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Analytic Linear Mappings and Vector Valued Hyperfunctions

By

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Contents

§0. Introduction

This paper is the detailed exposition of the report announced in [45].

In 1959 and 1960, M. Sato established the theory of hyperfunctions [31]. His idea was to consider "functions" in the generalized sense as boundary values of holomorphic functions. Hyperfunctions are the relative cohomology classes with coefficients in the sheaf of holomorphic functions. This generalizes the concept of functions more widely than L. Schwartz's distribution [37].

Recently, by the same method as that of M. Sato, P. D. F. Ion and T. Kawai [14] has extended the theory of hyperfunctions to the theory of hyperfunctions valued in a locally convex space, as has been done for distributions by L. Schwartz [35], [36].

On the other hand, A. Martineau [23] has shown that hyperfunctions are something of analytic functionals, and especially that hyperfunctions with compact support are nothing else but real analytic functionals.

In this direction, the extension of the theory of Fourier hyperfunctions by T. Kawai [17] to vector valued case can be found in Y. Ito and S. Nagamachi [15], [16], and S. Nagamachi and N. Mugibayashi [27], [28], [29]. Namely, vector valued Fourier hyperfunctions are something of continuous linear mappings of the space of rapidly decreasing holomorphic functions into a Hilbert space.

The extension of the theory of hyperfunctions of Sato-Martineau-Schapira to the vector valued case has not yet been seen.

So in this paper we established the theory of analytic linear mappings, that is, continuous linear mappings of the space of holomorphic (or analytic) functions into a locally convex space and apply it to the theory of vector valued hyperfunctions. Analytic linear mappings are so to speak "vector valued analytic functionals". Then vector valued hyperfunctions are realized as something of analytic linear mappings. Especially vector valued hyperfunctions with compact support are nothing else but real analytic linear mappings. They, by localization, forms a flabby sheaf and their section modules are realized as relative cohomology groups with coefficients in a sheaf of vector valued holomorphic functions, as in Sato-Ion-Kawai's theory.

In § 1, we introduce the spaces of holomorphic and analytic functions and mention the properties of their tensor products.

In § 2, we introduce the concepts of analytic linear mappings and real analytic linear mappings, and mention their properties. The structures of the spaces of analytic (or real analytic) linear mappings are clarified.

In § 3, we introduce some operations on analytic linear mappings such as multiplication by a holomorphic or an analytic function, tensor product of analytic linear mappings and convolution of analytic linear mappings.

In § 4, we introduce the concept of hyperfunctions valued in a Fréchet space and mention the properties of the sheaf of hyperfunctions valued in a Fréchet space. It will be shown that this sheaf is flabby and the space of vector valued hyperfunctions with support in a real compact set is the space of real analytic linear mappings. It will also be shown that the sheaf of vector valued distributions is a subsheaf of the sheaf of vector valued hyperfunctions.

In § 5, we introduce some operations on vector valued hyperfunctions such as multiplication by an analytic function, tensor product of vector valued hyperfunctions, convolution of vector valued hyperfunctions and transformation of a vector valued hyperfunction by an analytic isomorphism.

In § 6, we will prove the elliptic regularity and give the Dolbeault resolution of the sheaf of vector valued holomorphic functions by the flabby sheaves of differential forms with vector valued hyperfunctions as their coefficients.

In § 7, we mention Sato's theory of vector valued hyperfunctions as boundary values of vector valued holomorphic functions. We will prove some vanishing theorems of cohomology groups and relative cohomology groups with coefficients in a sheaf such as Malgrange's theorem and Martineau-Harvey's theorem and Sato's theorem. The last theorem implies that the real space $\boldsymbol{R}^n$ is purely n-codimensional with respect to the sheaf of vector valued holomorphic functions over $\boldsymbol{C}^n$.

Lastly, we will obtain the representation formula of analytic linear mappings as boundary values of vector valued holomorphic functions.

The author wishes to thank Professor M. Morimoto who informed the author of the work by A. Martineau [44] and allowed the author to correct some errors of the manuscript by his friendly advices.

§1. Holomorphic functions and analytic functions

Let $\mathcal{O}$ be the sheaf of holomorphic functions over $\boldsymbol{C}^n$. If Ω is an open set in $\boldsymbol{C}^n$, we set

$$\mathcal{O}(\Omega)=\Gamma(\Omega, \mathcal{O}),$$

the section module on Ω. This space has an FS-space topology for semi-norms

$$p_K(f)=\sup_K |f|,$$

where K runs over the family of compact subsets of Ω. It is known that $\mathcal{O}(\Omega)$ is a Fréchet nuclear space. Let K be a compact subset of $\boldsymbol{C}^n$. We put

$$\mathcal{O}(K)=\varinjlim_{\Omega\supset K} \mathcal{O}(\Omega).$$

We endow $\mathcal{O}(K)$ with the inductive limit topology. It is a nuclear space of type DFS (in particular, it is Hausdorff) and its dual $\mathcal{O}'(K)$ is a nuclear space of type FS.

Further, any bounded subset of $\mathcal{O}(K)$ is contained and bounded in a space $\mathcal{O}(\Omega)$ (cf. A. Martineau [24] or H. Komatsu [19]).

Let $\mathscr{A}$ be the sheaf of analytic functions over $\boldsymbol{R}^n$. If K is a compact subset of $\boldsymbol{R}^n$, we have an isomorphism

$$\mathscr{A}(K)=\mathcal{O}(K),$$

where $\mathscr{A}(K)$ denotes the space of analytic functions in a neighborhood of K in $\boldsymbol{R}^n$. $\mathscr{A}(K)$ is endowed with the topology of $\mathcal{O}(K)$. Then the space $\mathcal{O}(\boldsymbol{C}^n)$ is dense in $\mathscr{A}(K)$.

Proposition 1.1. *Let K_1, K_2 be two real compact sets. Then the mapping*

$$\mathscr{A}(K_1)\times\mathscr{A}(K_2)\longrightarrow\mathscr{A}(K_1\cap K_2),$$

$$(f_1, f_2)\longrightarrow f_1-f_2,$$

is a surjective homomorphism.

Proof. See P. Schapira [33], p. 45, Lemma 111. Q.E.D.

If Ω is an open set in $\boldsymbol{R}^n$, let $\mathscr{A}(\Omega)$ be the space of analytic functions on Ω equipped with the topology

$$\mathscr{A}(\Omega)=\varinjlim_{K\subset\Omega}\mathscr{A}(K).$$

Then $\mathscr{A}(\Omega)$ is a complete barreled nuclear space whose dual is a complete nuclear space.

We now state the properties of tensor product of spaces of holomorphic or analytic functions.

Proposition 1.2. *We have the following canonical isomorphisms:*

(i) $\mathcal{O}(\Omega_1)\hat{\otimes}\mathcal{O}(\Omega_2)\cong\mathcal{O}(\Omega_1\times\Omega_2)$,
($\Omega_1\subset\boldsymbol{C}^m$, $\Omega_2\subset\boldsymbol{C}^n$ *open sets*);

(ii) $\mathcal{O}(K_1)\hat{\otimes}\mathcal{O}(K_2)\cong\mathcal{O}(K_1\times K_2)$,
($K_1\subset\boldsymbol{C}^m$, $K_2\subset\boldsymbol{C}^n$ *compact sets*);

(iii) $\mathscr{A}(K_1)\hat{\otimes}\mathscr{A}(K_2)\cong\mathscr{A}(K_1\times K_2)$,
($K_1\subset\boldsymbol{R}^m$, $K_2\subset\boldsymbol{R}^n$ *compact sets*);

(iv) $\mathscr{A}(\Omega_1)\hat{\otimes}\mathscr{A}(\Omega_2)\cong\mathscr{A}(\Omega_1\times\Omega_2)$,
($\Omega_1\subset\boldsymbol{R}^m$, $\Omega_2\subset\boldsymbol{R}^n$ *open sets*).

Proof. (i) See F. Trèves [40], p. 530, Theorem 51.6.

(ii) Let $\{\Omega_{1j}\}_{j=1}^{\infty}$ and $\{\Omega_{2j}\}_{j=1}^{\infty}$ be fundamental systems of neighborhoods of K_1 and K_2, respectively. Then we have

$$\mathcal{O}(K_1\times K_2)=\varinjlim_{j}\mathcal{O}(\Omega_{1j}\times\Omega_{2j})\cong\varinjlim_{j}\mathcal{O}(\Omega_{1j})\hat{\otimes}\mathcal{O}(\Omega_{2j}).$$

On the other hand, since

$$\mathcal{O}(K_i)=\varinjlim_{j}\mathcal{O}(\Omega_{ij}),\qquad (i=1,2),$$

we have continuous injections

$$\mathcal{O}(\Omega_{ij})\longrightarrow\mathcal{O}(K_i),\qquad i=1,2.$$

Hence we have a continuous injection

$$\mathcal{O}(\Omega_{1j})\hat{\otimes}\mathcal{O}(\Omega_{2j})\longrightarrow\mathcal{O}(K_1)\hat{\otimes}\mathcal{O}(K_2).$$

Hence we have an isomorphism

$$\begin{aligned}\mathcal{O}(K_1)\hat{\otimes}\mathcal{O}(K_2)&\cong\varinjlim\mathcal{O}(\Omega_{1j})\hat{\otimes}\mathcal{O}(\Omega_{2j})\\&\cong\mathcal{O}(K_1\times K_2).\end{aligned}$$

(iii) Since $\mathscr{A}(K_1)\cong\mathcal{O}(K_1)$, $\mathscr{A}(K_2)\cong\mathcal{O}(K_2)$ and $\mathscr{A}(K_1\times K_2)\cong\mathcal{O}(K_1\times K_2)$, it is sufficient to apply (ii).

(iv) Let $\{K_{ij}\}_{j=1}^{\infty}$ be an increasing compact sets in Ω_i such that

$$\Omega_i=\bigcup_{j=1}^{\infty}K_{ij},\qquad (i=1,2).$$

Then we have

$$\mathcal{O}(\Omega_1 \times \Omega_2) = \varprojlim \mathscr{A}(K_{1j} \times K_{2j}) \cong \varprojlim \mathscr{A}(K_{1j}) \hat{\otimes} \mathscr{A}(K_{2j}).$$

On the other hand, since

$$\mathscr{A}(\Omega_i) = \varprojlim_{j} \mathscr{A}(K_{ij}), \qquad (i=1, 2),$$

we have natural continuous linear mappings

$$\mathscr{A}(\Omega_i) \longrightarrow \mathscr{A}(K_{ij}), \qquad (i=1, 2).$$

Hence we have a natural continuous linear mapping

$$\mathscr{A}(\Omega_1) \hat{\otimes} \mathscr{A}(\Omega_2) \longrightarrow \mathscr{A}(K_{1j}) \hat{\otimes} \mathscr{A}(K_{2j}).$$

Hence we have an isomorphism

$$\begin{aligned} \mathscr{A}(\Omega_1) \hat{\otimes} \mathscr{A}(\Omega_2) &\cong \varprojlim \mathscr{A}(K_{1j}) \hat{\otimes} \mathscr{A}(K_{2j}) \\ &\cong \mathscr{A}(\Omega_1 \times \Omega_2). \end{aligned}$$

Q. E. D.

§2. Analytic linear mappings

Definition 2.1. *Let E be a Fréchet space which is topologized by a non-decreasing countable basis $\{p_1, p_2, \cdots\}$ of continuous seminorms. Let Ω be an open subset of $\boldsymbol{C}^n$. Elements of $L(\mathcal{O}(\Omega); E)$ $(\equiv L_b(\mathcal{O}(\Omega); E))$ are called analytic linear mappings on Ω valued in E or simply analytic linear mappings on Ω. We say that $u \in L(\mathcal{O}(\Omega); E)$ is carried by a compact set K in Ω if for all open set ω which contains K, u can be extended to $\mathcal{O}(\omega)$, that is, if, for every p_j and for any ω which contains K, there exist a compact subset $K_{\omega j}$ of ω and a constant $C_{\omega j}$ such that*

$$p_j(u(f)) \leqq C_{\omega j} \sup_{K_{\omega j}} |f|.$$

We then call K the carrier of u. We denote by $\mathcal{O}'(\Omega; E)$ the space $L(\mathcal{O}(\Omega); E)$.

Proposition 2.1. *Let E be a Fréchet space. Then we have:*

(i) $\mathcal{O}'(\Omega; E) = L(\mathcal{O}(\Omega); E) \cong \mathcal{O}'(\Omega) \hat{\otimes} E$, ($\Omega$: *an open set in* $\boldsymbol{C}^n$).

(ii) $\mathcal{O}'(K; E) = L(\mathcal{O}(K); E) \cong \mathcal{O}'(K) \hat{\otimes} E$, ($K$: *a compact set in* $\boldsymbol{C}^n$).

(iii) $\mathscr{A}'(K; E) = L(\mathscr{A}(K); E) \cong \mathscr{A}'(K) \hat{\otimes} E$, ($K$: *a compact set in* $\boldsymbol{R}^n$).

(iv) $\mathscr{A}'(\Omega; E) = L(\mathscr{A}(\Omega); E) \cong \mathscr{A}'(\Omega) \hat{\otimes} E$, ($\Omega$: *an open set in* $\boldsymbol{R}^n$).

Proof. (i) We can evidently prove this by virtue of F. Trèves [40], Proposition 50.5, p. 522.

(ii) Since $\mathcal{O}(K)$ is a nuclear DFS-space, it also follows from F. Trèves [40], Proposition 50.5, p. 522.

(iii) Since $\mathscr{A}(K)=\mathcal{O}(K)$, it follows from (ii).

(iv) Since $\mathscr{A}(\Omega)=\varprojlim_{K\subset\Omega}\mathscr{A}(K)$ is a complete barreled nuclear space whose dual is a complete nuclear space, it follows also from F. Trèves [40], Proposition 50.5, p. 522. Q.E.D.

Proposition 2.2. *Let E be a Fréchet space, and K a polynomially convex compact subset of $\boldsymbol{C}^n$, and $u\in L(\mathcal{O}(\boldsymbol{C}^n); E)$. Then u is carried by K if and only if u can be extended to $L(\mathcal{O}(K); E)$.*

Proof. The condition is evidently sufficient. Conversely, let $\{\Omega_j\}_{j=1}^{\infty}$ be a fundamental system of Runge neighborhoods of K and $u_j\in L(\mathcal{O}(\Omega_j); E)$ the extension of u.

Since $\mathcal{O}(\boldsymbol{C}^n)$ is dense in $\mathcal{O}(\Omega_j)$, we have

$$u_j\,|\,\mathcal{O}(\Omega_{j'})=u_{j'}, \quad \text{if} \quad j>j'.$$

Hence u_j's define an element of

$$L(\varinjlim_j \mathcal{O}(\Omega_j); E)=L(\mathcal{O}(K); E). \qquad \text{Q.E.D.}$$

The elements of $L(\mathscr{A}(\boldsymbol{R}^n); E)$ are called real analytic linear mappings. They are analytic linear mappings on $\boldsymbol{C}^n$ which are carried by real compact set.

Theorem 2.1. *Let $u\in L(\mathscr{A}(\boldsymbol{R}^n); E)$, $u\neq 0$. There exists the least real compact set which carries u. We call it the support of u and denote it by* $\operatorname{supp}(u)$.

Proof. Let K_1 and K_2 be two real compact sets which carry u.

Let N be the kernel of the mapping

$$\mathscr{A}(K_1)\times\mathscr{A}(K_2)\longrightarrow\mathscr{A}(K_1\cap K_2),$$
$$(f_1, f_2)\longrightarrow f_1-f_2\,.$$

If $(f_1, f_2)\in N$, there exists $g\in\mathscr{A}(K_1\cup K_2)$ which extends f_1 and f_2.

Let $g_j\in\mathcal{O}(\boldsymbol{C}^n)$, g_j converging to g in $\mathscr{A}(K_1\cup K_2)$.

$$u(g)=\lim_j u(g_j)=u(f_i), \qquad i=1, 2.$$

Hence we can set, if $f\in\mathscr{A}(K_1\cap K_2)$ is of the form f_1-f_2,

$$u(f)=u(f_1)-u(f_2)\,.$$

This linear mapping is defined on $\mathscr{A}(K_1 \cap K_2)$ and continuous by virtue of Proposition 1.1.

Since $u \neq 0$, this implies that $K_1 \cap K_2 \neq \emptyset$. The passage to a certain family of compact sets is then evident. Q.E.D.

We remark that

$$\operatorname{supp}(u_1 + u_2) \subset \operatorname{supp}(u_1) \cup \operatorname{supp}(u_2),$$

$$\operatorname{supp}(\lambda u) \subset \operatorname{supp}(u), \ \lambda \in \boldsymbol{C}.$$

Proposition 2.3. *Let* $K = \bigcup_{i=1}^{p} K_i$ *be the union of real compact sets. Let* $u \in L(\mathscr{A}(\boldsymbol{R}^n); E)$, $\operatorname{supp}(u) \subset K$. *There exist* $u_i \in L(\mathscr{A}(\boldsymbol{R}^n); E)$ $(i = 1, \cdots, p)$ *with:*

$$u = \sum_{i=1}^{p} u_i, \quad \operatorname{supp}(u_i) \subset K_i.$$

Proof. We have to see that the mapping:

$$\prod_{i=1}^{p} L(\mathscr{A}(K_i); E) \longrightarrow L(\mathscr{A}(K); E),$$

$$(u_i)_{1 \leqq i \leqq p} \longrightarrow \sum_{i=1}^{p} u_i$$

is surjective, hence that the mapping:

$$\mathscr{A}(K) \longrightarrow \sum_{i=1}^{p} \mathscr{A}(K_i),$$

$$f \longrightarrow (f \mid K_i)_{1 \leqq i \leqq p}$$

is injective and of closed range, which is easy to verify. Q.E.D.

We now remark that the distributions with compact support valued in E are analytic linear mappings, for, by virtue of Stone-Weierstrass' theorem, the continuous injection

$$\mathscr{A}(\boldsymbol{R}^n) \longrightarrow C^\infty(\boldsymbol{R}^n)$$

is of dense range.

Analogously $\mathscr{A}(\Omega)$ is dense in $C^\infty(\Omega)$.

Proposition 2.4. *Let* $u \in \mathscr{E}'(\boldsymbol{R}^n; E)$. *We denote temporarily by* $\operatorname{supp}_{\mathscr{D}'}(u)$ *its support considering it as a distribution and by* $\operatorname{supp}(u)$ *its support considering it as an analytic linear mapping. We then have*

$$\operatorname{supp}(u) = \operatorname{supp}_{\mathscr{D}'}(u).$$

Proof. Let $K = \operatorname{supp}_{\mathscr{D}'}(u)$. For any open set $\Omega \supset K$, u can be extended to

$C^\infty(\Omega)$, hence to $\mathscr{A}(\Omega)$. Consequently

$$\operatorname{supp}(u) \subset K.$$

Conversely, let $u \in \mathscr{E}'(\boldsymbol{R}^n;\, E)$ such that u can be extended to $\mathscr{A}(K)$ and let $\phi \in \mathscr{D}(\boldsymbol{R}^n)$ such that

$$\operatorname{supp}(\phi) \cap K = \emptyset.$$

We must show that

$$u(\phi) = 0.$$

For this it is sufficient to construct the functions ϕ_ε having the properties:

$$\phi_\varepsilon \in \mathscr{A}(\boldsymbol{R}^n),$$

$$\phi_\varepsilon \to \phi \text{ in } C^\infty(\boldsymbol{R}^n), \quad (\varepsilon \to 0),$$

$$\phi_\varepsilon \to 0 \text{ in } \mathscr{A}(K), \quad (\varepsilon \to 0).$$

One verify that if

$$\rho_\varepsilon \in \mathscr{A}(\boldsymbol{R}^n),$$

$$\rho_\varepsilon \to \delta \text{ in } \mathscr{D}'(\boldsymbol{R}^n), \quad (\varepsilon \to 0),$$

$$\rho_\varepsilon \to 0 \text{ in } \mathscr{A}(\boldsymbol{R}^n - \{0\}), \quad (\varepsilon \to 0),$$

the functions

$$\phi_\varepsilon = \phi * \rho_\varepsilon$$

respond to the question.

We then put

$$\rho_\varepsilon(x) = \left(\frac{1}{\varepsilon\sqrt{\pi}}\right)^n \exp(-|x|^2/\varepsilon^2), \quad |x|^2 = x_1^2 + \cdots + x_n^2.$$

It is clear that ρ_ε tend to zero in $\mathscr{A}(\boldsymbol{R}^n - \{0\})$. We can also easily prove that ρ_ε tend to δ in $\mathscr{D}'(\boldsymbol{R}^n)$ [see F. Trèves [40], Lemma 15.1, p. 153]. Q. E. D.

Let now Ω be an open subset of $\boldsymbol{R}^n$ and K a compact subset of Ω. We call "envelope of K" (in Ω) and denote by $\tilde{K}$, the union of K and the relatively compact connected components (in Ω) of $\Omega - K$. It is again a compact set.

Proposition 2.5. *If $K = \tilde{K}$, $\mathscr{A}'(\partial\Omega;\, E)$ is dense in $\mathscr{A}'(\overline{\Omega - K};\, E)$.*

Proof. It is sufficient to see that the mapping of $\mathscr{A}(\overline{\Omega - K})$ into $\mathscr{A}(\partial\Omega)$ is injective. For this see the proof of Lemma 115 of P. Schapira [33], p. 51. Q. E. D.

§3. Operations on analytic linear mappings

In this section we now define several operations on analytic linear mappings.

a) Multiplication by a holomorphic or an analytic function

(i) Let Ω be an open set in $\boldsymbol{C}^n$. If $f \in \mathcal{O}(\Omega)$ and $u \in \mathcal{O}'(\Omega; E)$, we define

$$fu \in \mathcal{O}'(\Omega; E)$$

by the formula

$$(fu)(g) = u(fg) \qquad \text{for all} \quad g \in \mathcal{O}(\Omega).$$

$\mathcal{O}'(\Omega; E)$ is an $\mathcal{O}(\Omega)$-module.

(ii) Let K be a compact set in $\boldsymbol{C}^n$. If $f \in \mathcal{O}(K)$ and $u \in \mathcal{O}'(K; E)$, we define

$$fu \in \mathcal{O}'(K; E)$$

by the formula

$$(fu)(g) = u(fg) \qquad \text{for all} \quad g \in \mathcal{O}(K).$$

By this definition of multiplication by a holomorphic function, $\mathcal{O}'(K; E)$ becomes an $\mathcal{O}(K)$-module.

(iii) Let K be a compact set in $\boldsymbol{R}^n$. If $f \in \mathscr{A}(K)$ and $u \in \mathscr{A}'(K; E)$, we define

$$fu \in \mathscr{A}'(K; E)$$

by the formula

$$(fu)(g) = u(fg) \qquad \text{for all} \quad g \in \mathscr{A}(K).$$

Then we have

$$\operatorname{supp}(fu) \subset \operatorname{supp}(u).$$

By this definition of multiplication by an analytic function, $\mathscr{A}'(K; E)$ becomes an $\mathscr{A}(K)$-module.

(iv) Let Ω be an open set in $\boldsymbol{R}^n$. If $f \in \mathscr{A}(\Omega)$ and $u \in \mathscr{A}'(\Omega; E)$, we define

$$fu \in \mathscr{A}'(\Omega; E)$$

by the formula

$$(fu)(g) = u(fg) \qquad \text{for all} \quad g \in \mathscr{A}(\Omega).$$

Then we have

$$\operatorname{supp}(fu) \subset \operatorname{supp}(u).$$

By this definition of multiplication by an analytic function, $\mathscr{A}'(\Omega; E)$ becomes an $\mathscr{A}(\Omega)$-module.

b) Tensor product of analytic linear mappings

First we recall the tensor product of analytic functionals.

Proposition 3.1. *We have the following canonical isomorphisms:*

(i) $\mathcal{O}'(\Omega_1)\hat{\otimes}\mathcal{O}'(\Omega_2)\cong L(\mathcal{O}(\Omega_1); \mathcal{O}'(\Omega_2))\cong\mathcal{O}'(\Omega_1\times\Omega_2)$, $(\Omega_1\subset \boldsymbol{C}^m,\ \Omega_2\subset \boldsymbol{C}^n$ *open sets*$)$.

(ii) $\mathcal{O}'(K_1)\hat{\otimes}\mathcal{O}'(K_2)\cong L(\mathcal{O}(K_1); \mathcal{O}'(K_2))\cong\mathcal{O}'(K_1\times K_2)$, $(K_1\subset \boldsymbol{C}^m,\ K_2\subset \boldsymbol{C}^n$ *compact sets*$)$.

(iii) $\mathscr{A}'(K_1)\hat{\otimes}\mathscr{A}'(K_2)\cong L(\mathscr{A}(K_1); \mathscr{A}'(K_2))\cong\mathscr{A}'(K_1\times K_2)$, $(K_1\subset \boldsymbol{R}^m,\ K_2\subset \boldsymbol{R}^n$ *compact sets*$)$.

(iv) $\mathscr{A}'(\Omega_1)\hat{\otimes}\mathscr{A}'(\Omega_2)\cong L(\mathscr{A}(\Omega_1); \mathscr{A}'(\Omega_2))\cong\mathscr{A}'(\Omega_1\times\Omega_2)$, $(\Omega_1\subset \boldsymbol{R}^m,\ \Omega_2\subset \boldsymbol{R}^n$ *open sets*$)$.

Proof. (i) See F. Trèves [40], p. 531, Corollary to Theorem 51.6.

(ii) Let $\{\Omega_{1j}\}_{j=1}^{\infty}$ and $\{\Omega_{2j}\}_{j=1}^{\infty}$ be fundamental systems of neighborhoods of K_1 and K_2, respectively. Then we have

$$\mathcal{O}'(K_1\times K_2)=\varprojlim_j \mathcal{O}'(\Omega_{1j}\times\Omega_{2j})\cong \varprojlim_j \mathcal{O}'(\Omega_{1j})\hat{\otimes}\mathcal{O}'(\Omega_{2j}).$$

On the other hand, since

$$\mathcal{O}'(K_i)=\varprojlim_j \mathcal{O}'(\Omega_{ij}), \qquad (i=1, 2),$$

we have continuous linear mappings

$$\mathcal{O}'(K_i)\longrightarrow\mathcal{O}'(\Omega_{ij}), \qquad (i=1, 2).$$

Hence we have a continuous linear mapping

$$\mathcal{O}'(K_1)\hat{\otimes}\mathcal{O}'(K_2)\longrightarrow\mathcal{O}'(\Omega_{1j})\hat{\otimes}\mathcal{O}'(\Omega_{2j}).$$

Hence we can consider $\mathcal{O}'(K_1)\hat{\otimes}\mathcal{O}'(K_2)$ is a closed subspace of

$$\varprojlim_j \mathcal{O}'(\Omega_{1j})\hat{\otimes}\mathcal{O}'(\Omega_{2j})=\mathcal{O}'(K_1\times K_2).$$

But, since the mapping

$$\mathcal{O}(\Omega_{1j})\otimes\mathcal{O}(\Omega_{2j})\longrightarrow\mathcal{O}(K_1)\otimes\mathcal{O}(K_2)$$

is injective, the mapping

$$\mathcal{O}'(K_1)\otimes\mathcal{O}'(K_2)\longrightarrow\varprojlim_j \mathcal{O}'(\Omega_{1j})\otimes\mathcal{O}'(\Omega_{2j})$$

has a dense image in

$$\varinjlim_j \mathcal{O}'(\Omega_{1j})\otimes\mathcal{O}'(\Omega_{2j}),$$

which is dense in

$$\varinjlim_j \mathcal{O}'(\Omega_{1j})\hat{\otimes}\mathcal{O}'(\Omega_{2j})=\mathcal{O}'(K_1\times K_2).$$

Thus, since $\mathcal{O}'(K_1)\otimes\mathcal{O}'(K_2)$ is dense in $\mathcal{O}'(K_1)\hat{\otimes}\mathcal{O}'(K_2)$, $\mathcal{O}'(K_1)\hat{\otimes}\mathcal{O}'(K_2)$ is a dense closed subspace of $\mathcal{O}'(K_1\times K_2)$. Hence

$$\mathcal{O}'(K_1)\hat{\otimes}\mathcal{O}'(K_2)\cong\mathcal{O}'(K_1\times K_2).$$

(iii) Since $\mathscr{A}(K_i)=\mathcal{O}(K_i)$ $(i=1, 2)$, it is sufficient to apply (ii).

(iv) Since $\mathscr{A}(\Omega_i)=\varprojlim_j \mathscr{A}(K_{ij})$ where $\{K_{ij}\}_{j=1}^{\infty}$ is a increasing sequence of compact sets which exhaust Ω_i $(i=1, 2)$,

$$\begin{aligned}\mathscr{A}'(\Omega_1)\hat{\otimes}\mathscr{A}'(\Omega_2)&=\varinjlim_j \mathscr{A}'(K_{1j})\hat{\otimes}\mathscr{A}'(K_{2j})\\&=\varinjlim_j \mathscr{A}'(K_{1j}\times K_{2j})=\mathscr{A}'(\Omega_1\times\Omega_2).\end{aligned}$$

Q. E. D.

Next we consider tensor product of analytic linear mappings. In the following, we assume that E_1 and E_2 be two Fréchet spaces. ω stands for ε or π topology in the sense of F. Trèves [40].

Proposition 3.2. *We have the following canonical isomorphisms:*

(i) $\mathcal{O}'(\Omega_1; E_1)\hat{\otimes}_\omega\mathcal{O}'(\Omega_2; E_2)\cong\mathcal{O}'(\Omega_1\times\Omega_2; E_1\hat{\otimes}_\omega E_2)$, ($\Omega_1\subset \boldsymbol{C}^m$, $\Omega_2\subset\boldsymbol{C}^n$ *open sets*).

(ii) $\mathcal{O}'(K_1; E_1)\hat{\otimes}_\omega\mathcal{O}'(K_2; E_2)\cong\mathcal{O}'(K_1\times K_2; E_1\hat{\otimes}_\omega E_2)$, ($K_1\subset \boldsymbol{C}^m$, $K_2\subset\boldsymbol{C}^n$ *compact sets*).

(iii) $\mathscr{A}'(K_1; E_1)\hat{\otimes}_\omega\mathscr{A}'(K_2; E_2)\cong\mathscr{A}'(K_1\times K_2; E_1\hat{\otimes}_\omega E_2)$, ($K_1\subset \boldsymbol{R}^m$, $K_2\subset\boldsymbol{R}^n$ *compact sets*).

(iv) $\mathscr{A}'(\Omega_1; E_1)\hat{\otimes}_\omega\mathscr{A}'(\Omega_2; E_2)\cong\mathscr{A}'(\Omega_1\times\Omega_2; E_1\hat{\otimes}_\omega E_2)$, ($\Omega_1\subset \boldsymbol{R}^m$, $\Omega_2\subset\boldsymbol{R}^n$ *open sets*).

Proof. Since the tensor product of locally convex Hausdorff spaces is commutative and associative, it is sufficient to apply Propositions 2.1 and 3.1. Q. E. D.

Thus we have the following definition of the tensor product of analytic linear mappings.

Definition 3.1. *We use the notations of Proposition* 3.2.

(i) *Let* $u_i=\phi_i\otimes \boldsymbol{e}_i\in\mathcal{O}'(\Omega_i; E_i)$, $\phi_i\in\mathcal{O}'(\Omega_i)$, $\boldsymbol{e}_i\in E_i$ $(i=1, 2)$. *Then we define*

$$u_1\otimes_\omega u_2=(\phi_1\otimes\phi_2)\otimes(\boldsymbol{e}_1\otimes_\omega \boldsymbol{e}_2),$$

i.e.,

$$(u_1\otimes_\omega u_2)(f_1\otimes f_2)=\phi_1(f_1)\phi_2(f_2)(\boldsymbol{e}_1\otimes_\omega\boldsymbol{e}_2) \quad \textit{for} \quad f_i\in\mathcal{O}(\Omega_i), \quad (i=1, 2).$$

(ii) *Let* $u_i=\phi_i\otimes \boldsymbol{e}_i\in \mathscr{O}'(K_i;\, E_i)$, $\phi_i\in\mathscr{O}'(K_i)$, $\boldsymbol{e}_i\in E_i$ $(i=1, 2)$. *Then we define*

$$u_1\otimes_\omega u_2=(\phi_1\otimes\phi_2)\otimes(\boldsymbol{e}_1\otimes_\omega \boldsymbol{e}_2),$$

i.e.,

$$(u_1\otimes_\omega u_2)(f_1\otimes f_2)=\phi_1(f_1)\phi_2(f_2)(\boldsymbol{e}_1\otimes_\omega \boldsymbol{e}_2) \quad \textit{for} \quad f_i\in\mathscr{O}(K_i), \quad (i=1, 2).$$

(iii) *Let* $u_i=\phi_i\otimes \boldsymbol{e}_i\in \mathscr{A}'(K_i;\, E_i)$, $\phi_i\in\mathscr{A}'(K_i)$, $\boldsymbol{e}_i\in E_i$ $(i=1, 2)$. *Then we define*

$$u_1\otimes_\omega u_2=(\phi_1\otimes\phi_2)\otimes(\boldsymbol{e}_1\otimes_\omega \boldsymbol{e}_2),$$

i.e.,

$$(u_1\otimes_\omega u_2)(f_1\otimes f_2)=\phi_1(f_1)\phi_2(f_2)(\boldsymbol{e}_1\otimes_\omega \boldsymbol{e}_2) \quad \textit{for} \quad f_i\in\mathscr{A}(K_i), \quad (i=1, 2).$$

(iv) *Let* $u_i=\phi_i\otimes \boldsymbol{e}_i\in \mathscr{A}'(\Omega_i;\, E_i)$, $\phi_i\in\mathscr{A}'(\Omega_i)$, $\boldsymbol{e}_i\in E_i$ $(i=1, 2)$. *Then we define*

$$u_1\otimes_\omega u_2=(\phi_1\otimes\phi_2)\otimes(\boldsymbol{e}_1\otimes_\omega \boldsymbol{e}_2),$$

i.e.,

$$(u_1\otimes_\omega u_2)(f_1\otimes f_2)=\phi_1(f_1)\phi_2(f_2)(\boldsymbol{e}_1\otimes_\omega \boldsymbol{e}_2) \quad \textit{for} \quad f_i\in\mathscr{A}(\Omega_i), \quad (i=1, 2).$$

In all the real cases, we have

$$\operatorname{supp}(u_1\otimes_\omega u_2)\subset\operatorname{supp}(u_1)\times\operatorname{supp}(u_2).$$

c) Convolution of analytic linear mappings

We now define convolution of analytic linear mappings. In the following of this subsection, we assume that E_1 and E_2 be Fréchet spaces.

Proposition 3.3. *Let* $u\in\mathscr{O}'(\boldsymbol{C}^n;\, E_1)$ *and* $v\in\mathscr{O}'(\boldsymbol{C}^n;\, E_2)$. *Then there exists an analytic linear mapping, called the convolution product of u and v and denoted by $u*_\omega v$ or $v*_\omega u$, such that*

$$(u*_\omega v)_z(f(z))=(u_\xi\otimes_\omega v_\eta)(f(\xi+\eta)) \qquad \textit{for all} \quad f\in\mathscr{O}(\boldsymbol{C}^n),$$

where ω stands for ε or π topology.

Proof. Let $\phi\in\mathscr{O}'(\boldsymbol{C}^n)$ and $f\in\mathscr{O}(\boldsymbol{C}^n)$. We define

$$\phi*f\in\mathscr{O}(\boldsymbol{C}^n)$$

by the formula

$$(\phi*f)(z)=\phi_y(f(z-y)).$$

We put

$$\check{\phi}(f)=\phi(\check{f}),$$

where $\check{f}(z)=f(-z)$. If $\psi\in\mathscr{O}'(\boldsymbol{C}^n)$, we define $\phi*\psi$ by the formula

$$
\begin{aligned}
(\phi * \psi)(f) &= (\phi_x \otimes_\omega \psi_y)(f(x+y)) \\
&= \phi_x(\psi_y(f(x+y)) \\
&= \check{\varphi}(\psi * \check{f}) = \phi(\check{\psi} * f).
\end{aligned}
$$

By representing analytic functionals on $\boldsymbol{C}^n$ by measures with compact support, we see that, if ϕ and $\psi \in \mathcal{O}'(\boldsymbol{C}^n)$ are carried by the compact sets K and L, respectively, $\phi * \psi$ is carried by $K+L$. Then $\mathcal{O}'(\boldsymbol{C}^n)$ becomes a commutative algebra and $\mathscr{A}'(\boldsymbol{R}^n)$ is its subalgebra. If ϕ, $\psi \in \mathscr{A}'(\boldsymbol{R}^n)$, we have

$$
\operatorname{supp}(\phi * \psi) \subset \operatorname{supp}(\phi) + \operatorname{supp}(\psi).
$$

If $u = \phi \otimes \boldsymbol{e}$, $v = \psi \otimes \boldsymbol{f}$, ϕ, $\psi \in \mathcal{O}'(\boldsymbol{C}^n)$, $\boldsymbol{e} \in E_1$, and $\boldsymbol{f} \in E_2$, we have, by Definition 3.1, an analytic linear mapping in $\mathcal{O}'(\boldsymbol{C}^n; E_1 \hat{\otimes}_\omega E_2)$ defined by the formula

$$
\begin{aligned}
(u_x \otimes_\omega v_y)(f(x+y)) &= \phi_x(\psi_y(f(x+y)))(\boldsymbol{e} \otimes_\omega \boldsymbol{f}) \\
&= (\phi * \psi)(f)(\boldsymbol{e} \otimes_\omega \boldsymbol{f}).
\end{aligned}
$$

This defines an analytic linear mapping $u *_\omega v$, the convolution product of u and v, putting

$$
(u *_\omega v)(f) = (u_x \otimes_\omega v_y)(f(x+y)).
$$

This definition can be easily extended to arbitrary elements in $\mathcal{O}'(\boldsymbol{C}^n; E_1)$ and $\mathcal{O}'(\boldsymbol{C}^n; E_2)$. Q.E.D.

By definition of the carrier of analytic linear mapping, we see that, if $u \in \mathcal{O}'(\boldsymbol{C}^n; E_1)$ and $v \in \mathcal{O}'(\boldsymbol{C}^n; E_2)$ are carried by the compact sets K and L, respectively, $u *_\omega v$ is carried by $K+L$. If $u \in \mathscr{A}'(\boldsymbol{R}^n; E_1)$ and $v \in \mathscr{A}'(\boldsymbol{R}^n; E_2)$, we have

$$
\operatorname{supp}(u *_\omega v) \subset \operatorname{supp}(u) + \operatorname{supp}(v).
$$

§4. Hyperfunctions valued in a Fréchet space E

In the following we suppose that E is a Fréchet space.

First we consider hyperfunctions on a bounded open set in $\boldsymbol{R}^n$ valued in E.

Let Ω be a bounded open subset of $\boldsymbol{R}^n$. We put

$$
\mathscr{B}(\Omega; E) = \mathscr{A}'(\bar{\Omega}; E) / \mathscr{A}'(\partial\Omega; E).
$$

Definition 4.1. *The elements of $\mathscr{B}(\Omega; E)$ are called the hyperfunctions on Ω valued in a Fréchet space E or the E-valued hyperfunctions on Ω.*

Let K be a compact set containing Ω. Then we have

$$
K = (K - \Omega) \cup \bar{\Omega}.
$$

By virtue of Proposition 2.3, every element $u \in \mathscr{A}'(K; E)$ can be written as follows:

$$u = u_1 + u_2,\ u_1 \in \mathscr{A}'(K - \Omega; E) \quad \text{and} \quad u_2 \in \mathscr{A}'(\bar{\Omega}; E).$$

This shows that the canonical mapping:

$$\mathscr{A}'(\bar{\Omega}; E)/\mathscr{A}'(\partial\Omega; E) \longrightarrow \mathscr{A}'(K; E)/\mathscr{A}'(K - \Omega; E),$$

which is evidently injective, is also surjective. Hence, we have

$$\mathscr{B}(\Omega; E) \cong \mathscr{A}'(K; E)/\mathscr{A}'(K - \Omega; E), \quad K \supset \Omega.$$

Let now ω be an open set contained in Ω.

The mapping

$$\mathscr{A}'(\bar{\Omega}; E) \longrightarrow \mathscr{A}'(\bar{\Omega}; E)/\mathscr{A}'(\bar{\Omega} - \omega; E)$$

defines a mapping

$$\mathscr{B}(\Omega; E) \longrightarrow \mathscr{B}(\omega; E)$$

called the restriction.

If $T \in \mathscr{B}(\Omega; E)$, we denote by $T|\omega$ its image in $\mathscr{B}(\omega; E)$. It is clear that, if $\Omega_3 \subset \Omega_2 \subset \Omega_1$, and $T \in \mathscr{B}(\Omega_1; E)$, we have

$$(T|\Omega_2)|\Omega_3 = T|\Omega_3,$$

hence that the collection of $\mathscr{B}(\omega; E)$ defines a presheaf (of vector spaces) over Ω which we temporarily denote by ${}^E B|\Omega$.

Proposition 4.1. *Let Ω be a bounded open subset of $\boldsymbol{R}^n$.*

1) *The presheaf ${}^E B|\Omega$ is a sheaf.*
2) *This sheaf is flabby.*
3) *If K is a compact subset of Ω,*

$$\Gamma_K(\Omega, {}^E B|\Omega) = \mathscr{A}'(K; E).$$

4) *If $F = \bigcup_{i=1}^{p} F_i$ is a union of closed subsets of Ω and $T \in \Gamma_F(\Omega, {}^E B|\Omega)$, there exist $T_i \in \Gamma_{F_i}(\Omega, {}^E B|\Omega)$ such that*

$$T = \sum_{i=1}^{p} T_i.$$

5) *If ω is an open subset of Ω,*

$$({}^E B|\Omega)|\omega = {}^E B|\omega.$$

Proof. 1) (i) Let $\Omega = \bigcup_{i \in I} \Omega_i$ and $T \in \mathscr{B}(\Omega; E)$ such that $T|\Omega_i = 0$ for all $i \in I$. This is equivalent to say that, if $u_T \in \mathscr{A}'(\bar{\Omega}; E)$ is a representative of T, the image of u_T in $\mathscr{A}'(\bar{\Omega}; E)/\mathscr{A}'(\bar{\Omega}-\Omega_i; E)$ is zero for all i. From here we have

$$\operatorname{supp}(u_T) \cap \Omega_i = 0 \qquad \text{for all} \quad i,$$

hence,

$$\operatorname{supp}(u_T) \subset \partial\Omega,$$

that is, $T=0$.

(ii) Let $\Omega=\Omega_1 \cup \Omega_2$ and $T_i \in \mathscr{B}(\Omega_i; E)$ $(i=1, 2)$ with

$$T_1 \mid \Omega_1 \cap \Omega_2 = T_2 \mid \Omega_1 \cap \Omega_2 = T.$$

Let $u_T \in \mathscr{A}'(\overline{\Omega_1 \cap \Omega_2}; E)$ and $u_{T_i} \in \mathscr{A}'(\bar{\Omega}_i; E)$ be representatives of T and T_i $(i=1,2)$, respectively. Since

$$\operatorname{supp}(u_{T_i}-u_T) \subset \bar{\Omega}_i - \Omega_1 \cap \Omega_2$$

and since

$$\bar{\Omega}_i - \Omega_1 \cap \Omega_2 = (\overline{\Omega_i - \Omega_1 \cap \Omega_2}) \cup (\bar{\Omega}_i - \Omega_i),$$

we can, by replacing u_{T_i} with a equivalent u'_{T_i}, suppose that

$$u_{T_i} = u_T + v_i, \quad \operatorname{supp}(v_i) \subset \overline{\Omega_i - \Omega_1 \cap \Omega_2}.$$

We put

$$u'_T = u_T + v_1 + v_2 \in \mathscr{A}'(\overline{\Omega_1 \cup \Omega_2}; E).$$

Let T' be the image of u'_T in $\mathscr{B}(\Omega_1 \cup \Omega_2; E)$. We have $T' \mid \Omega_i = T_i$ for $\operatorname{supp}(u'_T - u_{T_i}) \cap \Omega_i = \operatorname{supp}(v_j) \cap \Omega_i$ (with $j \neq i$) and this set is contained in

$$(\overline{\Omega_j - \Omega_1 \cap \Omega_2}) \cap \Omega_i = \emptyset.$$

(iii) Let now $\Omega = \bigcup_{i \in I} \Omega_i$ and $T_i \in \mathscr{B}(\Omega_i; E)$, with

$$T_i \mid \Omega_i \cap \Omega_j = T_j \mid \Omega_i \cap \Omega_j.$$

We can suppose the covering is countable and by virtue of (ii) increasing. Thus we can suppose $\Omega_j \Subset \Omega_{j+1}$ and, since the envelope (in Ω) of a compact subset of Ω is a compact subset of Ω, we can suppose by (ii) that

$$\Omega = \bigcup_{j=0}^{\infty} \Omega_j,$$

$$\Omega_j \Subset \Omega_{j+1},$$

$$\tilde{\bar{\Omega}}_j = \bar{\Omega}_j \quad (\text{where } \tilde{\bar{\Omega}}_j \text{ is the envelope of } \bar{\Omega}_j \text{ in } \Omega),$$

$$T_j \in \mathscr{B}(\Omega_j; E), \quad T_{j+k} \mid \Omega_j = T_j.$$

Let $u_{T_j} \in \mathscr{A}'(\bar{\Omega}_j; E)$ be a representative of T_j. Let d_j be a metric defining the topology of $\mathscr{A}'(\bar{\Omega}-\Omega_j;\ E)$ and $v_j \in \mathscr{A}'(\partial\Omega;\ E)$ such that

$$d_i(u_{T_{j+1}} - v_{j+1} - (u_{T_j} - v_j)) \leqq 2^{-j}, \qquad \text{for all} \quad i \leqq j.$$

We construct v_j's by recurrence by virtue of Proposition 2.5. The sequence $u_{T_j} - v_j$ converges to an element $u_T \in \mathscr{A}'(\bar{\Omega};\ E)$. We have

$$\begin{aligned} u_T &= u_T - (u_{T_j} - v_j) + (u_{T_j} - v_j) \\ &= (u_{T_j} - v_j) + \lim_k \{u_{T_k} - v_k - (u_{T_j} - v_j)\}. \end{aligned}$$

Since the sequence

$$\{u_{T_k} - v_k - (u_{T_j} - v_j)\}_k$$

converges in $\mathscr{A}'(\bar{\Omega}-\Omega_j;\ E)$,

$$u_T = u_{T_j} - v_j + w_j, \quad w_j \in \mathscr{A}'(\bar{\Omega}-\Omega_j;\ E).$$

Hence, we have

$$T \mid \Omega_j = T_j,$$

where T is the image of u_T in $\mathscr{B}(\Omega;\ E)$.

2) The sheaf ${}^E B \mid \Omega$ is flabby for, if $\omega \subset \Omega$, $T \in \mathscr{B}(\omega;\ E)$, there exists $u_T \in \mathscr{A}'(\bar{\omega};\ E)$ and the image of u_T in $\mathscr{B}(\Omega;\ E)$ is an extension of T.

3) We have an injection if $K \subset \Omega$:

$$\mathscr{A}'(K;\ E) \longrightarrow \mathscr{A}'(\bar{\Omega};\ E)/\mathscr{A}'(\partial\Omega;\ E).$$

The image of $\mathscr{A}'(K;\ E)$ is the set of $T \in \mathscr{B}(\Omega;\ E)$ which are zero on $\Omega - K$, hence, is

$$\Gamma_K(\Omega,\ {}^E B \mid \Omega).$$

4) Let $\bar{F}$ and $\bar{F}_i$ be the closures of F and F_i in $\bar{\Omega}$, respectively, and u_T a representative of T in $\mathscr{A}'(\bar{\Omega};\ E)$ so that

$$\operatorname{supp}(u_T) \subset \partial\Omega \cup \bar{F}.$$

Hence, by applying Proposition 2.3, we can suppose

$$\operatorname{supp}(u_T) \subset \bar{F}.$$

Let $u_{T_i} \in \mathscr{A}'(\bar{F}_i;\ E)$

$$u_T = \sum_{i=1}^{p} u_{T_i}.$$

If T_i is the image of u_{T_i} in $\mathscr{B}(\Omega;\ E)$, we have

$$T = \sum_{i=1}^{p} T_i.$$

5) If $\omega' \subset \omega \subset \Omega$ are open sets, we have

$$\Gamma(\omega', {}^E B | \Omega) = \mathscr{B}(\omega'; E) = \Gamma(\omega', {}^E B | \omega). \qquad \text{Q.E.D.}$$

Next we consider hyperfunctions on $\boldsymbol{R}^n$ valued in a Fréchet space E. Let ${}^E B_1$ be the presheaf over $\boldsymbol{R}^n$ defined as follows:

If Ω is not bounded, $\mathscr{B}_1(\Omega; E) = \{0\}$.

If Ω is bounded, $\mathscr{B}_1(\Omega; E) = \mathscr{B}(\Omega; E)$.

The restrictions are defined by

$$\mathscr{B}_1(\Omega; E) \longrightarrow \mathscr{B}_1(\omega; E)$$

$$0 \longrightarrow 0 \qquad \text{if } \Omega \text{ is not bounded,}$$

$$T \longrightarrow T | \omega \qquad \text{if } \Omega \text{ is bounded.}$$

This presheaf satisfy the axiom (S1) of sheaves but not (S2) [cf. G. E. Bredon [2], p. 5, or R. Godement [5], p. 109].

We denote by ${}^E\mathscr{B}$ the sheaf associated to this presheaf ${}^E B_1$. It is a sheaf of vector space over $\boldsymbol{C}$.

Definition 4.2. *The sheaf ${}^E\mathscr{B}$ is called the sheaf of E-valued hyperfunctions over $\boldsymbol{R}^n$.*

If $T \in \Gamma(\Omega, {}^E\mathscr{B}) = \mathscr{B}(\Omega; E)$, T is an E-valued hyperfunction on Ω. Hence an E-valued hyperfunction on Ω is defined by the following:

a covering $\Omega = \bigcup_{i \in I} \Omega_i$ where Ω_i's are bounded open sets,

$T_i \in \mathscr{B}(\Omega_i; E)$ satisfying $T_i | \Omega_i \cap \Omega_j = T_j | \Omega_i \cap \Omega_j$.

Two such couples $(\Omega_i, T_i)_{i \in I}$ and $(\Omega_{i'}, T_{i'})_{i' \in I'}$ define the same E-valued hyperfunction if

$$T_i | \Omega_i \cap \Omega_{i'} = T_{i'} | \Omega_i \cap \Omega_{i'} \qquad \text{for all} \quad i \in I, \quad \text{all} \quad i' \in I'.$$

Theorem 4.1. 1) *For all bounded open sets*

$${}^E\mathscr{B} | \Omega = {}^E B | \Omega.$$

2) *The sheaf ${}^E\mathscr{B}$ is flabby.*

3) *If K is a compact subset of $\boldsymbol{R}^n$, we have*

$$\Gamma_K(\boldsymbol{R}^n, {}^E\mathscr{B}) = \mathscr{A}'(K; E).$$

4) *If $F = \bigcup_{i=1}^{p} F_i$ is a union of closed subsets of an open set Ω in $\boldsymbol{R}^n$ and if $T \in \Gamma_F(\Omega, {}^E\mathscr{B})$, there exist $T_i \in \Gamma_{F_i}(\Omega, {}^E\mathscr{B})$ with*

$$T=\sum_{i=1}^{p} T_i .$$

We write $\mathscr{B}_F(\Omega; E)$ *for* $\Gamma_F(\Omega, {}^E\mathscr{B})$. *We write* $\operatorname{supp}(T)$ *for the support of an E-valued hyperfunction T.*

Proof. 1) is evident.

2) Let $T_0 \in \mathscr{B}(\Omega_0; E)$ and Ω_0 an open set in $\boldsymbol{R}^n$. Let $\mathscr{F}$ be the family of couples (Ω, T) with

$$\Omega_0 \subset \Omega, \quad T \mid \Omega_0 = T_0 .$$

$\mathscr{F}$ is ordered and inductive for the relation

$$(\Omega, T) \prec (\Omega', T') \quad \text{if} \quad \Omega \subset \Omega',\ T' \mid \Omega = T.$$

Let (Ω, T) be a maximal element and we suppose that there exists $x_0 \notin \Omega$. Let ω be a bounded open set containing x_0. The E-valued hyperfunction $T \mid \Omega \cap \omega$ can be extended to $T_\omega \in \mathscr{B}(\omega; E)$ by virtue of Proposition 4.1. Hence there exists $\alpha \in \mathscr{B}(\Omega \cup \omega; E)$ with

$$S \mid \omega = T_\omega, \quad S \mid \Omega = T,$$

which is a contradiction.

3) follows from 1) and Proposition 4.1.

4) For simplification of notations we suppose that $\Omega = \boldsymbol{R}^n$ and $F = F_1 \cup F_2$. Let $\mathscr{F}$ be the family of triplets (Ω, T_1, T_2) such that

$$T_i \in \mathscr{B}_{F_i}(\Omega; E) \qquad (i=1, 2),$$

$$T_1 + T_2 = T \mid \Omega .$$

$\mathscr{F}$ is ordered and inductive for the relation of order of inclusion and extension.

Let (Ω, T_1, T_2) be a maximal element and suppose that there exists $x_0 \notin \Omega$. Let ω be a bounded open set containing x_0.

Let $T_i \mid \Omega \cap \omega \in \mathscr{B}_{F_i}(\Omega \cap \omega; E)$ can be extended to $T'_i \in \mathscr{B}_{\overline{F_i \cap \Omega}}(\omega; E)$ and

$$T \mid \omega - T'_1 - T'_2 \in \mathscr{B}_{\{F_1 \cup F_2 - (F_1 \cup F_2) \cap \Omega\}}(\omega; E).$$

Hence, by virtue of Proposition 4.1, there exist $S_i \in \mathscr{B}_{F_i - F_i \cap \Omega}(\omega; E)$ such that

$$T \mid \omega = T'_1 + T'_2 + S_1 + S_2 .$$

Since $(T'_i + S_i) \mid \Omega \cap \omega = T_i \mid \Omega \cap \omega$, there exist $T''_i \in \mathscr{B}(\Omega \cup \omega; E)$ such that

$$T''_i \mid \Omega = T_i, \quad T''_i \mid \omega = T'_i + S_i .$$

Hence we have

$$T''_i \in \mathscr{B}_{F_i}(\Omega \cup \omega; E) \quad \text{and}$$

$$T|\Omega\cup\omega=T_1''+T_2'',$$

which is a contradiction. Q. E. D.

Theorem 4.2. *The sheaf* ${}^E\mathscr{D}'$ *of E-valued distributions is a subsheaf of* ${}^E\mathscr{B}$.

Proof. Let Ω be an open set in $\boldsymbol{R}^n$. We define thus the mapping

$$\mathscr{D}'(\Omega;\ E)\longrightarrow\mathscr{B}(\Omega;\ E).$$

Let Ω_j be a sequence of open sets with

$$\Omega_j\Subset\Omega_{j+1},\quad\bigcup_{j=0}^{\infty}\Omega_j=\Omega.$$

Let $\phi_j\in\mathscr{D}(\Omega_{j+1})$ and $\phi_j=1$ in a neighborhood of $\bar{\Omega}_j$. Let $T\in\mathscr{D}'(\Omega;\ E)$ and put $T_j=\phi_jT$. Then $T_j\in\mathscr{E}'(\Omega;\ E)$, hence $T_j\in\mathscr{A}'(\Omega;\ E)$ and $T_j|\Omega_j\in\mathscr{B}(\Omega_j;\ E)$, where we denote by $T_j|\Omega_j$ the image of $T_j\in\mathscr{A}'(\bar{\Omega}_j;\ E)$ in $\mathscr{B}(\Omega_j;\ E)$. If $k>j$,

$$T_k-T_j\in\mathscr{E}'(\Omega_{k+1}-\bar{\Omega}_j).$$

Hence $\operatorname{supp}(T_k-T_j)\cap\Omega_j=\emptyset$ and

$$T_k|\Omega_j=T_j|\Omega_j\quad\text{in}\quad\mathscr{B}(\Omega_j;\ E).$$

The sequence $T_j|\Omega_j$ defines an E-valued hyperfunction $T'\in\mathscr{B}(\Omega;\ E)$. It is easy to verify that T' is independent of choices of $\{(\Omega_j,\ \phi_j)\}$ and that we have thus constructed a linear mapping of $\mathscr{D}'(\Omega;\ E)$ into $\mathscr{B}(\Omega;\ E)$ which commutes with restrictions.

If $T\in\mathscr{D}'(\Omega;\ E)$ is of image zero, it is equivalent to say that for all j

$$\operatorname{supp}(\phi_jT)\cap\Omega_j=\emptyset.$$

Hence, by virtue of Proposition 2.4, the restriction of T to $\mathscr{D}'(\Omega_j;\ E)$ is zero. Hence $T=0$. Q. E. D.

§ 5. Operations on hyperfunctions valued in a Fréchet space E

In this section we define several operations on E-valued hyperfunctions.

a) Multiplication by an analytic function

Let Ω be an open set in $\boldsymbol{R}^n$. If $f\in\mathscr{A}(\Omega)$ and $T\in\mathscr{B}(\Omega;\ E)$ and $\{\Omega_j\}_{j=0}^{\infty}$ be an open covering of Ω with $\Omega_j\Subset\Omega_{j+1}$, we shall define fT as follows. Let $u_{T_j}\in\mathscr{A}'(\bar{\Omega}_j;\ E)$ such that

$$u_{T_j}|\Omega_j=T|\Omega_j=T_j,$$

where $u_{T_j}|\Omega_j$ denotes the image of u_{T_j} in $\mathscr{B}(\Omega_j;\ E)$. Since $\mathscr{A}'(\bar{\Omega}_j;\ E)$ is an $\mathscr{A}(\bar{\Omega}_j)$-module, we have

$$fu_{T_{j+k}}\,|\,\Omega_j = fu_{T_j}\,|\,\Omega_j, \qquad \text{for} \quad k \geqq 0.$$

Hence, $fu_{T_j}\,|\,\Omega_j$'s define an E-valued hyperfunction which depends only on f and on T and which we denote by fT.

We verified that we have thus defined on $\mathscr{B}(\Omega; E)$ a structure of $\mathscr{A}(\Omega)$-module and at the same time that the sheaf ${}^E\mathscr{B}$ is an $\mathscr{A}$-module.

b) Tensor product of E-valued hyperfunctions

Let now E_1 and E_2 be two Fréchet spaces, and ω stands for ε or π topology. Let then Ω_1 and Ω_2 be open subsets of $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$, respectively. Let $T_1 \in \mathscr{B}(\Omega_1; E_1)$ and $T_2 \in \mathscr{B}(\Omega_2; E_2)$. Let

$$\Omega_1 = \bigcup_{j=1}^{\infty} \Omega_{1j} \quad \text{and} \quad \Omega_2 = \bigcup_{j=1}^{\infty} \Omega_{2j}$$

with

$$\Omega_{1j} \Subset \Omega_{1j+1} \quad \text{and} \quad \Omega_{2j} \Subset \Omega_{2j+1},$$

$$u_{T_{1j}} \in \mathscr{A}'(\bar{\Omega}_{1j}; E_1), \; u_{T_{1j}}\,|\,\Omega_{1j} = T\,|\,\Omega_{1j},$$

$$u_{T_{2j}} \in \mathscr{A}'(\bar{\Omega}_{2j}; E_2), \; u_{T_{2j}}\,|\,\Omega_{2j} = T\,|\,\Omega_{2j}.$$

We have

$$(u_{T_{1j+k}} \otimes_\omega u_{T_{2j+k}})\,|\,\Omega_{1j} \times \Omega_{2j} = (u_{T_{1j}} \otimes_\omega u_{T_{2j}})\,|\,\Omega_{1j} \times \Omega_{2j}$$

and the sequence $(u_{T_{1j}} \otimes_\omega u_{T_{2j}})\,|\,\Omega_{1j} \times \Omega_{2j}$ defines a hyperfunction on $\Omega_1 \times \Omega_2$ which is $T_1 \otimes_\omega T_2$.

We can verify that this product has properties of tensor products of vector valued distributions and extends them. In particular we have

$$\operatorname{supp}(T_1 \otimes_\omega T) \subset \operatorname{supp}(T_1) \times \operatorname{supp}(T_2).$$

c) Convolution of E-valued hyperfunctions

Let E, E_1 and E_2 be Fréchet spaces, ω stands for ε or π topology. Let now $T \in \mathscr{B}(\boldsymbol{R}^n; E_2)$, and $u \in \mathscr{A}'(\boldsymbol{R}^n; E_1)$. We will now define $u *_\omega T$, the convolution of an E_1-valued analytic linear mapping u and an E_2-valued hyperfunction T. Let Ω_j be an open ball with center at the origin and with radius j, and

$$u_{T_j} \in \mathscr{A}'(\bar{\Omega}_j; E_2), \quad u_{T_j}\,|\,\Omega_j = T\,|\,\Omega_j.$$

Let k be such that $u \in \mathscr{A}'(\Omega_k; E_1)$. If $j' \geqq j > k$, we have

$$(u *_\omega u_{T_j})\,|\,\Omega_{j-k} = (u *_\omega u_{T_{j'}})\,|\,\Omega_{j-k},$$

since

$$\operatorname{supp}(u *_\omega (u_{T_j} - u_{T_{j'}})) \subset \Omega_k + (\overline{\Omega_{j'} - \Omega_j}) \subset C\Omega_{j-k}.$$

The sequence $(u *_\omega u_{T_j}) \mid \Omega_{j-k}$ defines a hyperfunction which we denote by $u *_\omega T$.

We verify that we have thus defined a linear mapping of $\mathscr{B}(\boldsymbol{R}^n; E_2)$ into $\mathscr{B}(\boldsymbol{R}^n; E_1 \otimes_\omega E_2)$ which extends the convolution of analytic linear mappings, and, if $u \in \mathscr{E}'(\boldsymbol{R}^n; E_1)$, which extends the convolution of vector valued distributions.

The convolution product of several hyperfunctions, all but at most one with compact support, can be defined and commutative and distributive with respect to addition.

Let $u \in \mathscr{A}'(\boldsymbol{R}^n; E_1)$ and $T \in \mathscr{B}$ $(\boldsymbol{R}^n; E_2)$. We have

$$\operatorname{supp}(u *_\omega T) \subset \operatorname{supp}(u) + \operatorname{supp}(T).$$

In fact, let $u_{T_j} \in \mathscr{A}'(\bar{\Omega}_j; E_2)$ with $u_{T_j} \mid \Omega_j = T \mid \Omega_j$. We can, modifying u_{T_j} on $\partial\Omega_j$, suppose

$$\operatorname{supp}(u_{T_j}) \subset \operatorname{supp}(T).$$

Then we have

$$\begin{aligned} \operatorname{supp}(u *_\omega T) \cap \Omega_{j-k} &= \operatorname{supp}(u *_\omega u_{T_j}) \cap \Omega_{j-k} \\ &\subset (\operatorname{supp}(u) + \operatorname{supp}(u_{T_j})) \cap \Omega_{j-k} \\ &\subset (\operatorname{supp}(u) + \operatorname{supp}(T)) \cap \Omega_{j-k}. \end{aligned}$$

This permit us to extend our definition of the convolution.

If $T \in \mathscr{B}(\Omega; E_2)$ and $u \in \mathscr{A}'(\omega; E_1)$, where Ω and ω are open sets in $\boldsymbol{R}^n$, and if Ω' is an open set such that

$$(\omega + C\Omega) \cap \Omega' = \emptyset,$$

we set

$$(u *_\omega T) \mid \Omega' = (u *_\omega \bar{T}) \mid \Omega',$$

where $\bar{T}$ is an prolongation of T to $\mathscr{B}(\boldsymbol{R}^n; E_2)$. In particular, if $u \in \mathscr{A}'(\{0\})$, $u *$ defines a morphism of the sheaf ${}^E\mathscr{B}$.

Definition 5.1. *Let ${}^E\mathscr{F}$ be a subsheaf of ${}^E\mathscr{B}$. We call ${}^E\mathscr{F}$-support of an element $T \in \mathscr{B}(\Omega; E)$ and we denote by ${}^E\mathscr{F}$-*$\operatorname{supp}(T)$ *the smallest closed subset of Ω outside of which T belongs to ${}^E\mathscr{F}$. (Or again ${}^E\mathscr{F}$-*$\operatorname{supp}(T)$ *is the support of the image of T into the quotient sheaf ${}^E\mathscr{B}/{}^E\mathscr{F}$).*

If ${}^E\mathscr{F} = \{0\}$, we have hence

$$\{0\} - \operatorname{supp}(T) = \operatorname{supp}(T).$$

If ${}^E\mathscr{F} = {}^E\mathscr{B}$, we have

$${}^E\mathscr{B}\text{-}\operatorname{supp}(T) = \emptyset.$$

Theorem 5.1. *Let $T \in \mathscr{B}(\boldsymbol{R}^n; E_2)$ and $u \in \mathscr{A}'(\boldsymbol{R}^n; E_1)$. We have then*

$$ {}^{E_1\hat{\otimes}_\omega E_2}\mathscr{A}\text{-supp}\,(u *_\omega T) \subset {}^{E_1}\mathscr{A}\text{-supp}\,(u) + {}^{E_2}\mathscr{A}\text{-supp}\,(T). $$

Proof. i) It suffices to prove this formula for $T \in \mathscr{A}'(\boldsymbol{R}^n; E_2)$, for if Ω_j is the ball of radius j with center at the origin and if $u_{T_j} \in \mathscr{A}'(\bar{\Omega}_j; E_2)$ coincides with T in Ω_j, we have

$$ {}^{E_1\hat{\otimes}_\omega E_2}\mathscr{A}\text{-supp}\,(u *_\omega T) \subset ({}^{E_1}\mathscr{A}\text{-supp}\,(u) + {}^{E_2}\mathscr{A}\text{-supp}\,(T)) $$
$$ \cup\,(\text{supp}\,(u) + \text{supp}\,(T - u_{T_j})) $$

and, for a fixed k and a sufficiently large j, this set coincides on Ω_k with

$$ {}^{E_1}\mathscr{A}\text{-supp}\,(u) + {}^{E_2}\mathscr{A}\text{-supp}\,(T). $$

ii) Hence we suppose $T \in \mathscr{A}'(\boldsymbol{R}^n; E_2)$ and let K_1 and K_2 be compact neighborhoods of ${}^{E_1}\mathscr{A}$-supp (u) and ${}^{E_2}\mathscr{A}$-supp (T) respectively. We can write

$$ u = u_1 + u_2, $$
$$ T = v_1 + v_2 $$

with

$$ \text{supp}\,(u_1) \subset K_1, $$
$$ \text{supp}\,(v_1) \subset K_2, $$
$$ u_2 = \chi_{\omega_1} f_1, $$
$$ v_2 = \chi_{\omega_2} f_2, $$

where χ_{ω_i} is the characteristic function of $\omega_i = \boldsymbol{R}^n - K_i$ $(i = 1, 2)$ and f_i is an E_i-valued analytic function in a neighborhood of $\bar{\omega}_i$.

The theorem then follows from

Lemma 5.1. *Let Ω be an open subset of $\boldsymbol{R}^n$, $f \in \mathscr{A}(\bar{\Omega}; E_1)$ and $u \in \mathscr{A}'(\boldsymbol{R}^n; E_2)$. Then we have*

$$ {}^{E_1\hat{\otimes}_\omega E_2}\mathscr{A}\text{-supp}\,(u *_\omega \chi_\Omega f) \subset \partial\Omega + \text{supp}\,(u) $$

and, if $x_0 \notin \partial\Omega + \text{supp}\,(u)$,

$$ (\chi_\Omega f *_\omega u)(x_0) = \langle u_x, f(x_0 - x)\rangle_\omega, $$

where $\langle u_x, f(x)\rangle_\omega$ denotes the bilinear mapping which extends

$$ \langle u_x, f(x)\rangle_\omega = \tilde{u}(\tilde{f})(\boldsymbol{e}_1 \otimes_\omega \boldsymbol{e}_2) $$

for $u = \tilde{u} \otimes \boldsymbol{e}_2$ and $f = \tilde{f} \otimes \boldsymbol{e}_1$ and $\boldsymbol{e}_i \in E_i$ $(i = 1, 2)$, $\tilde{u} \in \mathscr{A}'(\boldsymbol{R}^n)$, $\tilde{f} \in \mathscr{A}(\boldsymbol{R}^n)$.

Proof. We can suppose that Ω is bounded. Let χ_ε be the characteristic function of the ball of radius ε with the center at x_0. Put

$$v=\chi_\Omega f *_\omega u-\chi_\varepsilon\langle u_t, f(x-t)\rangle_\omega.$$

We have to prove that, for ε small enough, x_0 does not belong to $\operatorname{supp}(v)$. Let $g_j\in\mathscr{A}(\boldsymbol{R}^n)$ be a sequence of analytic functions which tends to zero in $\mathscr{A}(\boldsymbol{R}^n-\{x_0\})$. It is sufficient to prove that

$$v(g_j)\to 0,$$

$$v(g_j)=\langle u_t, \int_{\boldsymbol{R}^n}\chi_\Omega(-x)f(-x)g_j(t-x)dx\rangle_\omega$$
$$-\int_{\boldsymbol{R}^n}\chi_\varepsilon(x)g_j(x)\langle u_t, f(x-t)\rangle_\omega\, dx.$$

We can then interchange the integrations and the bilinear mapping. Hence,

$$v(g_j)=\langle u_t, \int_{\boldsymbol{R}^n}\chi_\Omega(x)f(x)g_j(t+x)dx$$
$$-\int_{\boldsymbol{R}^n}\chi_\varepsilon(x)g_j(x)f(x-t)dx\rangle_\omega.$$

But we have

$$\int_{\boldsymbol{R}^n}\chi_\Omega(x)f(x)g_j(x+t)dx$$
$$=\int_{\boldsymbol{R}^n}\chi_\varepsilon(x)g_j(x)f(x-t)dx$$
$$+\int_{\boldsymbol{R}^n}(\chi_\Omega(x-t)-\chi_\varepsilon(x))g_j(x)f(x-t)dx.$$

Let $K=\operatorname{supp}(u)$. We have to see that

$$\int_{\boldsymbol{R}^n}(\chi_\Omega(x-t)-\chi_\varepsilon(x))g_j(x)f(x-t)dx$$

tends to zero in $\mathscr{A}(K; E_1)$. If t is in a neighborhood of K and if $|x-x_0|<\varepsilon$ and $x-t\in\Omega$, the integral is equal to

$$\int_{|x-x_0|>\varepsilon}\chi_\varepsilon(x-t)g_j(x)f(x-t)dx.$$

Suppose that $x_0=0$. We are reduced to prove that the mapping

$$g\to(\chi'_\varepsilon g)*(\chi_\Omega f)\quad\text{with}\quad \chi'_\varepsilon=1-\chi_\varepsilon$$

is a continuous linear mapping of $\mathscr{A}(\boldsymbol{R}^n-\{0\})$ into $\mathscr{A}(K; E_1)$ if $K\subset\Omega$ and ε is small enough. Let Ω_r be an open ball centered at the origin with radius r. It is sufficient to prove that our mapping transforms $\mathscr{A}(\bar{\Omega}_j-\Omega_{1/j})$ into $\mathscr{A}(K; E_1)$. The continuity then follows from Lebesgue's theorem. Q. E. D.

Hence we are reduced to prove Theorem 5.1 supposing that T and u are vector valued distributions. Let $\mathscr{E}^*(\Omega; E)$ be a non quasi-analytic class of E-valued functions (for example $\mathscr{E}^*(\Omega; E)=\mathscr{E}^{\{M_p\}}(\Omega; E)$ for a non quasi-analytic sequence $\{M_p\}$; [30], [32], [42], [43]). Since there exist partitions of unity in $\mathscr{E}^*(\Omega)=\mathscr{E}^{\{M_p\}}(\Omega)$, it is immediate that, T and u being distributions,

$$^{E_1\hat{\otimes}_\omega E_2}\mathscr{E}^*\text{-supp}\,(T*_\omega u)$$

$$\subset {}^{E_1}\mathscr{E}^*\text{-supp}\,(u)+{}^{E_2}\mathscr{E}^*\text{-supp}\,(T).$$

Let $F={}^{E_1}\mathscr{A}\text{-supp}\,(u)+{}^{E_2}\mathscr{A}\text{-supp}\,(T)$. Then we have

$$T*_\omega u\,|\,\boldsymbol{R}^n-F\in\mathscr{E}^*(\boldsymbol{R}^n-F:\,E_1\hat{\otimes}_\omega E_2)$$

and the theorem then follows from the fact ([41], and Appendix) that, if Ω is an open subset of $\boldsymbol{R}^n$, we have

$$\mathscr{A}(\Omega; E)=\cap\,\mathscr{E}^*(\Omega; E),$$

the intersection being taken over all the non quasi-analytic classes of E-valued functions. Q. E. D.

d) Image of an E-valued hyperfunction by an analytic isomorphism

Let Ω_1 and Ω_2 be open sets in $\boldsymbol{R}^n$ and y an analytic diffeomorphism of Ω_1 onto Ω_2;

$$y:\,\Omega_1\longrightarrow\Omega_2\,.$$

If $u\in\mathscr{A}'(\Omega_2; E)$, we define

$$u\circ y\in\mathscr{A}'(\Omega_1; E)$$

by the formula

$$(u\circ y)(f)=u((f\circ y^{-1})\,|J|),\qquad \text{for}\quad f\in\mathscr{A}(\Omega_1),$$

where $|J|$ is the Jacobian of the mapping y^{-1}. The mapping thus defined

$$y^*:\,\mathscr{A}'(\Omega_2; E)\longrightarrow\mathscr{A}'(\Omega_1; E)$$

is linear and verifies

$$\text{supp}\,(y^*u)=y^{-1}(\text{supp}\,(u))\qquad \text{for}\quad u\in\mathscr{A}'(\Omega_2; E).$$

Hence y^* can be prolonged to a morphism of sheaves

$$y*: {}^E\mathscr{B}\,|\,\Omega_2 \longrightarrow {}^E\mathscr{B}\,|\,\Omega_1 .$$

This permits us to define the sheaf ${}^E\mathscr{B}$ of E-valued hyperfunctions over a real analytic manifold M.

§6. Elliptic regularity and the resolution of the sheaf of holomorphic functions valued in a Fréchet space E

a) Elliptic regularity

Theorem 6.1. *Let P be an elliptic differential operator with constant coefficients. Let Ω be an open set in $\boldsymbol{R}^n$ and $u \in \mathscr{B}(\Omega; E)$ a solution of the equation*

$$Pu=0.$$

Then $u \in \mathscr{A}(\Omega; E)$.

Proof. We can suppose that Ω is a bounded open set. Let $\bar{u} \in \mathscr{A}'(\bar{\Omega}; E)$ be a prolongation of u such that

$$P\bar{u}=v \in \mathscr{A}'(\partial\Omega; E).$$

Let Y be an elementary solution of P. Y is analytic in $\boldsymbol{R}^n-\{0\}$. Hence, by virtue of Theorem 5.1,

$$\bar{u}=Y*P\bar{u}=Y*v$$

is analytic in the complement of $\partial\Omega$. Q.E.D.

b) The resolution of ${}^E\mathscr{O}$

Let Ω be an open set in $\boldsymbol{C}^n$, $\boldsymbol{C}^n$ being identified with $\boldsymbol{R}^{2n}$. Let ${}^E\mathscr{F}$ be one of the sheaves ${}^E\mathscr{A}$, ${}^E\mathscr{E}$, ${}^E\mathscr{D}'$, ${}^E\mathscr{B}$.

A differential form with coefficients in $\mathscr{F}(\Omega; E)$ is called of type (p, q) if we can write it as follows:

$$f=\sum_{|I|=p}\sum_{|J|=q} f_{I,J}dz_I \wedge d\bar{z}_J,$$

where $I=(i_1,\cdots, i_p)$, $J=(j_1,\cdots,j_q)$,

$$dz_I=dz_{i_1}\wedge\cdots\wedge dz_{i_p},$$

$$d\bar{z}_J=d\bar{z}_{j_1}\wedge\cdots\wedge d\bar{z}_{j_q},$$

$$f_{I,J}\in\mathscr{F}(\Omega; E).$$

We then define the sheaf ${}^E\mathscr{F}^{p,q}$ of differential forms of type (p, q) with coefficients in ${}^E\mathscr{F}$ and ∂ and $\bar{\partial}$ are the morphisms of sheaves:

$$\partial : {}^E\mathscr{F}^{p,q} \longrightarrow {}^E\mathscr{F}^{p+1,q},$$

$$\bar{\partial} : {}^E\mathscr{F}^{p,q} \longrightarrow {}^E\mathscr{F}^{p,q+1}$$

defined by

$$\partial f = \sum_{i=1}^{n} \sum_{|I|=p} \sum_{|J|=q} \frac{\partial}{\partial z_i} f_{I,J} dz_i \wedge dz_I \wedge d\bar{z}_J,$$

$$\bar{\partial} f = \sum_{i=1}^{n} \sum_{|I|=p} \sum_{|J|=q} \frac{\partial}{\partial \bar{z}_i} f_{I,J} d\bar{z}_i \wedge dz_I \wedge d\bar{z}_J.$$

We define in the same way the sheaf ${}^E\mathcal{O}^p$ of differential forms of type $(p, 0)$ with coefficients in ${}^E\mathcal{O}$. We then have a complex of sheaves:

$$0 \longrightarrow {}^E\mathcal{O}^p \longrightarrow {}^E\mathscr{F}^{p,0} \xrightarrow{\bar{\partial}} {}^E\mathscr{F}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} {}^E\mathscr{F}^{p,n} \longrightarrow 0,$$

for $\bar{\partial}\circ\bar{\partial}=0$, and, if $f\in {}^E\mathscr{F}^{p,0}(\Omega)$ has holomorphic coefficients, we have $\bar{\partial}f=0$.

If ${}^E\mathscr{F}$ is one of the sheaves ${}^E\mathscr{E}$ or ${}^E\mathscr{D}'$, it is well known that the complex is an exact sequence of sheaves, hence a resolution of ${}^E\mathcal{O}^p$ [cf. L. Hörmander [13], and P. D. F. Ion and T. Kawai [14]]. If ${}^E\mathscr{F}={}^E\mathscr{E}$, it is the resolution of E-Dolbeault-Grothendieck.

If ${}^E\mathscr{F}={}^E\mathscr{A}$, the complex is again a resolution of ${}^E\mathcal{O}^p$. In order to see this, we have only to take into account the resolution of $\mathcal{O}^p$ by the sheaves $\mathscr{A}^{p,q}$ and the argument of P. D. F. Ion and T. Kawai [14].

Theorem 6.2. *The sequence*

$$0 \longrightarrow {}^E\mathcal{O}^p \longrightarrow {}^E\mathscr{B}^{p,0} \xrightarrow{\bar{\partial}} {}^E\mathscr{B}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} {}^E\mathscr{B}^{p,n} \longrightarrow 0$$

is an exact sequence of sheaves.

Proof. Let $u\in\mathscr{B}^{p,0}(\Omega; E)$ satisfy $\bar{\partial}u=0$. If

$$u=\sum_{|I|=p} u_I dz_I,$$

$\bar{\partial}u=0$ implies that

$$\frac{\partial}{\partial \bar{z}_i} u_I=0 \qquad \text{for} \quad i=1,\cdots, n,$$

from which we have

$$\sum_{i=1}^{n} \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar{z}_i} u_I=0.$$

Since the operator (on $\boldsymbol{R}^{2n}$)

$$\sum_{i=1}^{n} \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar{z}_i}$$

is elliptic, it follows from Theorem 6.1 that u_I's are analytic on $\boldsymbol{R}^{2n}$ and hence holomorphic.

Next we have to prove the exactness of the above sequence in the latter steps.

We will reason as in L. Hörmander [13], p. 32. In order to do so, we have only to note the following

Lemma. *Let Ω be a relatively compact open set in $\boldsymbol{C}^n$. Let $u \in \mathscr{B}^{p,q+1}(\Omega; E)$ $(p, q \geqq 0)$ satisfy the condition $\bar{\partial}u=0$. If Ω' is a relatively compact open subset of Ω, we can find $v \in \mathscr{B}^{p,q}(\Omega'; E)$ with $\bar{\partial}v=u$ in Ω'.*

Proof. We shall prove inductively that the lemma is true if u does not involve $d\bar{z}_{k+1}, \cdots, d\bar{z}_n$. This is trivial if $k=0$, for u must then be zero since every term in u is of degree $q+1>0$ with respect to $d\bar{z}$. For $k=n$, the statement is identical to the theorem.

Assuming that it has already been proved when k is replaced by $k-1$, we write

$$u = d\bar{z}_k \wedge g + h,$$

where $g \in \mathscr{B}^{p,q}(\Omega; E)$, $h \in \mathscr{B}^{p,q+1}(\Omega; E)$, and g and h are independent of $d\bar{z}_k, \cdots, d\bar{z}_n$. Write

$$g = \sum_{|I|=p}{}' \sum_{|J|=q}{}' g_{I,J} dz_I \wedge d\bar{z}_J,$$

where $\sum'$ means that we sum only over increasing multi-indices. Since $\bar{\partial}u=0$, we obtain

$$\frac{\partial g_{I,J}}{\partial \bar{z}_j}=0, \qquad j>k.$$

Thus $g_{I,J}$ is holomorphic in these variables.

We now choose a solution $G_{I,J}$ of the equation

$$\frac{\partial G_{I,J}}{\partial \bar{z}_k} = g_{I,J}\,.$$

To do so, we choose

$$Y = \delta'_{k-1} \otimes \frac{1}{\pi z_k} \otimes \delta'_{n-k},$$

where δ'_{k-1} and δ'_{n-k} are the Dirac measures at the origin in $\boldsymbol{C}^{k-1}$ and $\boldsymbol{C}^{n-k}$ respectively. Let Ω'' be such that $\Omega' \Subset \Omega'' \Subset \Omega$ and $\chi_{\bar{\Omega}''}$ the characteristic function of $\bar{\Omega}''$. Set

$$G_{I,J} = Y * (\chi_{\bar{\Omega}''} g_{I,J}) \,|\, \Omega.$$

Then $G_{I,J} \in \mathscr{B}(\Omega; E)$ and it satisfies in Ω''

$$\frac{\partial G_{I,J}}{\partial \bar{z}_k} = g_{I,J},$$

$$\frac{\partial G_{I,J}}{\partial \bar{z}_j} = 0, \quad j > k.$$

If we set

$$G = \sum_{|I|=p}' \sum_{|J|=q}' G_{I,J} dz_I \wedge d\bar{z}_J,$$

then we obtain

$$\bar{\partial} G = d\bar{z}_k \wedge g + h_1,$$

where h_1 does not involve $d\bar{z}_k, \cdots, d\bar{z}_n$. Hence $h - h_1 = f - \bar{\partial} G$ does not involve $d\bar{z}_k, \cdots, d\bar{z}_n$, so by induction hypothesis we can find $w \in \mathscr{B}^{p,q}(\Omega'; E)$ so that $\bar{\partial} w = f - \bar{\partial} G$ there. But then $v = w + G$ satisfies the equation $\bar{\partial} v = u$, which completes the proof. Q.E.D.

§7. Boundary values of holomorphic functions valued in a Fréchet space E

1. Sato's theory

a) Cohomology groups with coefficients in the sheaf ${}^E\mathscr{A}$

Let ${}^E\mathscr{A}$ be the sheaf of E-valued analytic functions over $\boldsymbol{R}^n$ and ${}^E\mathscr{O}$ the sheaf of E-valued holomorphic functions over $\boldsymbol{C}^n$, the complexification of $\boldsymbol{R}^n$. If $x \in \boldsymbol{R}^n$, we have an isomorphism

$${}^E\mathscr{A}_x \cong {}^E\mathscr{O}_x.$$

Hence, for all open subset Ω of $\boldsymbol{R}^n$, we have

$${}^E\mathscr{A} \mid \Omega = {}^E\mathscr{O} \mid \Omega.$$

Since every open set in $\boldsymbol{C}^n$ is paracompact, it follows from Theorem B42 of P. Schapira [33], p. 38, that

$$\mathscr{A}(\Omega; E) = \varinjlim_{\tilde{\Omega} \cap \boldsymbol{R}^n = \Omega} \mathscr{O}(\tilde{\Omega}; E),$$

where $\tilde{\Omega}$ is an open neighborhood in $\boldsymbol{C}^n$ of an open set Ω in $\boldsymbol{R}^n$ such that $\tilde{\Omega} \cap \boldsymbol{R}^n = \Omega$ and $\mathscr{A}(\Omega; E)$ is the section module of ${}^E\mathscr{A}$ on Ω and $\mathscr{O}(\tilde{\Omega}; E)$ is the section module of ${}^E\mathscr{O}$ on $\tilde{\Omega}$.

In the same way, in the following of this section, $\tilde{\Omega}$ denotes an open set in $\boldsymbol{C}^n$ and Ω denotes an open set in $\boldsymbol{R}^n$ as far as the contrary is not explicitly mentioned.

Theorem 7.1. *Let Ω be an arbitrary open set in $\boldsymbol{R}^n$. Then we have*

$$H^p(\Omega, {}^E\mathscr{A}) = 0$$

for every positive integer p.

Proof. We know, by virtue of Grauert's Theorem [cf. H. Grauert [6] or H. Komatsu [19], Theorem V.2.5, p. 194], that Ω has a fundamental system of Stein open neighborhoods. Then, it follows, from Oka-Cartan Theorem B [cf. P. D. F. Ion and T. Kawai [14], Theorem 2.1, p. 11] and Theorem B42 of P. Schapira [33], p. 38, that for $p>0$, we have

$$H^p(\Omega, {}^E\mathscr{A}) = \varinjlim_{\tilde{\Omega}\cap \boldsymbol{R}^n=\Omega} H^p(\tilde{\Omega}, {}^E\mathcal{O}) = 0. \qquad \text{Q.E.D.}$$

b) Malgrange's Theorem

Theorem 7.2 (Malgrange's Theorem). *Let $\tilde{\Omega}$ be an open set in $\boldsymbol{C}^n$ and F a closed subset of $\tilde{\Omega}$. Then we have*

(i) $H^p_F(\tilde{\Omega}, {}^E\mathcal{O}) = 0$, *for* $p>n$.

(ii) $H^p(\tilde{\Omega}, {}^E\mathcal{O}) = 0$, *for* $p \geqq n$.

Proof. (i) By virtue of Theorem 6.2, and Theorem B32 of P. Schapira [33], p. 27, the cohomology group $H^p_F(\tilde{\Omega}, {}^E\mathcal{O})$ is isomorphic to the p-th cohomology group of the complex:

$$0 \longrightarrow \mathscr{B}^{0,0}_F(\tilde{\Omega}; E) \xrightarrow{\bar{\partial}} \mathscr{B}^{0,1}_F(\tilde{\Omega}; E) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{B}^{0,n}_F(\tilde{\Omega}; E) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots.$$

Hence, for $p>n$, this cohomology group is zero.

(ii) We apply this result to $\boldsymbol{C}^n$ with $F = \boldsymbol{C}^n - \tilde{\Omega}$ and have

$$H^p_{\boldsymbol{C}^n-\tilde{\Omega}}(\boldsymbol{C}^n, {}^E\mathcal{O}) = 0, \qquad \text{for} \quad p>n.$$

We write the exact sequence of cohomology groups with support in $\boldsymbol{C}^n - \tilde{\Omega}$ [cf. P. Schapira [33], Corollary 1 of Theorem B35, p. 32]:

$$\cdots \longrightarrow H^p(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow H^p(\tilde{\Omega}, {}^E\mathcal{O}) \longrightarrow H^{p+1}_{\boldsymbol{C}^n-\tilde{\Omega}}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow H^{p+1}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow \cdots.$$

The theorem then follows from the fact that

$$H^p(\boldsymbol{C}^n, {}^E\mathcal{O}) = 0, \qquad \text{for} \quad p>0. \qquad \text{Q.E.D.}$$

c) Cohomology groups with support in a compact subset of $\boldsymbol{C}^n$. Martineau-Harvey's theorem

Theorem 7.3. *Let K be a compact subset of $\boldsymbol{C}^n$ such that*

$$H^p(K, \mathcal{O}) = 0, \qquad \textit{for} \quad p>0.$$

Then we have

$$H_K^p(\boldsymbol{C}^n, {}^E\mathcal{O})=0, \qquad \textit{for} \quad p\neq n,$$

and there exists an isomorphism ρ:

$$\rho\colon H_K^n(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow \mathcal{O}'(K; E).$$

Further, if $K_1 \subset K_2$ *satisfy the hypotheses of this theorem, it follows from the proof that the diagram*

$$\begin{array}{ccc} H_{K_1}^n(\boldsymbol{C}^n, {}^E\mathcal{O}) & \longrightarrow & H_{K_2}^n(\boldsymbol{C}^n, {}^E\mathcal{O}) \\ \downarrow & & \downarrow \\ \mathcal{O}'(K_1; E) & \longrightarrow & \mathcal{O}'(K_2; E) \end{array}$$

is commutative.

Proof. We consider the resolutions of $\mathcal{O}$ and ${}^E\mathcal{O}$:

(1) $$0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{A}^{0,0} \xrightarrow{\bar\partial} \mathscr{A}^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{A}^{0,n} \longrightarrow 0,$$

(2) $$0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{B}^{0,0} \xrightarrow{\bar\partial} \mathscr{B}^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{B}^{0,n} \longrightarrow 0,$$

(3) $$0 \longrightarrow {}^E\mathcal{O} \longrightarrow {}^E\mathscr{B}^{0,0} \xrightarrow{\bar\partial} {}^E\mathscr{B}^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {}^E\mathscr{B}^{0,n} \longrightarrow 0.$$

Since we have

$$H^p(K, \mathscr{A}^{0,q}) = \operatorname*{lim.ind.}_{\Omega\supset K} H^p(\Omega, \mathscr{A}^{0,q})=0, \quad \text{for} \quad p>0,\ q\geqq 0,$$

by virtue of Theorem 4.11.1 of R. Godement [5], p. 193 and Lemma 411 of P. Schapira [33], p. 118, the cohomology groups $H^p(K, \mathcal{O})$ are isomorphic to the cohomology groups of the complex:

(4) $$0 \longrightarrow \mathscr{A}^{0,0}(K) \xrightarrow{\bar\partial} \mathscr{A}^{0,1}(K) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{A}^{0,n}(K) \longrightarrow 0,$$

and since the sheaves ${}^E\mathscr{B}^{0,p}$ are flabby, the relative cohomology groups $H_K^p(\boldsymbol{C}^n, {}^E\mathcal{O})$ with support in K are isomorphic to the cohomology groups of the complex:

(5) $$0 \longrightarrow {}^E\mathscr{B}_K^{0,0} \xrightarrow{\bar\partial} {}^E\mathscr{B}_K^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {}^E\mathscr{B}_K^{0,n} \longrightarrow 0$$

(where ${}^E\mathscr{B}_K^{0,p}$ stands for $\mathscr{B}_K^{0,p}(\boldsymbol{C}^n; E)$). The hypotheses $H^p(K, \mathcal{O})=0$, $p>0$, imply that the sequence (4) is exact. Hence, the operator $\bar\partial$ are homomorphisms, since they are of closed range, for the spaces $\mathscr{A}^{0,p}(K)$ are DFS-spaces. [cf. A. Grothendieck [11], Chapter 4, § 2, Theorem 3, p. 218.]

If we denote $\mathscr{B}_K^{0,p}(\boldsymbol{C}^n)$ by $\mathscr{B}_K^{0,p}$, the spaces $\mathscr{A}^{0,p}(K)$ and $\mathscr{B}_K^{0,n-p}$ are DFS- and FS-spaces by the duality pairing:

$$\langle \sum_{|J|=n-p} \phi_J d\bar{z}_J, \sum_{|I|=p} f_I d\bar{z}_I \rangle$$
$$= \sum_{I \cup J=(1,\cdots,n)} \varepsilon_{I,J} \langle \phi_J, f_I \rangle,$$

where $\varepsilon_{I,J}$ denotes the signature of the permutation $(1,\cdots, n) \to (I, J)$. Further, the transpose of the operator

$$\bar{\partial}: \mathscr{A}^{0,p}(K) \longrightarrow \mathscr{A}^{0,p+1}(K)$$

is (aside from sign) the operator

$$\bar{\partial}: \mathscr{B}_K^{0,n-p-1} \longrightarrow \mathscr{B}_K^{0,n-p}.$$

By virtue of Serre's lemma [cf. H. Komatsu [18], Theorem 19, p. 381 or P. Schapira [33], Lemma 413, p. 121] the sequence

$$0 \longrightarrow \mathscr{B}_K^{0,0} \xrightarrow{\bar{\partial}} \mathscr{B}_K^{0,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{B}_K^{0,n}$$

is exact. Hence the sequence

$$0 \longrightarrow {}^E\mathscr{B}_K^{0,0} \xrightarrow{\bar{\partial}} {}^E\mathscr{B}_K^{0,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} {}^E\mathscr{B}_K^{0,n}$$

is also exact by virtue of Theorem 1.10 of P. D. F. Ion and T. Kawai [14], p. 9, since the spaces $\mathscr{B}_K^{0,p}$ are all nuclear FS-spaces and ${}^E\mathscr{B}_K^{0,p} \cong \mathscr{B}_K^{0,p} \hat{\otimes} E$. Consequently, we have

$$H_K^p(\boldsymbol{C}^n, {}^E\mathscr{O}) = 0, \qquad \text{for} \quad p < n.$$

At last we consider two exact sequences:

$$0 \longrightarrow \mathscr{O}(K) \longrightarrow \mathscr{A}^{0,0}(K) \xrightarrow{\bar{\partial}} \mathscr{A}^{0,1}(K)$$
$$0 \longleftarrow \mathscr{B}_K^{0,n}/\bar{\partial}\mathscr{B}_K^{0,n-1} \longleftarrow \mathscr{B}_K^{0,n} \xleftarrow{\bar{\partial}} \mathscr{B}_K^{0,n-1}.$$

The mapping

$$\bar{\partial}: \mathscr{B}_K^{0,n-1} \longrightarrow \mathscr{B}_K^{0,n}$$

is of closed range since it is the transpose of a mapping of closed range. Hence

$$\mathscr{B}_K^{0,n}/\bar{\partial}\mathscr{B}_K^{0,n-1} \text{ (which is isomorphic to } H_K^n(\boldsymbol{C}^n, \mathscr{O}))$$

is isomorphic to $\mathscr{O}'(K)$. Since we have

$${}^E\mathscr{B}_K^{0,p} \cong \mathscr{B}_K^{0,p} \hat{\otimes} E \quad \text{and} \quad \mathscr{O}'(K; E) \cong \mathscr{O}'(K) \hat{\otimes} E,$$

we conclude that

$$H_K^n(\boldsymbol{C}^n, {}^E\mathscr{O}) \cong H_K^n(\boldsymbol{C}^n, \mathscr{O}) \hat{\otimes} E \cong \mathscr{O}'(K; E). \qquad \text{Q.E.D.}$$

d) The relative cohomology groups with support in $\boldsymbol{R}^n$ (Sato's theorem)

Theorem 7.4. *Let Ω be an open subset of $\boldsymbol{R}^n$.*

(i) *The relative cohomology groups $H^p_{\bar\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O})$ are zero for $p \neq n$.*

(ii) *The presheaf over $\boldsymbol{R}^n$*

$$\Omega \longrightarrow H^n_{\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O})$$

is a sheaf.

(iii) *This sheaf is isomorphic to the sheaf ${}^E\mathcal{B}$ of E-valued hyperfunctions.*

Proof. (i), (ii) Let Ω be a bounded open subset of $\boldsymbol{R}^n$. We have the exact sequence

$$\cdots \longrightarrow H^p_{\partial\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow H^p_{\bar\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow H^p_{\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow H^{p+1}_{\partial\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow \cdots$$

[cf. H. Komatsu [19], Theorem II.3.2, p. 77, or P. Schapira [33], Theorem B35, p. 31]. Since $\bar\Omega$ and $\partial\Omega$ are real compact sets which consequently satisfy the hypotheses of Theorem 411 of P. Schapira [33], p. 118, we have

$$H^p_{\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) = 0, \quad p < n-1,$$

and we have the exact sequence

$$\begin{aligned} 0 \longrightarrow H^{n-1}_{\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) &\longrightarrow H^n_{\partial\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) \\ \longrightarrow H^n_{\bar\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) &\longrightarrow H^n_{\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow 0. \end{aligned}$$

Since the morphism

$$H^n_{\partial\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow H^n_{\bar\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O})$$

is isomorphic to the morphism

$$\mathcal{A}'(\partial\Omega; E) \longrightarrow \mathcal{A}'(\bar\Omega; E)$$

which is injective, we have

$$H^{n-1}_{\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) = 0.$$

Then we consider the sheaves over $\boldsymbol{R}^n$:

$$\mathcal{H}^p_{\boldsymbol{R}^n}({}^E\mathcal{O})$$

associated with the presheaves

$$\Omega \longrightarrow H^p_{\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}).$$

These sheaves are zero for $p < n$ and since $H^p_{\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) = 0$, if $p > n$, by virtue of Theorem 7.2. The parts (i) and (ii) of the theorem follow from Theorem II.3.18 of H.

Komatsu [19], p. 89, or Theorem B36 of P. Schapira [33], p. 34.

(iii) Let Ω be an open subset of $\boldsymbol{R}^n$. From the exact sequence

$$0 \longrightarrow H^n_{\partial\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow H^n_{\bar{\Omega}}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow H^n_{\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow 0,$$

we deduce that the sheaf

$$\mathscr{H}^n_{\boldsymbol{R}^n}({}^E\mathcal{O})$$

is flabby.

If K is a real compact set, the relative cohomology group $H^n_K(\boldsymbol{C}^n, {}^E\mathcal{O})$ is isomorphic to $\mathscr{A}'(K; E)$ by virtue of Theorem 7.3. Hence for all bounded open set Ω, the relative cohomology groups

$$H^n_{\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O}) = H^n_{\bar{\Omega}}(\boldsymbol{C}^n, {}^E\mathcal{O}) / H^n_{\partial\Omega}(\boldsymbol{C}^n, {}^E\mathcal{O})$$

and

$$\mathscr{B}(\Omega; E) = \mathscr{A}'(\bar{\Omega}; E)/\mathscr{A}'(\partial\Omega; E)$$

are isomorphic. Consequently the sheaves

$$\mathscr{H}^n_{\boldsymbol{R}^n}({}^E\mathcal{O}) \quad \text{and} \quad {}^E\mathscr{B}$$

are isomorphic. Q.E.D.

Let ρ be the isomorphism

$$\mathscr{H}^n_{\boldsymbol{R}^n}({}^E\mathcal{O}) \xrightarrow{\rho} {}^E\mathscr{B}.$$

Let Ω be an open set in $\boldsymbol{R}^n$ and $\tilde{\Omega}$ an open set in $\boldsymbol{C}^n$ with $\tilde{\Omega} \cap \boldsymbol{R}^n = \Omega$. Using the resolution of ${}^E\mathcal{O}$ by ${}^E\mathscr{B}^{0,p}$'s we see that we have an isomorphism (again denoted by ρ)

$$\rho\colon \mathscr{B}^{0,n}_{\Omega}(\tilde{\Omega}; E)/\bar{\partial}\mathscr{B}^{0,n-1}_{\Omega}(\tilde{\Omega}; E) \xrightarrow{\sim} \mathscr{B}(\Omega; E).$$

The inverse isomorphism is that which assigns to $T \in \mathscr{B}(\Omega; E)$ the class of

$$T \otimes \delta_y d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

modulo $\bar{\partial}\mathscr{B}^{0,n-1}_{\Omega}(\tilde{\Omega}; E)$.

In order to see this it suffices to prove this on sections with compact support. We denote by $\mathscr{A}(K \times \{0\})$ the space of analytic functions in a neighborhood of K in $\boldsymbol{R}^{2n}$ (i.e.: the dual of $\mathscr{B}_K(\boldsymbol{R}^{2n})$). The isomorphism

$$\mathscr{B}^{0,n}_K(\boldsymbol{R}^{2n}; E)/\bar{\partial}\mathscr{B}^{0,n-1}_K(\boldsymbol{R}^{2n}; E) \xrightarrow{\rho} \mathscr{A}'(K; E)$$

is the transpose of

$$\mathscr{A}(K) \longrightarrow \{f \in \mathscr{A}(K \times \{0\});\ \bar{\partial}f = 0\},$$

that is,

$$\mathscr{A}(K) \longrightarrow \mathscr{O}(K) \longrightarrow \mathscr{A}^{0,0}(K \times \{0\}),$$

and the transpose of the mapping

$$\mathscr{O}(K) \longrightarrow \mathscr{A}(K)$$

is the mapping

$$T \longrightarrow \text{class of } (T \otimes \delta_y) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \text{ modulo } [\bar{\partial}\mathscr{B}_K^{0,n-1}(\boldsymbol{R}^{2n}; E)].$$

2. Utilization of the Čech's cohomology groups

a) The mapping of "boundary value"

Let Ω be an open set in $\boldsymbol{R}^n$ and $\tilde{\Omega}$ a Stein neighborhood of Ω in $\boldsymbol{C}^n$ with

$$\tilde{\Omega} \cap \boldsymbol{R}^n = \Omega.$$

We have the exact sequence

$$\cdots \longrightarrow H^{n-1}(\tilde{\Omega}, {}^E\mathscr{O}) \longrightarrow H^{n-1}(\tilde{\Omega} - \Omega, {}^E\mathscr{O}) \xrightarrow{\delta} H^n_\Omega(\tilde{\Omega}, {}^E\mathscr{O}) \longrightarrow H^n(\tilde{\Omega}, {}^E\mathscr{O}) = 0.$$

If $n = 1$, we find

$$H^1_\Omega(\tilde{\Omega}, {}^E\mathscr{O}) = \mathscr{O}(\tilde{\Omega} - \Omega; E)/\mathscr{O}(\tilde{\Omega}; E),$$

and, if $n > 1$, δ is an isomorphism

$$H^{n-1}(\tilde{\Omega} - \Omega, {}^E\mathscr{O}) \xrightarrow{\sim} H^n_\Omega(\tilde{\Omega}, {}^E\mathscr{O}).$$

Using the resolution of ${}^E\mathscr{O}$ by the sheaves ${}^E\mathscr{B}^{0,p}$, we recall the construction of δ.

We consider the double complex below with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0_\Omega(\tilde{\Omega}, {}^E\mathscr{O}) = 0 & \longrightarrow & \mathscr{O}(\tilde{\Omega}; E) & \longrightarrow & \mathscr{O}(\tilde{\Omega} - \Omega; E) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{B}^{0,0}_\Omega(\tilde{\Omega}; E) & \longrightarrow & \mathscr{B}^{0,0}(\tilde{\Omega}; E) & \longrightarrow & \mathscr{B}^{0,0}(\tilde{\Omega} - \Omega; E) & \longrightarrow & 0 \\
 & & \bar{\partial}\downarrow & & \bar{\partial}\downarrow & & \bar{\partial}\downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \bar{\partial}\downarrow & & \bar{\partial}\downarrow & & \bar{\partial}\downarrow & & \\
0 & \longrightarrow & \mathscr{B}^{0,n-1}_\Omega(\tilde{\Omega}; E) & \longrightarrow & \mathscr{B}^{0,n-1}(\tilde{\Omega}; E) & \longrightarrow & \mathscr{B}^{0,n-1}(\tilde{\Omega} - \Omega; E) & \longrightarrow & 0 \\
 & & \bar{\partial}\downarrow & & \bar{\partial}\downarrow & & \bar{\partial}\downarrow & & \\
0 & \longrightarrow & \mathscr{B}^{0,n}_\Omega(\tilde{\Omega}; E) & \longrightarrow & \mathscr{B}^{0,n}(\tilde{\Omega}; E) & \longrightarrow & \mathscr{B}^{0,n}(\tilde{\Omega} - \Omega; E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let $\dot{T} \in H^{n-1}(\tilde{\Omega}-\Omega, {}^E\mathcal{O})$ and $T \in \mathscr{B}^{0,n-1}(\tilde{\Omega}-\Omega; E)$ a representative of $\dot{T}$ (hence $\bar{\partial}T=0$). Let $\bar{T} \in \mathscr{B}^{0,n-1}(\tilde{\Omega}; E)$ be a prolongation of T. $\bar{\partial}\bar{T} \in \mathscr{B}^{0,n}_{\Omega}(\tilde{\Omega}; E)$ shall be a representative of $\delta\dot{T}$ in $\mathscr{B}^{0,n}_{\Omega}(\tilde{\Omega}; E)/\bar{\partial}\mathscr{B}^{0,n-1}_{\Omega}(\tilde{\Omega}; E) \cong H^n_{\Omega}(\tilde{\Omega}, {}^E\mathcal{O})$.

Let now $\mathscr{U}$ be a Stein covering of $\tilde{\Omega}-\Omega$, i.e., a covering by domains of holomorphy. Since a finite intersection of domains of holomorphy is a domain of holomorphy [cf. L. Hörmander [13], Corollary 2.5.7, p. 40] and, if ω is such an open set, we have $H^p(\omega, {}^E\mathcal{O})=0$ for any $p>0$, the covering $\mathscr{U}$ shall be "acyclic" [cf. P. Schapira [33], Definition B51, p. 42]. By virtue of Leray's theorem [cf. H. Komatsu [19], Theorem II.3.29, p. 98, or P. Schapira [33], Theorem B52, p. 43], there exists an isomorphism

$$\lambda\colon H^{n-1}(\mathscr{U}, {}^E\mathcal{O}) \longrightarrow H^{n-1}(\tilde{\Omega}-\Omega, {}^E\mathcal{O}).$$

We shall consider a special covering $\mathscr{U}$. We set

$$\tilde{\Omega}_i = \tilde{\Omega} \cap \{z \in \boldsymbol{C}^n;\ \mathrm{Im}\, z_i \neq 0\},$$

$$\mathscr{U} = \{\tilde{\Omega}_i\}_{i=1}^n .$$

$\mathscr{U}$ is an acyclic covering of $\tilde{\Omega}-\Omega$ by n open sets (but by 2^n connected open sets). Hence we have

$$C^n(\mathscr{U}, {}^E\mathcal{O}) = \{0\}$$

(since the n-cochains are alternate elements of $n+1$ indices taken in the set $\{1,\cdots, n\}$). We set

$$\tilde{\Omega}\sharp\Omega = \tilde{\Omega}_{1,\cdots,n} = \bigcap_{i=1}^n \tilde{\Omega}_i,$$

$$\tilde{\Omega}^i = \tilde{\Omega}_{1,\cdots,\hat{i},\cdots,n} = \bigcap_{j\neq i} \tilde{\Omega}_j .$$

We have an isomorphism

$$\begin{array}{ccc} C^{n-1}(\mathscr{U}, {}^E\mathcal{O}) & \longrightarrow & \mathcal{O}(\tilde{\Omega}\sharp\Omega; E) \\ f & \longrightarrow & f_{1,\cdots,n} \end{array}$$

and an isomorphism

$$\begin{array}{ccc} C^{n-2}(\mathscr{U}, {}^E\mathcal{O}) & \longrightarrow & \prod_{i=1}^n \mathcal{O}(\tilde{\Omega}^i; E) \\ f & \longrightarrow & (f_{1,\cdots,\hat{i},\cdots,n})_{i=1}^n \end{array}$$

(with the convention that this two last groups are zero for $n=1$). The image of the mapping

$$\delta\colon C^{n-2}(\mathscr{U}, {}^E\mathcal{O}) \longrightarrow C^{n-1}(\mathscr{U}, {}^E\mathcal{O})$$

by these isomorphisms is the mapping which we denote by σ:

$$\prod_{i=1}^{n} \mathcal{O}(\tilde{\Omega}^i; E) \longrightarrow \mathcal{O}(\tilde{\Omega}\sharp\Omega; E),$$

$$(f_{1,\cdots,\hat{i},\cdots,n})_{i=1}^{n} \longrightarrow \sum_{i=1}^{n} (-1)^{i-1} f'_{1,\cdots,\hat{i},\cdots,n},$$

where $f'_{1,\cdots,\hat{i},\cdots,n}$ is the restriction of $f_{1,\cdots,\hat{i},\cdots,n}$ to $\tilde{\Omega}\sharp\Omega$. We denote by

$$\sum_{i} \mathcal{O}(\tilde{\Omega}^i; E)$$

the image of the mapping σ. It is equal to the image of the mapping

$$(f_{1,\cdots,\hat{i},\cdots,n})_{i=1}^{n} \longrightarrow \sum_{i=1}^{n} f'_{1,\cdots,\hat{i},\cdots,n}.$$

Hence we have an isomorphism

$$\mathcal{O}(\tilde{\Omega}\sharp\Omega; E)/\sum_{i} \mathcal{O}(\tilde{\Omega}^i; E) \xrightarrow{\sim} H^{n-1}(\mathcal{U}, {}^{E}\mathcal{O}).$$

We denote by μ the mapping

$$\mathcal{O}(\tilde{\Omega}\sharp\Omega; E) \longrightarrow H^{n-1}(\mathcal{U}, {}^{E}\mathcal{O})$$

thus defined.

We consider the mapping of

$$\mathcal{O}(\tilde{\Omega}\sharp\Omega; E) \text{ into } \mathcal{B}(\Omega; E):$$

$$\mathcal{O}(\tilde{\Omega}\sharp\Omega; E) \xrightarrow{\mu} H^{n-1}(\mathcal{U}, {}^{E}\mathcal{O}) \xrightarrow{\lambda} H^{n-1}(\tilde{\Omega}-\Omega, {}^{E}\mathcal{O})$$
$$\xrightarrow{\delta} H^{n}_{\Omega}(\tilde{\Omega}, {}^{E}\mathcal{O}) \xrightarrow{\rho} \mathcal{B}(\Omega; E).$$

Definition 7.1. *We put*

$$b = \left(\frac{2}{i}\right)^{n} \rho\circ\delta\circ\lambda\circ\mu.$$

If $f \in \mathcal{O}(\tilde{\Omega}\sharp\Omega; E)$, $b(f)$ is called the boundary value of f.

b) Properties of the operator b

In order to study the mapping b we have to return to the Leray's isomorphism λ. We recall how it is constructed. We consider the double complex:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \to & \mathcal{O}(\tilde{\Omega}-\Omega;E) & \to & \mathscr{B}^{0,0}(\tilde{\Omega}-\Omega;E) & \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} & & & \mathscr{B}^{0,n-1}(\tilde{\Omega}-\Omega;E) & \xrightarrow{\bar{\partial}} & \mathscr{B}^{0,n}(\tilde{\Omega}-\Omega;E) \to 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \to & C^0(\mathscr{U}, {}^E\mathcal{O}) & \to & C^0(\mathscr{U}, {}^E\mathscr{B}^{0,0}) & \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} & & & C^0(\mathscr{U}, {}^E\mathscr{B}^{0,n-1}) & \xrightarrow{\bar{\partial}} & C^0(\mathscr{U}, {}^E\mathscr{B}^{0,n}) \to 0 \\
 & & \delta\downarrow & & \delta\downarrow & & & & \delta\downarrow & & \delta\downarrow \\
0 & \to & C^1(\mathscr{U}, {}^E\mathcal{O}) & \to & C^1(\mathscr{U}, {}^E\mathscr{B}^{0,0}) & \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} & & & C^1(\mathscr{U}, {}^E\mathscr{B}^{0,n-1}) & \xrightarrow{\bar{\partial}} & C^1(\mathscr{U}, {}^E\mathscr{B}^{0,n}) \to 0 \\
 & & \delta\downarrow & & \delta\downarrow & & & & \delta\downarrow & & \delta\downarrow \\
 & & \vdots & & \vdots & & & & \vdots & & \vdots \\
 & & \delta\downarrow & & \delta\downarrow & & & & \delta\downarrow & & \delta\downarrow \\
0 & \to & C^{n-2}(\mathscr{U}, {}^E\mathcal{O}) & \to & C^{n-2}(\mathscr{U}, {}^E\mathscr{B}^{0,0}) & \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} & & & C^{n-2}(\mathscr{U}, {}^E\mathscr{B}^{0,n-1}) & \xrightarrow{\bar{\partial}} & C^{n-2}(\mathscr{U}, {}^E\mathscr{B}^{0,n}) \to 0 \\
 & & \delta\downarrow & & \delta\downarrow & & & & \delta\downarrow & & \delta\downarrow \\
0 & \to & C^{n-1}(\mathscr{U}, {}^E\mathcal{O}) & \to & C^{n-1}(\mathscr{U}, {}^E\mathscr{B}^{0,0}) & \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} & & & C^{n-1}(\mathscr{U}, {}^E\mathscr{B}^{0,n-1}) & \xrightarrow{\bar{\partial}} & C^{n-1}(\mathscr{U}, {}^E\mathscr{B}^{0,n}) \to 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & & & 0 & & 0 \quad .
\end{array}$$

Let $f \in \mathcal{O}(\tilde{\Omega} \# \Omega; E)$, $n > 1$. We define

$$f_p \in C^p(\mathscr{U}, {}^E\mathscr{B}^{0,n-p-1}),$$

putting

$$(f_{n-1})_{1,\dots,n} = f,$$

$$f_p = \bar{\partial} g_p, \qquad p < n-1,$$

where $g_p \in C^p(\mathscr{U}, {}^E\mathscr{B}^{0,n-p-2})$ is a solution of

$$\delta g_p = f_{p+1},$$

$$\begin{array}{ccccc}
C^p(\mathscr{U}, {}^E\mathscr{B}^{0,n-p-2}) & \xrightarrow{\bar{\partial}} & C^p(\mathscr{U}, {}^E\mathscr{B}^{0,n-p-1}), & g_p \xrightarrow{\bar{\partial}} & f_p \\
\delta\Big\downarrow & & & & \delta\Big\downarrow \\
C^{p+1}(\mathscr{U}, {}^E\mathscr{B}^{0,n-p-2}) & & & & f_{p+1}.
\end{array}$$

The solutions g_p exist since $\delta f_p = 0$ for any p and the columns of the double complex are all exact except for the first. Finally we obtain

$$f_0 \in C^0(\mathscr{U}, {}^E\mathscr{B}^{0,n-1}), \quad \delta f_0 = 0.$$

Hence f_0 defines an element of $\mathscr{B}^{0,n-1}(\tilde{\Omega}-\Omega; E)$ (again denoted by f_0) such that $\bar{\partial} f_0 = 0$. The class of f_0 modulo $\bar{\partial}\mathscr{B}^{0,n-2}(\tilde{\Omega}-\Omega; E)$ will be $\lambda(f)$.

This process of construction is called "Weil's process".

Proposition 7.1. *Let E_1 and E_2 be two Fréchet spaces. Let $\tilde{\Omega}$ and $\tilde{\Omega}'$ be open sets in $\boldsymbol{C}^n$ and $\boldsymbol{C}$, respectively, with*

$$\tilde{\Omega} \cap \boldsymbol{R}^n = \Omega \quad \textit{and} \quad \tilde{\Omega}' \cap \boldsymbol{R} = \Omega',$$

and $f \in \mathcal{O}(\tilde{\Omega} \# \Omega; E_1)$ and $f' \in \mathcal{O}(\tilde{\Omega}' - \Omega'; E_2)$. If $f \otimes_\omega f' \in \mathcal{O}(\tilde{\Omega} \times \tilde{\Omega}' \# \Omega \times \Omega'; E_1 \hat{\otimes}_\omega E_2)$ is defined by

$$(f\otimes_{\omega}f')(z, z')=f(z)\otimes_{\omega}f'(z'),$$

we have

$$b(f\otimes_{\omega}f')=b(f)\otimes_{\omega}b(f'),$$

where ω *stands for the* ε *or* π *topology.*

Proof. Put

$$\tilde{\Omega}_i=\tilde{\Omega}\cap\{z\in \boldsymbol{C}^n;\ \mathrm{Im}\, z_i\neq 0\}.$$

Let $\mathscr{U}=\{\tilde{\Omega}_i\}_{i=1}^n$, $\mathscr{U}'=\{\tilde{\Omega}'-\Omega'\}$ and $\mathscr{U}''$ be the covering of $\tilde{\Omega}''-\Omega''$ (where $\tilde{\Omega}''=\tilde{\Omega}\times\tilde{\Omega}'$, $\Omega''=\Omega\times\Omega'$) by the open sets

$$\tilde{\Omega}''_i=\tilde{\Omega}_i\times\tilde{\Omega}', \qquad 1\leqq i\leqq n,$$

$$\tilde{\Omega}''_{n+1}=\tilde{\Omega}\times(\tilde{\Omega}'-\Omega').$$

We denote again by ${}^{E_1\hat{\otimes}_{\omega}E_2}\mathscr{B}$ the sheaf of $E_1\hat{\otimes}_{\omega}E_2$-valued hyperfunctions over $\boldsymbol{R}^{2(n+1)}$ and similarly we denote by $\bar{\partial}$ and δ the differential (in $d\bar{z}$) and the coboundary operator "in" $\boldsymbol{C}^{n+1}$.

We temporarily suppose $n>1$.

Let (f_p, g_p) be the elements of a Weil's process departing from f and reaching $\lambda(f)$:

$$f_p\in C^p(\mathscr{U}, {}^{E_1}\mathscr{B}^{0,n-p-1}),$$

$$g_p\in C^p(\mathscr{U}, {}^{E_1}\mathscr{B}^{0,n-p-2}),$$

$$\bar{\partial}g_p=f_p, \quad \delta g_p=f_{p+1},$$

$$(f_{n-1})_{1\cdots n}=f, \quad f_0\in\mathscr{B}^{0,n-1}(\tilde{\Omega}-\Omega;\ E_1).$$

We define

$$f''_p\in C^p(\mathscr{U}'', {}^{E_1\hat{\otimes}_{\omega}E_2}\mathscr{B}^{n+1-p-1}),$$

$$g''_p\in C^p(\mathscr{U}'', {}^{E_1\hat{\otimes}_{\omega}E_2}\mathscr{B}^{n+1-p-1})$$

by

$$f''_{i_0\cdots i_p}=0 \quad \text{if} \quad n+1\notin(i_0,\cdots, i_p),$$

$$f''_{i_0\cdots i_{p-1},n+1}=f_{i_0\cdots i_{p-1}}\otimes_{\omega}f',$$

$$g''_{i_0\cdots i_p}=0 \quad \text{if} \quad n+1\notin(i_0,\cdots, i_p),$$

$$g''_{i_0\cdots i_{p-1},n+1}=g_{i_0\cdots i_{p-1}}\otimes_{\omega}f'.$$

We have

$$\bar{\partial} g''_p = f''_p,$$

$$\delta g''_p = f''_{p+1}$$

since

$$\bar{\partial}(g_{i_0 \cdots i_{p-1}} \otimes_\omega f') = (\bar{\partial} g_{i_0 \cdots i_{p-1}}) \otimes_\omega f'$$

and

$$(\delta g''_p)_{i_0 \cdots i_p, n+1} = \sum_{j=0}^{p} (-1)^j g''_{i_0 \cdots \hat{i}_j \cdots i_p n+1},$$

since $g''_{i_0 \cdots i_p} = 0$ if $(n+1) \notin (i_0, \cdots, i_p)$, and the last term of this equality becomes

$$(\delta g_{p-1})_{i_0 \cdots i_p} \otimes_\omega f'.$$

We finally find

$$f''_1 \in C^1(\mathscr{U}'', {}^{E_1 \hat{\otimes}_\omega E_2}\mathscr{B}^{0,n-1}),$$

$$(f''_1)_{i,j} = 0 \quad \text{if} \quad n+1 \notin (i, j),$$

$$(f''_1)_{i n+1} = f_i \otimes_\omega f' \quad 1 \leqq i < n+1$$

with $f_i = f_j$ on $\tilde{\Omega}_i \cap \tilde{\Omega}_j$. Let $f_0 \in \mathscr{B}^{0,n-1}(\tilde{\Omega} - \Omega; E_1)$ such that $f_0 \mid \tilde{\Omega}_i = f_i$. From now on we can suppose $n \geqq 1$. If $n = 1$, the proof begins now. We define g''_0 thus: let $\bar{f}_0$ be a prolongation of f_0 to $\mathscr{B}^{0,n-1}(\tilde{\Omega}; E_1)$. Put

$$(g''_0)_i = 0 \quad (1 \leqq i < n+1), \qquad (g''_0)_{n+1} = \bar{f}_0 \otimes_\omega f'.$$

Then

$$(\delta g''_0)_{i,n+1} = f_i \otimes_\omega f', \qquad i < n+1,$$

$$(\delta g''_0)_{i,j} = 0, \quad i, j < n+1.$$

Hence

$$\delta g''_0 = f''_1.$$

Let

$$f''_0 = \bar{\partial} g''_0.$$

f''_0 defines an element of $\mathscr{B}^{0,n}(\tilde{\Omega}'' - \Omega''; E_1 \hat{\otimes}_\omega E_2)$, $\bar{\partial} \bar{f}_0 \otimes_\omega f'$, the class of which modulo $\bar{\partial} \mathscr{B}^{0,n-1}(\tilde{\Omega}'' - \Omega''; E_1 \hat{\otimes}_\omega E_2)$ is $\lambda(f'')$ (with $f'' = f \otimes_\omega f'$) for the (f''_p, g''_p) are the elements of the Weil's process issued from f''.

Let $\bar{f}' \in \mathscr{B}(\tilde{\Omega}'; E_2)$ be a prolongation of f' and

$$\bar{f}''_0 = \bar{\partial} \bar{f}_0 \otimes_\omega \bar{f}' \in \mathscr{B}^{0,n}(\tilde{\Omega}''; E_1 \hat{\otimes}_\omega E_2).$$

$\bar{f}''_0$ is a prolongation of f''_0.

By virtue of the definition of b we have

$$\bar{\partial} \tilde{f}_0 = \left[\left(\frac{i}{2} \right)^n b(f) \otimes \delta_y + \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_i} T_i \right] d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n,$$

$$\bar{\partial} \tilde{f}' = \left[\frac{i}{2} b(f') \otimes \delta_{y'} + \frac{\partial}{\partial \bar{z}'} T \right] d\bar{z}'$$

with $T_i \in \mathscr{B}_\Omega(\tilde{\Omega}; E_1)$ and $T' \in \mathscr{B}_{\Omega'}(\tilde{\Omega}'; E_2)$. Hence we have

$$\bar{\partial} \tilde{f}''_0 = \bar{\partial} \tilde{f}_0 \otimes_\omega \bar{\partial} \tilde{f}'$$

$$= \left[\left(\frac{i}{2} \right)^{n+1} b(f) \otimes_\omega b(f') \otimes \delta_y \otimes \delta_{y'} + \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_i} \tilde{T}_i + \frac{\partial}{\partial \bar{z}'} \tilde{T}' \right] d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \wedge d\bar{z}'$$

with $\tilde{T}_i$, $\tilde{T}' \in \mathscr{B}_{\Omega''}(\tilde{\Omega}''; E_1 \hat{\otimes}_\omega E_2)$. Consequently we have

$$b(f'') = b(f) \otimes_\omega b(f'). \qquad \text{Q.E.D.}$$

Proposition 7.2. *Let E_1 and E_2 be two Fréchet spaces. Let $\tilde{\Omega}$ be an open set in $\boldsymbol{C}^n$ with*

$$\tilde{\Omega} \cap \boldsymbol{R}^n = \Omega.$$

Let $h \in \mathcal{O}(\tilde{\Omega}; E_1)$ and $f \in \mathcal{O}(\tilde{\Omega} \# \Omega; E_2)$. We denote again by h the restriction of h to Ω. Then we have

$$b(h \otimes_\omega f) = h \otimes_\omega b(f),$$

where ω stands for the ε or π topology.

Proof. Let (f_p, g_p) be the elements of a Weil's process departing from f and reaching $\lambda(f)$. In order to see that $(h \otimes_\omega f_p, h \otimes_\omega g_p)$ are the elements of a Weil's process departing from $h \otimes_\omega f$ and reaching $h \otimes_\omega \lambda(f)$, we have to prove that

$$\bar{\partial}(h \otimes_\omega g_p) = h \otimes_\omega \bar{\partial} g_p,$$

$$\delta(h \otimes_\omega g_p) = h \otimes_\omega \delta g_p,$$

which is evident. For ρ we have only to remark that

$$\rho^{-1}(h(x) \otimes_\omega T) = \text{the class of } ((h(x) \otimes_\omega T) \otimes \delta_y) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

$$= \text{the class of } h(z) \otimes_\omega (T \otimes \delta_y) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n. \qquad \text{Q.E.D.}$$

We shall prove in the same way that, if $P(D_x)$ is a differential operator, $P(D_z)$ its complexification, we have

$$b(P(D_z)f) = P(D_x) b(f).$$

Hence if $P(x, D_x)$ is a differential operator whose coefficients can be prolonged to holomorphic functions on $\tilde{\Omega}$, we have

$$b(P(z, D_z)f) = P(x, D_x)b(f).$$

In general, we have the following:

Proposition 7.3. *Suppose that* E_1 *and* E_2 *be two Fréchet spaces and* $\tilde{\Omega}$ *be a convex tube domain:* $\tilde{\Omega} = \boldsymbol{R}^n \times i\omega$. *Let* Ω *be* $\tilde{\Omega} \cap \boldsymbol{R}^n = \boldsymbol{R}^n$. *Let* $u \in \mathscr{A}'(\boldsymbol{R}^n; E_1)$ *and* $f \in \mathscr{O}(\tilde{\Omega}\sharp\Omega; E_2)$. *Then we have*

$$b(u *_\omega f) = u *_\omega b(f),$$

where ω *stands for the* ε *or* π *topology.* (*If* $u = \phi \otimes \boldsymbol{e}, f = \tilde{f} \otimes \boldsymbol{f}$, $\phi \in \mathscr{A}'(\boldsymbol{R}^n)$, $\tilde{f} \in \mathscr{O}(\tilde{\Omega}\sharp\Omega)$, $\boldsymbol{e} \in E_1$ *and* $\boldsymbol{f} \in E_2$,

$$(u *_\omega f)(z) = \langle \phi_t, \tilde{f}(z-t) \rangle \boldsymbol{e} \otimes_\omega \boldsymbol{f}.$$

If $f \in \mathscr{O}(\tilde{\Omega}\sharp\Omega; E_2)$, $u *_\omega f \in \mathscr{O}(\tilde{\Omega}\sharp\Omega; E_1 \hat{\otimes}_\omega E_2)$.)

Proof. Here again we show only that $u *_\omega$ commutes with λ. Let (f_p, g_p) be the elements of a Weil's process issued from f. We have to prove that

$$\bar{\partial}(u *_\omega g_p) = u *_\omega \bar{\partial} g_p,$$

$$\delta(u *_\omega g_p) = u *_\omega \delta g_p.$$

The first equality is evident and the second follows from the fact that if $\tilde{\Omega}_1 \subset \tilde{\Omega}_2$ are two tube domains:

$$\tilde{\Omega}_1 = \boldsymbol{R}^n \times i\omega_1,$$

$$\tilde{\Omega}_2 = \boldsymbol{R}^n \times i\omega_2,$$

$$\omega_1 \subset \omega_2,$$

and if $h \in \mathscr{A}(\tilde{\Omega}_2; E_2)$, $u \in \mathscr{A}'(\boldsymbol{R}^n; E_1)$, we have

$$(u *_\omega h) \mid \tilde{\Omega}_1 = u *_\omega (h \mid \tilde{\Omega}_1).$$

Then we have

$$(\delta(u *_\omega g))_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j (u *_\omega g)_{i_0 \cdots \hat{i}_j \cdots i_{p+1}} \mid \tilde{\Omega}_{i_0 \cdots i_{p+1}}$$

$$= u *_\omega \Big[\sum_{j=0}^{p+1} (-1)^j g_{i_0 \cdots \hat{i}_j \cdots i_{p+1}}\Big] \mid \tilde{\Omega}_{i_0 \cdots i_{p+1}}. \qquad \text{Q.E.D.}$$

Theorem 7.5. (i) *Let* $\tilde{\Omega}$ *be a domain of holomorphy and* $\Omega = \tilde{\Omega} \cap \boldsymbol{R}^n$. *Put*

$$\tilde{\Omega}_i = \tilde{\Omega} \cap \{z \in \boldsymbol{C}^n; \operatorname{Im} z_i \neq 0\},$$

$$\tilde{\Omega}\sharp\Omega = \bigcap_{i=1}^{n} \tilde{\Omega}_i,$$

$$\tilde{\Omega}^i = \bigcap_{j \neq i} \tilde{\Omega}_j.$$

The mapping

$$b: \mathcal{O}(\tilde{\Omega}\sharp\Omega;\, E) \longrightarrow \mathcal{B}(\Omega;\, E)$$

is surjective and its kernel is $\sum_i \mathcal{O}(\tilde{\Omega}^i;\, E)$ *if* $n>1$ *or* $\mathcal{O}(\tilde{\Omega};\, E)$ *if* $n=1$.

(ii) *Let* $u \in \mathscr{A}'(\boldsymbol{R}^n;\, E)$. *Put*

$$\tilde{u}(z) = \left(\frac{1}{2i\pi}\right)^n u_t\left(\frac{1}{(t_1-z_1)\cdots(t_n-z_n)}\right).$$

Then $\tilde{u} \in \mathcal{O}(\boldsymbol{C}^n\sharp\boldsymbol{R}^n;\, E)$ *and* $b(\tilde{u})=u$.

(iii) *Let* $g \in \mathscr{A}(\Omega;\, E)$. *There exists a domain of holomorphy* $\tilde{\Omega}$ *such that* g *can be prolonged to* $\mathcal{O}(\tilde{\Omega};\, E)$ *and* $\tilde{\Omega} \cap \boldsymbol{R}^n = \Omega$. *Let then*

$$\tilde{\Omega}_\sigma = \{z \in \tilde{\Omega};\, \sigma\, \mathrm{Im}\, z > 0\}$$

where $\sigma=(\sigma_1,\cdots,\sigma_n)$, $\sigma_i=\pm 1$, *and* g_σ *be the function in* $\mathcal{O}(\tilde{\Omega}\sharp\Omega;\, E)$ *which is equal to zero on all the connected components of* $\tilde{\Omega}\sharp\Omega$ *except for* $\tilde{\Omega}_\sigma$ *where it is equal to* $g_\sigma | \tilde{\Omega}_\sigma$. *Put*

$$\mathrm{sgn}\,(\sigma) = \sigma_1 \cdots \sigma_n.$$

Then

$$b(g_\sigma) = \mathrm{sgn}\,(\sigma) g.$$

Proof. (i) has been proved at the paragraph 2, a).

(ii) By virtue of Proposition 7.3, it is sufficient to prove that

$$b\left(\frac{1}{z_1\cdots z_n}\right) = (-2i\pi)^n \delta_{x_1} \otimes \cdots \otimes \delta_{x_n},$$

and since

$$\frac{1}{z_1\cdots z_n} = \frac{1}{z_1} \otimes \cdots \otimes \frac{1}{z_n},$$

it is sufficient by virtue of Proposition 7.1 for $E_1=E_2=\boldsymbol{C}$ to prove this result for $n=1$. But since $\frac{1}{z}$ is locally integrable in $\boldsymbol{R}^2$, $\frac{1}{z}$ can be prolonged to a distribution on $\boldsymbol{R}^2$ and we have

$$\bar{\partial}\left(\frac{-1}{2i\pi}\,\frac{1}{z}\right) = \frac{i}{2}(\delta_x \otimes \delta_y)\, d\bar{z}$$

(cf. P. Schapira [33], Corollary of Theorem A31, p. 6), from which

$$b\left(\frac{-1}{2i\pi}\,\frac{1}{z}\right) = \delta_x.$$

(iii) By virtue of Proposition 7.2, it is sufficient to prove this result for $g=1$, but this has been proved in P. Schapira [33], p. 139. Q.E.D.

c) Representation of analytic linear mappings

Let K be a compact set in $\boldsymbol{C}^n$ of the form

$$K=K_1\times\cdots\times K_n.$$

Since K admits a fundamental system of neighborhoods which are domains of holomorphy,

$$H^i(K, \mathcal{O})=0, \qquad i>0.$$

(cf. R. Godement [5], Theorem 4.11.1, p. 193 and P. D. F. Ion and T. Kawai [14], Theorem 2.1, p. 11.) By virtue of Theorem 7.3, there exists an isomorphism

$$H_K^n(\boldsymbol{C}^n, {}^E\mathcal{O}) \longrightarrow \mathcal{O}'(K; E).$$

Let Ω be a domain of holomorphy containing K and put

$$\Omega_i=\Omega\cap\{z\in\boldsymbol{C}^n;\ z_i\notin K_i\}.$$

$\mathcal{U}=\{\Omega_i\}_{i=1}^n$ forms an acyclic covering of $\Omega-K$ since Ω_i are domains of holomorphy. Put

$$\Omega^i=\bigcap_{j\neq i}\Omega_j,$$

$$\Omega\sharp K=\bigcap_{i=1}^n\Omega_i.$$

Let $\sum_i \mathcal{O}(\Omega^i; E)$ be the image in $\mathcal{O}(\Omega\sharp K; E)$ of $\prod_{i=1}^n \mathcal{O}(\Omega^i; E)$ by the mapping

$$(f_i)_{i=1}^n \longrightarrow \sum_{i=1}^n(-1)^{i-1}f_i'$$

where f_i' denotes the restriction of f_i to $\Omega\sharp K$. We can define as in the paragraph 2a the mappings

$$\mathcal{O}(\Omega\sharp K; E) \xrightarrow{\mu} H^{n-1}(\mathcal{U}, {}^E\mathcal{O}) \xrightarrow{\lambda} H^{n-1}(\Omega-K, {}^E\mathcal{O})$$
$$\xrightarrow{\delta} H_K^n(\boldsymbol{C}^n, {}^E\mathcal{O}) \xrightarrow{\rho} \mathcal{O}'(K; E),$$

and put

$$b=\left(\frac{2}{i}\right)^n\rho\circ\delta\circ\lambda\circ\mu.$$

Theorem 7.6. (i) *The mapping*

$$b\colon \mathcal{O}(\Omega\sharp K; E)\longrightarrow\mathcal{O}'(K; E)$$

is surjective and its kernel is $\sum_{i=1}^{n} \mathcal{O}(\Omega^i; E)$ *if* $n>1$ *or* $\mathcal{O}(\Omega; E)$ *if* $n=1$.

(ii) *Let* $u \in \mathcal{O}'(K; E)$. *Put*

$$\tilde{u}(z) = \left(\frac{1}{2i\pi}\right)^n u_\xi\left(\frac{1}{(\xi_1 - z_1)\cdots(\xi_n - z_n)}\right).$$

Then $\tilde{u} \in \mathcal{O}(\Omega \# K; E)$ *and*

$$b(\tilde{u}) = u.$$

(iii) *Let* $f \in \mathcal{O}(\Omega \# K; E)$ *and* $g \in \mathcal{O}(K)$. *Let* $\omega = \omega_1 \times \cdots \times \omega_n$ *be an open set containing* K *with* $\omega \subset \Omega$ *and* $g \in \mathcal{O}(\omega)$. *Let* Γ_i $(i=1,\cdots, n)$ *be regular contours in* ω_i *enclosing once* K_i *and oriented in the usual sense. We have*

$$b(f)(g) = (-1)^n \int_{\Gamma_1} \cdots \int_{\Gamma_n} f(z) g(z) dz_1 \cdots dz_n.$$

Proof. (i) and (ii) can be proved as Theorem 7.5 by modifying slightly Propositions 7.1 and 7.3.

(iii) The integral

$$\int_{\Gamma_1} \cdots \int_{\Gamma_n} f(z) g(z) dz$$

does not depend on the chosen contours and defines a linear mapping:

$$b' : \mathcal{O}(\Omega \# K; E) \longrightarrow \mathcal{O}'(K; E)$$

which is zero on $\sum_i \mathcal{O}(\Omega^i; E)$. Hence it is sufficient by virtue of (i) and (ii) to prove that if $u \in \mathcal{O}'(K; E)$ we have

$$b'(\tilde{u}) = u.$$

But

$$(-1)^n \left(\frac{1}{2i\pi}\right)^n \int_{\Gamma_1} \cdots \int_{\Gamma_n} u_\xi\left(\frac{1}{\xi - z}\right) g(z) dz$$
$$= u_\xi\left(\left(\frac{1}{2i\pi}\right)^n \int_{\Gamma_1} \cdots \int_{\Gamma_n} \frac{g(z)}{z - \xi} dz\right) = u(g).$$

Thus we have proved the theorem. Q.E.D.

3. Representation of distributions valued in a Fréchet space E

We preserve the notations of the paragraph 2. $\tilde{\Omega}$ is a domain of holomorphy which encounters $\boldsymbol{R}^n$ along Ω.

Let $f \in \mathcal{O}(\tilde{\Omega} \# \Omega; E)$, $\phi \in \mathcal{D}(\Omega)$, and $\sigma = (\sigma_1, \cdots, \sigma_n)$, $\sigma_i = \pm 1$, $\operatorname{sgn}(\sigma) = \sigma_1 \cdots \sigma_n$. We set

$$C_y^\sigma(f, \phi)=\int_\Omega f(x+i\sigma y)\phi(x)dx.$$

This integral is defined for sufficiently small $|y|>0$.

We suppose that, for an arbitrary $\phi \in \mathscr{D}(\Omega)$, $C_y^\sigma(f, \phi)$ has a limit $C^\sigma(f, \phi)$ when y tends to zero "by positive values" (i.e.: $y_i>0$, $i=1,\cdots, n$). It follows from the Banach-Steinhaus theorem that there exists an E-valued distribution $T_\sigma \in \mathscr{D}'(\Omega; E)$ such that

$$T_\sigma(\phi)=C^\sigma(f, \phi)$$

(cf. F. Trèves [40], Corollary to Theorem 33.1, p. 348).

We denote by $\mathcal{O}(\tilde{\Omega}\#\Omega; E; b')$ the subspace of $\mathcal{O}(\tilde{\Omega}\#\Omega; E)$ of those f such that $C_y^\sigma(f, \phi)$ has a limit for an arbitrary $\phi \in \mathscr{D}(\Omega)$ and for an arbitrary σ. We put then

$$b'(f)=\sum_\sigma \operatorname{sgn}(\sigma)T_\sigma.$$

Now let $\mathcal{O}(\tilde{\Omega}\#\Omega; E; \mathscr{D}')$ (resp. $\mathcal{O}(\tilde{\Omega}\#\Omega; E; C^0)$) the subspace of $\mathcal{O}(\tilde{\Omega}\#\Omega; E)$ of E-valued functions whose restriction to each connected component of $\tilde{\Omega}\#\Omega$ can be prolonged to an E-valued distribution (resp. to an E-valued continuous function up to the boundary) in the neighborhood of Ω.

We evidently have

$$\mathcal{O}(\tilde{\Omega}\#\Omega; E; C^0)\subset\mathcal{O}(\tilde{\Omega}\#\Omega; E; b').$$

Proposition 7.4. *Let $f\in\mathcal{O}(\tilde{\Omega}\#\Omega; E; C^0)$. Then $b(f)\in C^0(\Omega; E)$ and $b(f)=b'(f)$.*

Proof. Let $\tilde{\Omega}^+=\tilde{\Omega}\cap\{y_1>0,\cdots, y_n>0\}$. We can for simplicity suppose that f is zero on all the connected components of $\tilde{\Omega}\#\Omega$ except for $\tilde{\Omega}^+$. That is, $f=(f^+, 0,\cdots, 0)$. Let $\tilde{1}=(1, 0,\cdots, 0)\in\mathcal{O}(\tilde{\Omega}\#\Omega)$. Let $\phi_{z_i}^+$ be the characteristic function of $\{y_i>0\}$ and ϕ_{x_i} that of $\{y_i=0\}$ in $\boldsymbol{C}$. By virtue of the proof of Lemma 431 of P. Schapira [33], p. 145, there exist the elements (f_p, g_p) of a Weil's process departing from $\tilde{1}$ and reaching

$$\left(\frac{i}{2}\right)^{n-1}[\phi_{x_1}\otimes\delta_{y_1}\otimes\phi_{x_2}\otimes\delta_{y_2}\otimes\cdots\otimes\phi_{x_{n-1}}\otimes\delta_{y_{n-1}}\otimes\phi_{z_n}^+]d\bar{z}_1\wedge\cdots\wedge d\bar{z}_{n-1} \in\mathscr{B}^{0,n-1}(\tilde{\Omega}-\Omega),$$

and such that

$$\bar{\partial}g_p=f_p,\quad \delta g_p=f_{p+1}$$
$$(f_{n-1})_{1\cdots n}=1,\quad f_0\in\mathscr{B}^{0,n-1}(\tilde{\Omega}-\Omega).$$

In fact,

$$(f_{n-1})_{1\cdots n}=1,$$

$$g_{1\cdots n-1}=\phi^+_{z_1}\otimes\cdots\otimes\phi^+_{z_{n-1}},$$

$$g_{i_0\cdots i_{n-2}}=0,\quad \text{if}\quad (i_0,\cdots,i_{n-2})\neq(1,\cdots,n-1),$$

$$f_{1\cdots n-1}=\bar\partial g_{1\cdots n-1}=\frac{i}{2}\sum_{i=1}^{n-1}\phi^+_{z_1}\otimes\cdots\otimes\phi_{x_i}\otimes\delta_{y_i}\otimes\cdots\otimes\phi^+_{z_{n-1}}d\bar z_i,$$

$$f_{i_0\cdots i_{n-2}}=0,\quad \text{if}\quad (i_0,\cdots,i_{n-2})\neq(1,\cdots,n-1),$$

$$g_{1\cdots n-2}=\frac{i}{2}\sum_{i=1}^{n-2}\phi^+_{z_1}\otimes\cdots\otimes\phi_{x_i}\otimes\delta_{y_i}\otimes\cdots\otimes\phi^+_{z_{n-2}}d\bar z_i$$

$$+\frac{i}{2}\phi^+_{z_1}\otimes\cdots\otimes\phi^+_{z_{n-2}}\otimes\phi_{x_{n-1}}\otimes\delta_{y_{n-1}}d\bar z_{n-1},$$

$$g_{i_0\cdots i_{n-3}}=0,\quad \text{if}\quad (i_0,\cdots,i_{n-3})\neq(1,\cdots,n-2),$$

$$f_{1\cdots n-2}=\bar\partial g_{1\cdots n-2}$$

$$=\left(\frac{i}{2}\right)^2\sum_{i,j=1}^{n-2}\phi^+_{z_1}\otimes\cdots\otimes\phi_{x_i}\otimes\delta_{y_i}\otimes\cdots\otimes\phi_{x_j}\otimes\delta_{y_j}\otimes\cdots\otimes\phi^+_{z_{n-2}}d\bar z_i\wedge d\bar z_j$$

$$+\left(\frac{i}{2}\right)^2\left(\sum_{i=1}^{n-2}\phi^+_{z_1}\otimes\cdots\otimes\phi_{x_i}\otimes\delta_{y_i}\otimes\cdots\otimes\phi^+_{z_{n-2}}\otimes\phi_{x_{n-1}}\otimes\delta_{y_{n-1}}d\bar z_i\right)\wedge d\bar z_{n-1},$$

$$f_{i_0\cdots i_{n-3}}=0,\quad \text{if}\quad (i_0,\cdots,i_{n-3})\neq(1,\cdots,n-2),$$

thus f_p and g_p are linear combinations of tensor products of

$$\phi^+_{z_i}\quad\text{and}\quad\frac{i}{2}(\phi_{x_i}\otimes\delta_{y_i})d\bar z_i.$$

Hence we can set

$$f'_p=f\otimes f_p,\quad g'_p=g\otimes g_p.$$

Then (f'_p, g'_p)'s are the elements of a Weil's process issued from f, for

$$\delta g_p=f_{p+1}\quad\text{and}\quad\bar\partial g_p=f_p$$

follows from

$$\frac{\partial}{\partial\bar z_i}d\bar z_i[[f\otimes\cdots\otimes\phi_{x_i}\otimes\delta_{y_i}\otimes\cdots]\cdots\wedge d\bar z_i\wedge\cdots]=0$$

$$\frac{\partial}{\partial\bar z_i}d\bar z_i[[f\otimes\cdots\otimes\phi^+_{z_i}\otimes\cdots]\cdots]$$

$$=\left[\frac{i}{2}f\otimes\cdots\otimes\phi_{x_i}\otimes\delta_{y_i}\otimes\cdots\right]d\bar z_i\wedge\cdots=f\otimes d\bar z_i\wedge[\cdots].$$

Hence

$$f\otimes\left(\frac{i}{2}\right)^{n-1}[\phi_{x_1}\otimes\delta_{y_1}\otimes\cdots\otimes\phi_{x_{n-1}}\otimes\delta_{y_{n-1}}\otimes\phi^+_{z_n}]d\bar z_1\wedge\cdots\wedge d\bar z_{n-1}$$

will be a representative of $\lambda(f)$ in $\mathscr{B}^{0,n-1}(\tilde{\Omega}-\Omega;\, E)$. This element is naturally prolongable to $\tilde{\Omega}$ and if we apply $\bar{\partial}$ we find

$$\left(\frac{i}{2}\right)^n f\otimes[\phi_{x_1}\otimes\delta_{y_1}\otimes\cdots\otimes\phi_{x_n}\otimes\delta_{y_n}]d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n$$
$$=\left(\frac{i}{2}\right)^n f(x)\otimes\delta_y d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n.$$

Hence $b(f)=f|\Omega=b'(f)$. Q.E.D.

Theorem 7.7. *We have $\mathscr{O}(\tilde{\Omega}\sharp\Omega;\, E;\, \mathscr{D}')=\mathscr{O}(\tilde{\Omega}\sharp\Omega;\, E;\, b')$. If $f\in\mathscr{O}(\tilde{\Omega}\sharp\Omega;\, E;\, \mathscr{D}')$, then for any $x\in\Omega$ there exist a neighborhood $\tilde{\omega}$ of x in $\tilde{\Omega}$, and $p\in\boldsymbol{N}^n$ and $g\in\mathscr{O}(\tilde{\omega}\sharp\omega;\, E;\, C^0)$ such that*

$$D_z^p g=f,$$

where $\omega=\tilde{\omega}\cap\boldsymbol{R}^n$.

Proof. See L. Schwartz [35]. Q.E.D.

Theorem 7.8. *Let $f\in\mathscr{O}(\tilde{\Omega}\sharp\Omega;\, E;\, b')$. Then we have $b(f)\in\mathscr{D}'(\Omega;\, E)$ and $b(f)=b'(f)$, that is, we have, for any $\phi\in\mathscr{D}(\Omega)$,*

$$b(f)(\phi)=\lim_{y_\sigma\to+0}\sum_\sigma \operatorname{sgn}(\sigma)\int_\Omega f(x+i\sigma y_\sigma)\phi(x)dx$$

where $\sigma=(\sigma_1,\cdots,\sigma_n)$, $\sigma_i=\pm1$, and $\operatorname{sgn}(\sigma)=\sigma_1\cdots\sigma_n$.

Proof. It suffices to prove that all point x of Ω has a neighborhood ω such that

$$b(f|\tilde{\omega}\sharp\omega)\in\mathscr{D}'(\omega;\, E),$$
$$b(f|\tilde{\omega}\sharp\omega)=b'(f|\tilde{\omega}\sharp\omega)$$

where $\tilde{\omega}$ is a neighborhood of x in $\tilde{\Omega}$ such that $\tilde{\omega}\cap\boldsymbol{R}^n=\omega$.

Let then $g\in\mathscr{O}(\tilde{\omega}\sharp\omega;\, E;\, C^0)$ and $p\in\boldsymbol{N}^n$, $D_z^p g=f$. Such g exists for a sufficiently small neighborhood $\tilde{\omega}$ by virtue of Theorem 7.7. By virtue of Proposition 7.4 we have

$$b(g)=b'(g).$$

The theorem follows from this, for we have

$$b(f)=b(D_z^p g)=D_x^p b(g) \quad\text{and}\quad b'(D_z^p g)=D_x^p b'(g)$$

since

$$\int_\omega D_z^p g(x+iy)\phi(x)dx=\int_\omega D_x^p g(x+iy)\phi(x)dx$$
$$=(-1)^{|p|}\int_\omega g(x+iy)D_x^p\phi(x)dx$$

holds. Q.E.D.

Appendix. A characterization of vector valued analytic functions

In this appendix we give a characterization of vector valued analytic functions by non quasi-analytic classes of vector valued functions.

Let E be a Fréchet space. We denote by $\alpha=(\alpha_1,\cdots,\alpha_n)$ an n-tuple of non-negative integers and $|\alpha|=\alpha_1+\cdots+\alpha_n$. We also denote $x=(x_1,\cdots,x_n)\in \boldsymbol{R}^n$ and

$$D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

We denote by $\mathscr{M}$ the set of sequences $\{M_p\}_{p=0}^\infty$ of positive numbers satisfying the following conditions:

(1) logarithmic convexity,

$$M_p^2\leqq M_{p-1}M_{p+1},\quad p=1,2,\cdots;$$

(2) there are constants A and h such that

$$M_pM_q\leqq Ah^{p+q}M_{p+q}\qquad \text{for all}\quad p \text{ and } q,$$
$$M_{p+1}\leqq Ah^pM_p;$$

(3) non quasi-analyticity,

$$\sum_{p=1}^\infty \frac{M_{p-1}}{M_p}<\infty.$$

Let Ω be an open subset of $\boldsymbol{R}^n$. We denote by $\mathscr{E}^{\{M_p\}}(\Omega;E)$ the set of indefinitely differentiable functions defined in Ω valued in E such that, on every compact subset $K\subset\Omega$, there exist constants A and h satisfying

$$\sup_{x\in K}\|D^\alpha f(x)\|\leqq Ah^{|\alpha|}M_{|\alpha|}\qquad \text{for all}\quad \alpha,$$

where $\|\ \|$ denotes continuous seminorms defining the topology of E.

We denote by $\mathscr{E}^{\{M_p\}}(\Omega)$ the space $\mathscr{E}^{\{M_p\}}(\Omega;\boldsymbol{C})$. Then the space $\mathscr{E}^{\{M_p\}}(\Omega;E)$ is stable by multiplication by a function of $\mathscr{E}^{\{M_p\}}(\Omega)$ and by differentiation and it contains functions with arbitrarily small support.

We have the following theorem characterizing E-valued analytic functions by non quasi-analytic classes of E-valued functions. For $E=\boldsymbol{C}$, this is already known (cf. Chou [41]).

Theorem.

$$\bigcap_{\{M_p\}\in\mathscr{M}}\mathscr{E}^{\{M_p\}}(\Omega;E)=\mathscr{A}(\Omega;E)$$

$\equiv$ *the set of E-valued analytic functions.*

Proof. $\mathscr{A}(\Omega; E) \subset \cap \mathscr{E}^{\{M_p\}}(\Omega; E)$ is evident. Hence let $f \not\in \mathscr{A}(\Omega; E)$. Then there exist a compact subset K of Ω and a continuous seminorm $\| \ \|$ such that

$$\overline{\lim_{|\alpha|}} \left(\sup_{x \in K} \frac{\| D^\alpha f(x) \|}{|\alpha|!} \right)^{1/|\alpha|} = +\infty .$$

Hence we can extract a subsequence $\{\alpha^{(m)}\}$ ($|\alpha^{(m)}|$ strictly increasing) such that

$$\sup_{x \in K} \left\| \frac{D^{\alpha^{(m)}} f(x)}{|\alpha^{(m)}|!} \right\|^{1/|\alpha^{(m)}|} \geqq m^3 .$$

We put $M'_p = (m^2 |\alpha^{(m)}|)^p$ for $|\alpha^{(m-1)}| < p \leqq |\alpha^{(m)}|$ and $M'_0 = 1$. Then we can see that there exists $M_p \in \mathscr{M}$ with $M_p \leqq M'_p$. From here we have $f \not\in \mathscr{E}^{\{M_p\}}(\Omega; E)$.

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References

[1] Bishop, E.: Analytic functions with values in a Fréchet space. Pacific J. Math., **12** (1962), 1177–1192.

[2] Bredon, G. E.: Sheaf Theory. McGraw-Hill, New Yrok/St. Louis/San Francisco/Toronto/London/Sydney, 1967.

[3] Bungart, L.: Holomorphic functions with values in locally convex spaces and applications to integral formulas. Trans. Amer. Math. Soc., **111** (1964), 317–344.

[4] Dieudonné, J. and L. Schwartz: La dualité dans les espaces (F) et (LF). Ann. Inst. Fourier, **1** (1949), 61–101.

[5] Godement, R.: Topologie algébrique et théorie des faisceaux. Hermann, Paris, 1964.

[6] Grauert, H.: On Levi's problem and the imbedding of real analytic manifolds. Ann. of Math., **68** (1958), 460–472.

[7] Grothendieck, A.: Sur certains espaces de fonctions holomorphes I, II. Jour. f. reine u. angew. Math., **192** (1953), 35–60, 77–95.

[8] ———: Opérations algébriques sur les distributions à valuers vectorielles — Théorème de Künnteh. Séminaire Schwartz, 1953/54. n°24.

[9] ———: Sur les espaces (F) et (DF). Summa Brasil. Math. **3** (1954), 57–123.

[10] ———: Produits tensoriels topologiques et espaces nucléaires. Mem. Amer. Math. Soc., **16** (1955).

[11] ———: Espaces Vectoriels Topologiques. São Paulo, 1964.

[12] Harvey, F.: Hyperfunctions and Linear Partial Differential Equations. Stanford University, Ph. D., 1966.

[13] Hörmander, L.: An Introduction to Complex Analysis in Several Variables. North-Holland/American Elsevier, Amsterdam/London/New Yrok, 1973.

[14] Ion, P. D. F. and T. Kawai: Theory of vector valued hyperfunctions. Publ. RIMS Kyoto Univ., **11** (1975), 1–19.

[15] Ito, Y. and S. Nagamachi: On the Theory of Vector Valued Fourier Hyperfunctions. J. Math. Tokushima Univ., **9** (1975), 1–33.

[16] ——— and ———: Theory of H-valued Fourier Hyperfunctions. Proc. Japan Acad., **51** (1975), 558–561.

[17] Kawai, T.: On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients. J. Fac. Sci. Univ. Tokyo, Sect. IA, **17** (1970), 467–517.

[18] Komatsu, H.: Projective and injective limits of weakly compact sequences of locally convex spaces. J. Math. Soc. Japan, **19** (1967), 366–383.

[19] ———: Theory of hyperfunctions and partial differential operators with constant coefficients. Lecture Note of Univ. of Tokyo, **22**, 1968 (in Japanese).

[20] Lelong, P.: Fonctionnelles Analytiques et Fonctions Entières (n Variables). Les Presses de l'Université de Montréal, Montréal, 1968.

[21] Malgrange, B.: Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. Ann. Inst. Fourier, **6** (1955/1956), 271–355.

[22] ———: Faisceaux sur les variétés analytiques réelles. Bull. Soc. Math. France, **85** (1957), 231–237.

[23] Martineau, A.: Les hyperfonctions de M. Sato. Séminaire Bourbaki, **13** (1960/1961), n°214

[24] ———, Sur les Fonctionnelles Analytiques et la Transformation de Fourier-Borel. Journ. Anal. Math. Jérusalem, **11** (1963), 1–164.

[25] ———: Distributions et valeurs au bord des fonctions holomorphes. Proc. Intern. Summer Course on the Theory of Distributions, Lisbon, 1964, pp. 195–326.

[26] Morimoto, M.: Introduction to Sato's Hyperfunctions. Kyoritsu Shuppan, Tokyo, 1976 (in Japanese).

[27] Nagamachi, S. and N. Mugibayashi: Hyperfunction Quantum Field Theory. Commun. math. Phys., **46** (1976), 119–134.

[28] ——— and ———: Hyperfunction Quantum Fields Theory II, Eculidean Green's Functions. Commun. math. Phys., **49** (1976), 257–275.

[29] ——— and ———: Quantum Fields Theory in Terms of Fourier Hyperfunctions. Publ. RIMS Kyoto Univ., **12**, Supplement (1977) 309–341.

[30] Roumieu, C.: Sur quelques extensions de la notion de distribution. Ann. scient. Éc. Norm. Sup., 3e série, **77** (1960), 41–121.

[31] Sato, M.: Theory of hyperfunctions I & II. J. Fac. Sci., Univ. Tokyo, Sect. I, **8** (1959/60), 139–193, 387–436.

[32] Schapira, P.: Sur les ultra-distributions. Ann. scient. Éc. Norm Sup., 4e série, **1** (1968), 395–415.

[33] ———: Théorie des Hyperfonctions. Springer-Verlag, Berlin/Heidelberg/New York, 1970.

[34] Schwartz, L.: Espaces de Fonctions Différentiables à Valeurs Vectorielles. J. d'Anal. Math., **4** (1952), 88–148.

[35] ———: Théorie des Distributions à Valeurs Vectorielles I. Ann. Inst. Fourier, Grenoble, **7** (1957), 1–141.

[36] ———: Théorie des Distributions à Valeurs Vectorielles II. Ann. Inst. Fourier, Grenoble, **8** (1958), 1–209.

[37] ———: Théorie des distributions. Hermann, Paris, 1973.

[38] Serre, J. -P.: Un théorème de dualité. Comm. Math. Hel., **29** (1955), 9–26.

[39] Swan, R. G.: The Theory of Sheaves. Chicago Lectures in Mathematics, The University of Chicago Press, Chicago and London, 1964.

[40] Trèves, F.: Topological Vector Spaces, Distributions and Kernels. Academic Press, New York/London, 1967.

[41] Chou, C. -C.: Problème de régularité universelle. C. R. Acad. Sc., Paris, **260** (1965), 4397–4399.

[42] Komatsu, H.: Ultradistributions, I. Structure theorems and a characterization. J. Fac. Sci., Univ. Tokyo, Sec. I. A. **20** (1973), 25–105.

[43] Lions, J. L. et E. Magenes: Problèmes aux Limites Non Homogènes et Applications, Vol. **3**. Dunod, Paris, 1970.

[44] Martineau, A.: Sur la topologie des espaces de fonctions holomorphes. Math. Annalen, **163** (1966), 62–88.

[45] Ito, Y.: On Analytic Linear Mappings and Vector Valued Hyperfunctions. Surikaiseki Kenkyusho Kokyuroku, R. I. M. S., Kyoto Univ., **355** (1979), 98–110 (in Japanese).