\mathcal{O}$ and ${}^E\mathcal{O}$ (cf. Y. Ito [18]):

$$(1) \quad 0 \longrightarrow \mathcal{O} \longrightarrow \mathfrak{a}^{0,0} \xrightarrow{\bar\partial} \mathfrak{a}^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathfrak{a}^{0,n} \longrightarrow 0,$$

$$(2) \quad 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{B}^{0,0} \xrightarrow{\bar\partial} \mathcal{B}^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{B}^{0,n} \longrightarrow 0,$$

$$(3) \quad 0 \longrightarrow {}^E\mathcal{O} \longrightarrow {}^E\mathcal{B}^{0,0} \xrightarrow{\bar\partial} {}^E\mathcal{B}^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {}^E\mathcal{B}^{0,n} \longrightarrow 0.$$

Since we have

$$H^p(K, \mathfrak{a}^{0,q}) = \lim_{\Omega \supset K}\text{ind.}\, H^p(\Omega, \mathfrak{a}^{0,q}) = 0, \qquad \text{for} \quad p>0,\ q \geqq 0,$$

by virtue of Theorem 4.11.1 of R. Godement [8], p. 193, the cohomology groups $H^p(K, \mathcal{O})$ are isomorphic to the cohomology groups of the complex:

$$(4) \quad 0 \longrightarrow \mathfrak{a}^{0,0}(K) \xrightarrow{\bar\partial} \mathfrak{a}^{0,1}(K) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathfrak{a}^{0,n}(K) \longrightarrow 0,$$

and since the sheaves ${}^E\mathscr{B}^{0,p}$ are flabby, the relative cohomology groups $H^p_K(V, {}^E\mathcal{O})$ with support in K are isomorphic to the cohomology groups of the complex:

$$(5) \quad 0 \longrightarrow {}^E\mathscr{B}^{0,0}_K \xrightarrow{\bar\partial} {}^E\mathscr{B}^{0,1}_K \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {}^E\mathscr{B}^{0,n}_K \longrightarrow 0$$

(where ${}^E\mathscr{B}^{0,p}_K$ stands for $\mathscr{B}^{0,p}_K(Y; E)$). The hypotheses $H^p(K, \mathcal{O})=0$, $p>0$, imply that the sequence (4) is exact. Hence, the operators $\bar\partial$ are homomorphisms, since they are of closed range, for the spaces $\mathfrak{a}^{0,p}(K)$ are DFS-spaces (cf. A. Grothendieck [12], Chapter 4, §2, Theorem 3, p. 218).

If we denote $\mathscr{B}^{0,p}_K(Y)$ by $\mathscr{B}^{0,p}_K$, the spaces $\mathfrak{a}^{0,p}(K)$ and $\mathscr{B}^{0,n-p}_K$ are DFS- and FS-spaces by the duality pairing:

$$\begin{aligned}&\Big\langle \sum_{|J|=n-p} \phi_J d\bar z_J,\ \sum_{|I|=p} f_I d\bar z_I \Big\rangle \\ &\quad = \sum_{I \cup J=(1,\dots,n)} \varepsilon_{I,J} \langle \phi_J, f_I \rangle,\end{aligned}$$

where $\varepsilon_{I,J}$ denotes the signature of the permutation $(1,\dots, n) \to (I, J)$, and other notations are usual ones. Further, the transpose of the operator

$$\bar\partial\colon \mathfrak{a}^{0,p}(K) \longrightarrow \mathfrak{a}^{0,p+1}(K)$$

is (aside from sign) the operator

$$\bar\partial\colon \mathscr{B}^{0,n-p-1}_K \longrightarrow \mathscr{B}^{0,n-p}_K.$$

By virtue of Serre's lemma (cf. H. Komatsu [24], Theorem 19, p. 381 and others) the sequence

$$0 \longrightarrow \mathscr{B}^{0,0}_K \xrightarrow{\bar\partial} \mathscr{B}^{0,1}_K \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{B}^{0,n}_K$$

is exact. Hence the sequence

$$0 \longrightarrow {}^E\mathscr{B}^{0,0}_K \xrightarrow{\bar\partial} {}^E\mathscr{B}^{0,1}_K \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {}^E\mathscr{B}^{0,n}_K$$

is also exact by virtue of Theorem 1.10 of P.D.F. Ion and T. Kawai [17], p. 9, since the spaces $\mathscr{B}^{0,p}_K$ are all nuclear FS-spaces and ${}^E\mathscr{B}^{0,p}_K \cong \mathscr{B}^{0,p}_K \hat\otimes E$. Consequently, we have

$$H^p_K(V, {}^E\mathcal{O}) = 0 \qquad \text{for} \quad p<n.$$

Next we consider two exact sequences:

$$0 \longrightarrow \mathcal{O}(K) \longrightarrow \mathfrak{a}^{0,0}(K) \xrightarrow{\bar\partial} \mathfrak{a}^{0,1}(K)$$

$$0 \longleftarrow \mathscr{B}_K^{0,n}/\bar\partial\mathscr{B}_K^{0,n-1} \longleftarrow \mathscr{B}_K^{0,n} \xleftarrow{\bar\partial} \mathscr{B}_K^{0,n-1}.$$

The mapping

$$\bar\partial\colon \mathscr{B}_K^{0,n-1} \longrightarrow \mathscr{B}_K^{0,n}$$

is of closed range since it is the transpose of a mapping of closed range. Hence

$$\mathscr{B}_K^{0,n}/\bar\partial\mathscr{B}_K^{0,n-1} \text{ (which is isomorphic to } H_K^n(V, \mathcal{O}))$$

is isomorphic to $\mathcal{O}'(K)$. Since we have

$${}^E\mathscr{B}_K^{0,p} \cong \mathscr{B}_K^{0,p}\hat\otimes E \quad \text{and} \quad L_b(\mathcal{O}(K); E) \cong \mathcal{O}'(K)\hat\otimes E,$$

we conclude that

$$H_K^n(V, {}^E\mathcal{O}) \cong H_K^n(V, \mathcal{O})\hat\otimes E \cong L_b(\mathcal{O}(K); E).$$

At last we will prove the first isomorphism in the statement (ii). We now assume that V is a Stein neighborhood. The exact sequence of relative cohomology groups can be written,

$$0 = H^{n-1}(V, {}^E\mathcal{O}) \longrightarrow H^{n-1}(V-K, {}^E\mathcal{O}) \longrightarrow H_K^n(V, {}^E\mathcal{O}) \longrightarrow H^n(V, {}^E\mathcal{O}) = 0.$$

(cf. H. Komatsu [25], Theorem II.3.2, p. 77, and P. Schapira [42], Corollary 1 of Theorem B.35, p. 32). Hence $H_K^n(V, {}^E\mathcal{O})$ is isomorphic to $H^{n-1}(V-K, {}^E\mathcal{O})$.

Q. E. D.

This theorem shows that an E-valued analytic linear mapping can be represented as a relative cohomology class or a cohomology class. This generalizes the theory of J. S. Silva-G. Köthe-A. Grothendieck-M. Morimoto [49], [28], [10], [37].

2. Cauchy-Weil transformation and Cauchy-Hilbert transformation of E-valued analytic linear mappings

In the following we assume that $V = \boldsymbol{C}^n$ and E is a Fréchet space.

We will first establish the Cauchy-Weil's integral formula following M. Morimoto [37].

We say that $\Omega \subset \boldsymbol{C}$ is a ring domain if, for some $r, R \in \boldsymbol{R}$, $0 \leqq r < R < \infty$ and for some $z_0 \in \boldsymbol{C}$, we can write

$$\Omega = \{z \in \boldsymbol{C};\ r < |z - z_0| < R\}.$$

(We consider that an open disc in $\boldsymbol{C}$ is a ring domain). z_0 is said to be the center of Ω. In the following of this section the letter Ω denotes a ring domain.

We put $V=\boldsymbol{C}_z^n$, $V^*=\boldsymbol{C}_\zeta^n$. A compact set L in V is said to be a special polyhedron if, for a finite number of ring domains Ω_j and $\zeta_j \in V^*$, $j=1, 2,\dots, m$, L is represented as

$$L=\bigcap_{j=1}^{m} \zeta_j^{-1}(\bar{\Omega}_j)=\{z \in V;\ r_j \leqq |\zeta_j(z)| \leqq R_j,\ j=1, 2,\dots, m\}.$$

Since L is compact, ζ_j, $j=1, 2,\dots, m$ must generate V^* when $m \geqq n$.

Now, we assume that ζ_j is in a generic position, that is, each n of them generate V^*. We put

$$S_j=L \cap \zeta_j^{-1}(\partial\bar{\Omega}_j),$$

and call it a surface of L. We give the surface S_j the orientation as a subdomain of $\zeta_j^{-1}(\partial\bar{\Omega}_j)=\partial(\zeta_j^{-1}(\bar{\Omega}_j))$ oriented as the boundary of $\zeta_j^{-1}(\bar{\Omega}_j)$. We put

$$S_{j_0 j_1 \cdots j_k}=S_{j_0} \cap S_{j_1} \cap \cdots \cap S_{j_k}.$$

Then this is a surface with boundary, of dimension $2n-k-1$. We define the orientation of $S_{j_0 \cdots j_k}$ by induction with respect to k:

Assuming that $S_{j_0 \cdots j_{k-1}}$ is already oriented, we give $S_{j_0 \cdots j_{k-1} j_k}$ the orientation as the boundary of the subdomain $S_{j_0 \cdots j_{k-1}} \cap \zeta_{j_k}^{-1}(\bar{\Omega}_{j_k})$ of $S_{j_0 \cdots j_{k-1}}$.

If we exchange the indices of $S_{j_0 \cdots j_k}$, its orientation is reversed although the two surface is identical as sets. Until now we defined $S_{j_0 \cdots j_k}$ assuming that ζ_{j_0}, $\dots, \zeta_{j_k}$ are in a generic position, but we agree to define $S_{j_0 \cdots j_k}=\phi$ if $\zeta_{j_0},\dots, \zeta_{j_k}$, $1 \leqq k \leqq n-1$, are linearly dependent (especially, the same indice appears more than twice). Evidently, we have

$$\partial S_{j_0 \cdots j_k}=\sum_{i=1}^{m} S_{j_0 \cdots j_k i}$$

taking into consideration the above mentioned orientation.

We can now formulate the Cauchy-Weil's integral formula.

Theorem 7.2 (Cauchy-Weil). *Let L be a special polyhedron represented as*

$$L=\bigcap_{j=1}^{m} \zeta_j^{-1}(\bar{\Omega}_j)=\{z \in V;\ r_j \leqq |\zeta_j(z)| \leqq R_j,\ \zeta_j \in V^*,\ j=1, 2,\dots, m\},$$

and S_j its surface. Then, for a holomorphic function f in a neighborhood of L, we have the formula

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \sum_{i_0, i_1,\dots, i_{n-1}} \int_{S_{i_0 i_1 \cdots i_{n-1}}} f(w) \bigwedge_{j=0}^{n-1} \frac{d\zeta_{i_j}(w)}{\zeta_{i_j}(w)-\zeta_{i_j}(z)}$$

$$=\begin{cases} f(z), & z \in \overset{\circ}{L}(=\text{the interior of } L), \\ 0, & z \in V \backslash (L \cup M), \end{cases}$$

where $\sum$ is taken over all possible combinations of $i_0, i_1, \dots, i_{n-1}$, and we put

$$M = \bigcup_{j=1}^{m} \zeta_j^{-1}(\partial \bar{\Omega}_j).$$

Proof. See M. Morimoto [37], Theorem 6.1.2, p. 147. Q. E. D.

We now introduce the notion of linearly convex subsets of $\boldsymbol{C}^n$.

Definition 7.2 (M. Morimoto). *A subset X of $\boldsymbol{C}^n$ is said to be linearly convex if, for any $w \notin X$, there exists a complex hyperplane S such that*

$$w \in S, \quad \text{and} \quad S \cap X = \phi,$$

in other words, there exists $\zeta \in V^*$ such that

$$\zeta(w) \neq \zeta(z) \qquad \text{for any} \quad z \in X.$$

The necessary and sufficient condition that a compact subset K of V is linearly convex is that the family

$$\mathfrak{W}(V \backslash K) = \{W_\zeta \equiv \zeta^{-1}(\boldsymbol{C} \backslash \zeta(K));\ \zeta \in V^* \backslash \{0\}\}$$

becomes an open covering of $V \backslash K$.

Then we have

Lemma 7.1. *Let K be a linearly convex compact subset of V, and W an open neighborhood of K. Then there exists a special polyhedron L such that*

$$K \subset \mathring{L}, \quad \text{and} \quad L \subset W.$$

(Here we say that L is placed between K and W).

Proof. See M. Morimoto [37], Lemma 6.2.1, p. 150. Q. E. D.

Hence, for a linearly convex compact set K, an arbitrary element of $H(K)$ has an integral representation by the Cauchy-Weil's integral formula.

Now we consider the space $H'(K; E)$ of E-valued local analytic linear mappings defined on K when K is a linearly convex compact subset of V. We will represent $H'(K; E)$ as the space of certain cocycles.

Let K be a linearly convex compact subset of V. For $\zeta \in V^* \backslash \{0\}$, we put

$$W_\zeta = \zeta^{-1}(\boldsymbol{C} \backslash \zeta(K)).$$

If $\zeta = \lambda \zeta'$ for a complex number $\lambda \neq 0$, we have $W_\zeta = W_{\zeta'}$. Hence, if we denote by $[\zeta]$ of the class of ζ in $(V^* \backslash \{0\}) / \boldsymbol{C}^*$, we can put $W_{[\zeta]} = W_\zeta$. By the assumption of linear convexity of K, the family of open subsets

$$\mathfrak{W}(V \backslash K) = \{W_\zeta;\ \zeta \in V^* \backslash \{0\}\}$$

becomes an open covering of $V\backslash K$. ${}^{E}\mathcal{O}^{(n,0)}$ denotes the sheaf of differential forms of type $(n, 0)$ with E-valued holomorphic functions as coefficients. Then $Z^{n-1}(\mathfrak{W}(V\backslash K); {}^{E}\mathcal{O}^{(n,0)})$ denotes the space of $(n-1)$-cocycles of $\mathfrak{W}(V\backslash K)$ with coefficients in ${}^{E}\mathcal{O}^{(n,0)}$.

Definition 7.3. *$\psi \in Z^{n-1}(\mathfrak{W}(V\backslash K); {}^{E}\mathcal{O}^{(n,0)})$ is said to be zero at infinity if the following two conditions hold:*

(i) *if $\zeta_0, \zeta_1, \ldots, \zeta_{n-1}$ are linearly dependent over $\boldsymbol{C}$, $\psi_{\zeta_0\zeta_1\cdots\zeta_{n-1}}=0$.*

(ii) *In the case where $\zeta_0, \zeta_1, \ldots, \zeta_{n-1}$ are linearly independent over $\boldsymbol{C}$, for*

$$\psi_{\zeta_0\zeta_1\cdots\zeta_{n-1}}=\vartheta_{\zeta_0\zeta_1\cdots\zeta_{n-1}}d\zeta_0\wedge d\zeta_1\wedge\cdots\wedge d\zeta_{n-1},$$

$$\vartheta_{\zeta_0\zeta_1\cdots\zeta_{n-1}}\in {}^{E}\mathcal{O}(W_{\zeta_0\zeta_1\cdots\zeta_{n-1}}),$$

and for an arbitrary open neighborhood U_j of $\zeta_j(K)$, we have $\vartheta_{\zeta_0\zeta_1\cdots\zeta_{n-1}}\to 0$ if $\zeta_j(z)\notin U_j$, $j=0, 1, 2,\ldots, n-1$, and $\sum_{j=0}^{n-1}|\zeta_j(z)|\to\infty$. We denote by

$$Z_0^{n-1}(\mathfrak{W}(V\backslash K); {}^{E}\mathcal{O}^{(n,0)})$$

the set of all cocycles $\psi\in Z^{n-1}(\mathfrak{W}(V\backslash K); {}^{E}\mathcal{O}^{(n,0)})$ which are zero at infinity.

$Z_0^{n-1}(\mathfrak{W}(V\backslash K); {}^{E}\mathcal{O}^{(n,0)})$, being considered as a subspace of $\oplus {}^{E}\mathcal{O}(W_{\zeta_0\zeta_1\cdots\zeta_{n-1}})$, has the FS*-space structure as the closed subspace of FS*-space (cf. H. Komatsu [24], Theorem 2, p. 370).

When $\zeta_0, \zeta_1,\ldots, \zeta_{n-1}\in V^*$ are given, by the correspondence of the function of z, $\prod_{j=0}^{n-1}(\zeta_j(w)-\zeta_j(z))^{-1}$, to w, a continuous mapping of $W_{\zeta_0\zeta_1\cdots\zeta_{n-1}}$ into $H(K)$ is determined. Hence, for $T\in H'(K; E)$,

$$w\longrightarrow T_z\Big(\prod_{j=0}^{n-1}(\zeta_j(w)-\zeta_j(z))^{-1}\Big)$$

determines an E-valued continuous function on $W_{\zeta_0\zeta_1\cdots\zeta_{n-1}}$. It is easily seen by the representation of T by a measure that this is holomorphic with respect to $w\in W_{\zeta_0\zeta_1\cdots\zeta_{n-1}}$. We put

$$\begin{aligned}&\psi_{\zeta_0\zeta_1\cdots\zeta_{n-1}}(w)\\&=T_z\Big(\prod_{j=0}^{n-1}(\zeta_j(w)-\zeta_j(z))^{-1}\Big)d\zeta_0(w)\wedge d\zeta_1(w)\wedge\cdots\wedge d\zeta_{n-1}(w).\end{aligned}$$

$\{\psi_{\zeta_0\zeta_1\cdots\zeta_{n-1}}\}$ determines an element of $Z_0^{n-1}(\mathfrak{W}(V\backslash K); {}^{E}\mathcal{O}^{(n,0)})$. We call the mapping which associates $T\in H'(K; E)$ with $\psi=\{\psi_{\zeta_0\zeta_1\cdots\zeta_{n-1}}\}\in Z_0^{n-1}(\mathfrak{W}(V\backslash K); {}^{E}\mathcal{O}^{(n,0)})$ the Cauchy-Weil transformation, and denote it by $\psi=\mathbf{CW}T$.

Theorem 7.3. *Let K be a linearly convex compact subset of V, $\mathfrak{W}(V\backslash K)$ an open covering*

$$\mathfrak{W}(V\backslash K)=\{W_\zeta;\ \zeta\in V^*\backslash\{0\}\}.$$

Then the Cauchy-Weil transformation **CW** *is an isomorphism of $H'(K;E)$ onto $Z_0^{n-1}(\mathfrak{W}(V\backslash K);\ {}^E\mathcal{O}^{(n,0)})$ as topological vector spaces.*

Proof. It is evident that the Cauchy-Weil transformation is a continuous linear map. If we show that **CW** is bijective, it follows from the closed graph theorem that **CW** is a topological isomorphism.

Proof of injectivity. Let L be a special polyhedron such that $K\subset\mathring{L}$. These $\mathring{L}$'s form a fundamental system of neighborhoods of K by virtue of Lemma 7.1. For a holomorphic function f in a neighborhood of L, we have a Cauchy-Weil's formula

$$f(z)=\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\sum_{i_0,\dots i_{n-1}}\int_{S_{i_0i_1\cdots i_{n-1}}}f(w)\frac{d\zeta_{i_0}(w)\wedge\cdots\wedge d\zeta_{i_{n-1}}(w)}{\prod_{j=0}^{n-1}(\zeta_{i_j}(w)-\zeta_{i_j}(z))},\quad z\in L.$$

Since this integral converges in the topology of $H(K)$, for $T\in H'(K;E)$, we can interchange T and the integral and have

$$T(f)=\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\sum_{i_0,i_1,\dots,i_{n-1}}\int_{S_{i_0i_1\cdots i_{n-1}}}f(w)\psi_{i_0i_1\cdots i_{n-1}}(w).$$

Here $\psi_{i_0i_1\cdots i_{n-1}}$ is given as above. Hence if $\psi=0$, we must have $T=0$.

Proof of surjectivity. Assume that $\psi\in Z_0^{n-1}(\mathfrak{W}(V\backslash K);\ {}^E\mathcal{O}^{(n,0)})$ and $f\in H(K)$ are given. Now, let W be an open neighborhood of K such that $f\in H(W)$ and L a special polyhedron placed between K and W. Then, if we put

$$T_\psi(f)=\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\sum_{i_0,\dots,i_{n-1}}\int_{S_{i_0i_1\cdots i_{n-1}}}f(w)\psi_{i_0i_1\cdots i_{n-1}}(w),$$

the right hand side does not depend on the choice of W and L and $T_\psi\in H'(K;E)$ is determined. Then we have $\mathbf{CW}T_\psi=\psi$. In the following, we will show this step by step.

First, in order to show that the right hand side of the above equation does not depend on the choice of W and L, we have only to show that, for special polyhedra L_1 and L_2 placed between K and W, the integral in the right hand side has the same value. Since $L_1\cap L_2$ is also a special polyhedron placed between K and W, we may assume $L_2\subset L_1$. By induction, we may consider only the case where

$$L_1=\bigcap_{j=1}^N\zeta_j^{-1}(\bar{\Omega}_j),$$

$$L_2=L_1\cap\zeta_{N+1}^{-1}(\bar{\Omega}_{N+1}).$$

Here, of course, Ω_{N+1} is a ring domain such that $\zeta_{N+1}(K)\subset\Omega_{N+1}$.

Let D be an open disc cocentered with Ω_{N+1} such that $\zeta_{N+1}(L_1)\subset D$. Since

$D\backslash\bar{\Omega}_{N+1}$ is a union of two ring domains, $L_2'=\zeta_{N+1}^{-1}(\boldsymbol{C}\backslash\Omega_{N+1})\cap L_1=\zeta_{N+1}^{-1}(D\backslash\Omega_{N+1})\cap L_1$ being a union of two special polyhedra, we have $L_1=L_2\cup L_2'$. Writing in short the right hand side by $\int_L f\psi$, we have

$$\int_{L_1} f\psi=\int_{L_2} f\psi+\int_{L_2'} f\psi.$$

Hence we have to prove

$$\int_{L_2'} f\psi=0.$$

Denoting in general by S_j' the surface of L_2', we can calculate as follows.

$$\int_{L_2'} f\psi=\sum_{i_0,i_1,\dots,i_{n-1}<N+1}\int_{S'_{i_0\cdots i_{n-1}}} f\psi_{i_0\cdots i_{n-1}}$$
$$+\sum_{i_0,i_1,\dots,i_{n-2}<N+1}\int_{S'_{i_0\cdots i_{n-2}N+1}} f\psi_{i_0\cdots i_{n-2}N+1}.$$

In case $i_0,\dots,i_{n-2}<N+1$, $\psi_{i_0\cdots i_{n-2}N+1}$ is defined in a neighborhood of $S'_{i_0\cdots i_{n-2}}$ and is holomorphic there. In fact, since $W_{N+1}=\zeta_{N+1}^{-1}(\boldsymbol{C}\backslash\zeta_{N+1}(K))$ contains L_2', we have

$$S'_{i_0\cdots i_{n-2}}\subset L_2'\cap W_{i_0i_1\cdots i_{n-2}}\subset W_{i_0\cdots i_{n-2}N+1}.$$

In particular, for $i_0, i_1,\dots, i_{n-1}<N+1$, we have

$$\psi_{i_0i_1\cdots i_{n-1}}=(-1)^{n+1}\sum_{h=0}^{n-1}(-1)^h\psi_{i_0\cdots \hat{i}_h\cdots i_{n-1}N+1}$$

in a neighborhood of $S'_{i_0\cdots i_{n-1}}$. Hence we can write the first term of the right hand side into

$$\sum_{i_0,\dots,i_{n-1}<N+1}(-1)^{(n+1)}\sum_{h=0}^{n-1}(-1)^h\int_{S'_{i_0\cdots i_{n-1}}} f\psi_{i_0\cdots \hat{i}_h\cdots i_{n-1}N+1}$$

Arranging this with respect to the term $\psi_{i_0\cdots i_{n-2}N+1}$, we have

$$\int_{L_2'} f\psi=\sum_{i_0,\dots,i_{n-2}<N+1}\sum_{i=1}^{N+1}\int_{S'_{i_0\cdots i_{n-2}i}} f\psi_{i_0i_1\cdots i_{n-2}N+1}$$
$$=\sum_{i_0,\dots,i_{n-2}<N+1}\int_{\partial S'_{i_0\cdots i_{n-2}}} f\psi_{i_0i_1\cdots i_{n-2}N+1}.$$

Hence, since the integrand form is holomorphic in a neighborhood of $S'_{i_0\cdots i_{n-2}}$, we have

$$\int_{L_2'} f\psi=0$$

by virtue of the vector valued variant of the Cauchy-Poincaré's theorem (cf. M. Morimoto [37], Theorem 6.1.1, p. 145).

By the definition of the topology of $H(K)$, T_ψ is evidently a continuous linear map of $H(K)$ into E.

Now we will show $\mathbf{CW}T_\psi=\psi$.

Let $\zeta_0, \zeta_1,\dots, \zeta_{n-1}\in V^*$ be linearly independent. For $\psi\in Z_0^{n-1}(\mathfrak{W}(V\backslash K); {}^E\mathcal{O}^{(n,0)})$, we put

$$\psi_{\zeta_0\zeta_1\cdots\zeta_{n-1}}=\vartheta_{\zeta_0\zeta_1\cdots\zeta_{n-1}}d\zeta_0\wedge d\zeta_1\wedge\cdots\wedge d\zeta_{n-1}.$$

Putting $\mathbf{CW}T_\psi=\tilde{\psi}$, we calculate $\tilde{\psi}_{\zeta_0\zeta_1\cdots\zeta_{n-1}}$. Let Ω_j be a ring domain containing $\zeta_j(K)$, and $w\in V$ satisfy $\zeta_j(w)\notin\bar{\Omega}_j$, $j=0, 1,\dots, n-1$. Since the integral defining T_ψ does not depend on the choice of L, we have, by the definition of $\mathbf{CW}$,

$$\tilde{\psi}_{\zeta_0\zeta_1\cdots\zeta_{n-1}}(w)=\tilde{\vartheta}_{\zeta_0\zeta_1\cdots\zeta_{n-1}}(w)d\zeta_0(w)\wedge\cdots\wedge d\zeta_{n-1}(w),$$

$$\tilde{\vartheta}_{\zeta_0\zeta_1\cdots\zeta_{n-1}}(w)$$

$$=\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\int_{\partial\zeta_0^{-1}(\Omega_0)\wedge\cdots\wedge\partial\zeta_{n-1}^{-1}(\Omega_{n-1})}\frac{\vartheta_{\zeta_0\zeta_1\cdots\zeta_{n-1}}(z)}{\prod_{j=0}^{n-1}(\zeta_j(w)-\zeta_j(z))}d\zeta_0(z)\wedge\cdots\wedge d\zeta_{n-1}(z)$$

Since $\zeta_j(w)\notin\Omega_j$ and $\vartheta_{\zeta_0\zeta_1\cdots\zeta_{n-1}}$ is zero at infinity, we have, by the usual Cauchy integral formula,

$$\tilde{\vartheta}_{\zeta_0\zeta_1\cdots\zeta_{n-1}}(w)=\vartheta_{\zeta_0\zeta_1\cdots\zeta_{n-1}}(w).$$

The choice of a ring domain Ω_j containing $\zeta_j(K)$ being arbitrary, this equality holds for an arbitrary $w\in W_{\zeta_0\zeta_1\cdots\zeta_{n-1}}$. Hence we have

$$\tilde{\psi}_{\zeta_0\zeta_1\cdots\zeta_{n-1}}=\psi_{\zeta_0\zeta_1\cdots\zeta_{n-1}}. \qquad \text{Q.E.D.}$$

Now, if, for a finite number of $\zeta_1, \zeta_2,\dots, \zeta_N\in V^*$, a compact subset K of V can be represented as

$$K=\bigcap_{j=1}^{N}\zeta_j^{-1}(\zeta_j(K)),$$

we call temporarily K a compact polyhedron. A compact polyhedron is evidently linearly convex. Let K be a compact polyhedron of the above form. If we put

$$\mathfrak{W}'(V\backslash K)=\{W_j=\zeta_j^{-1}(\mathbf{C}\backslash\zeta_j(K));\, j=1, 2,\dots, N\},$$

$\mathfrak{W}'(V\backslash K)$ is a Stein open covering of $V\backslash K$ and $\mathfrak{W}'(V\backslash K)\subset\mathfrak{W}(V\backslash K)$ holds. The restriction map acts as

$$Z_0^{n-1}(\mathfrak{W}(V\backslash K); {}^E\mathcal{O}^{(n,0)})\longrightarrow Z_0^{n-1}(\mathfrak{W}'(V\backslash K); {}^E\mathcal{O}^{(n,0)}).$$

Now we consider a compact set L_j such that $\zeta_j(K)\Subset L_j\Subset\mathbf{C}$ and ∂L_j is piecewise smooth. By Lemma 2.1.1 of M. Morimoto [37], p. 27, such L_j's form a fundamental system of neighborhoods of $\zeta_j(K)$. Hence a polyhedron of the form

$$L=\bigcap_{j=1}^{N}\zeta_j^{-1}(L_j)$$

whose boundary is piecewise smooth can be taken as a fundamental system of neighborhoods of K. As the integral domain of the right hand side of the defining equation of T_ψ, we have only to take into consideration the distinguished boundary of such L. Hence, for $\psi\in Z_0^{n-1}(\mathfrak{W}'(V\backslash K);\ {}^E\mathcal{O}^{(n,0)})$, by the formula

$$T_\psi(f)=\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\sum_{i_0\cdots i_{n-1}}\int_{S_{i_0\cdots i_{n-1}}}f(w)\psi_{i_0\cdots i_{n-1}}(w)\,,$$

$T_\psi\in H'(K;\ E)$ is determined. We will write this as $\mathbf{T}\colon\psi\to T_\psi$. Then we can show the following corollary 1 by the calculation of the proof of Theorem 7.3.

Corollary 1. *Let K be a compact polyhedron of the form*

$$K=\bigcap_{j=1}^{N}\zeta_j^{-1}(\zeta_j(K)),\quad \zeta_j\in V^*\quad j=1,\ 2,\dots,\ N.$$

Then the following diagram is commutative, and all the mappings in the diagram are isomorphisms of FS*-*spaces.*

$$\begin{tikzcd}
H'(K;\ E) \arrow[r, "\mathbf{CW}"] & Z_0^{n-1}(\mathfrak{W}(V\backslash K);\ {}^E\mathcal{O}^{(n,0)}) \arrow[d] \\
 & Z_0^{n-1}(\mathfrak{W}'(V\backslash K);\ {}^E\mathcal{O}^{(n,0)}). \arrow[ul, "\mathbf{T}"']
\end{tikzcd}$$

Next, we consider a more special case. Let $V=\boldsymbol{C}^n$ and K be of product type:

$$K=K_1\times K_2\times\cdots\times K_n,\quad K_j\subset\boldsymbol{C}\ (\text{compact}),\quad j=1,\ 2,\dots,\ n.$$

Then, $\mathfrak{W}'(V\backslash K)$ is nothing else but

$$\mathfrak{U}=\{U_1,\dots,\ U_n\},\quad U_j=\{z=(z_1,\ z_2,\dots,\ z_n)\in\boldsymbol{C}^n;\ z_j\notin K_j\}\,.$$

By the correspondence of $f(z)$ with $f(z)dz_1\cdots dz_n$, we can identify ${}^E\mathcal{O}$ and ${}^E\mathcal{O}^{(n,0)}$. And we have

$$\begin{aligned}Z^{n-1}(\mathfrak{U};\ {}^E\mathcal{O})&=C^{n-1}(\mathfrak{U};\ {}^E\mathcal{O})\\&=H(U_1\cap\cdots\cap U_n;\ E)\\&=H(\boldsymbol{C}^n\#K;\ E)\,,\end{aligned}$$

where

$$\boldsymbol{C}^n\#K=(\boldsymbol{C}\backslash K_1)\times(\boldsymbol{C}\backslash K_2)\times\cdots\times(\boldsymbol{C}\backslash K_n)\,.$$

Now, if, for $T\in H'(K;\ E)$, we put

$$\check{T}(w)=T_z\Big(\frac{1}{(w_1-z_1)(w_2-z_2)\cdots(w_n-z_n)}\Big),$$

we have $\check{T}\in H(\boldsymbol{C}^n\sharp K;\, E)$, and $\check{T}$ is zero at infinity. We call $\check{T}$ the Cauchy-Hilbert transform of T and denote it by $\check{T}=\mathbf{CH}T$. If we denote by $H_0(\boldsymbol{C}^n\sharp K;\, E)$ the subspace of $H(\boldsymbol{C}^n\sharp K;\, E)$ consisting of E-valued functions which are zero at infinity, **CH** acts as

$$\mathbf{CH}\colon H'(K;\, E)\longrightarrow H_0(\boldsymbol{C}^n\sharp K;\, E).$$

Conversely, if we put, for $\vartheta\in H_0(\boldsymbol{C}^n\sharp K;\, E)$,

$$T_\vartheta(f)=\Big(\frac{1}{2\pi\sqrt{-1}}\Big)^n\int_{\partial L_1\times\cdots\times\partial L_n}\cdots\int f(w)\vartheta(w)dw_1\cdots dw_n$$

$(K_j\Subset L_j\Subset \boldsymbol{C})$, $T_\vartheta(f)$ is determined independently of L as long as f is holomorphic in a neighborhood of $L=L_1\times\cdots\times L_n$ and we have $T_\vartheta\in H'(K;\, E)$. Then we have the following corollary 2.

Corollary 2. *Let K be a product compact subset of $\boldsymbol{C}^n$:*

$$K=K_1\times K_2\times\cdots\times K_n;\quad K_j\subset \boldsymbol{C}\ \text{(compact)},\quad j=1, 2,\dots, n.$$

Then the above defined Cauchy-Hilbert transformation **CH** *is an isomorphism of* FS*-*spaces, and* **T**: $\vartheta\to T_\vartheta$ *gives the inverse.*

The results of this subsection also generalize those of J. S. Silva-G. Köthe-A. Grothendieck-M. Morimoto [49], [28] [10], [37].

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