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On the Abstract Cauchy Problems in the Sense of Fourier Hyperfunctions

By

Yoshifumi ITO

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In this paper we will study the Fourier hyperfunction solution of the abstract Cauchy problem

$$\begin{cases} du(t)/dt = Au(t), \\ u(0) = a, \end{cases}$$

where A is a closed linear operator in a complex Banach space X and $a \in X$.

The abstract Cauchy problem has been studied by many authors ([1], [2], [7], [8], [9] and the others quoted there). By modifying Ōuchi's method, our method of using vector valued Fourier hyperfunctions simplifies the necessary and sufficient conditions for the well-posedness of the abstract Cauchy problem in the generalized sense.

As for the notions of Fourier hyperfunctions and vector valued Fourier hyperfunctions we refer [3], [4], [5], [6] and [10].

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§1. Fourier hyperfunctions with values in a Banach space

In the later sections we shall use Fourier hyperfunctions of one variable with values in a Banach space. So that we recall their notion and properties following Junker [3], [4]. Let E be a complex Banach space. Let $\tilde{\boldsymbol{C}}$ be the space $\boldsymbol{D} \times \sqrt{-1}\boldsymbol{R}$ with the product topology where $\boldsymbol{D} = [-\infty, \infty]$ is the radial compactification of the space $\boldsymbol{R}$ in the sense of Kawai [5], [6]. Let Ω be an open set in $\tilde{\boldsymbol{C}}$. Consider the space $\tilde{\mathscr{O}}(\Omega; E)$ of all E-valued slowly increasing holomorphic functions defined on Ω. Let I be an open set in $\boldsymbol{D}$. We can now characterize an E-valued Fourier hyperfunction on I as an element of the quotient space

$$\mathscr{R}(I; E) = \tilde{\mathscr{O}}(D \backslash I; E) / \tilde{\mathscr{O}}(D; E),$$

where D is an open neighborhood in $\tilde{\boldsymbol{C}}$ of I containing I as a closed set. If $f \in \mathscr{R}(I; E)$ is defined by $\tilde{f}(z) \in \tilde{\mathcal{O}}(D \backslash I; E)$, it is denoted by $f=[\tilde{f}(z)]$ and $\tilde{f}(z)$ is called a defining function of f.

Further results for vector valued Fourier hyperfunctions can be found in the articles cited above.

§2. Existence and uniqueness of Fourier hyperfunction solutions

Let $L(E, F)$ be the totality of bounded linear operators from E into F, where E and F are complex Banach spaces with norms $\|\cdot\|_E$ and $\|\cdot\|_F$ respectively. Then $L(E, F)$ is a complex Banach space with the operator norm denoted by $\|\cdot\|=\|\cdot\|_{E,F}$. The space $L(E, E)$ is written as $L(E)$ for short.

Let X be a complex Banach space. For any linear operator A in X, we denote its domain by $D(A)$. If A is closed, $D(A)$ becomes a complex Banach space with the graph norm, which we denote by $[D(A)]$.

The resolvent set $\rho(A)$ of A is defined as

$$\rho(A)=\{\lambda \in \boldsymbol{C};\ (\lambda-A)^{-1} \in L(X)\}.$$

Now we define the well-posedness of the abstract Cauchy problem in the sense of Fourier hyperfunctions.

Definition 2.1. *Let X be a complex Banach space and A a closed linar operator in X. Then A is said to be well-posed for the abstract Cauchy problem at $t=0$ in the sense of Fourier hyperfunctions (well-posed for short), if there exists $T \in \mathscr{R}(\boldsymbol{D}; L(X, [D(A)]))$ satisfying the following conditions:*

(α) support of $T \subset [0, \infty]$

(β) $(\delta'(t) \otimes I-\delta(t) \otimes A) * T=\delta(t) \otimes I_X$, $T *(\delta'(t) \otimes I-\delta(t) \otimes A)=\delta(t) \otimes I_{[D(A)]}$,

where I is the identity mapping of $[D(A)]$ into X, and I_X and $I_{[D(A)]}$ are the identities on X and on $[D(A)]$ respectively, and $*$ means convolution and $\otimes$ denotes tensor product.

We shall call T a Fourier hyperfunction fundamental solution.

From Definition 2.1 we deduce the following

Proposition 2.2. *If a closed linear operator A is well-posed, then the fundamental solution T is unique in $\mathscr{R}(\boldsymbol{D}; L(X, [D(A)]))$.*

Proof. This result easily follows from the facts that the support of T is contained in $[0, \infty]$ and that T is a two-sided fundamental solution. Q.E.D.

Now we give a criterion for the existence of the Fourier hyperfunction fundamental solution of the abstract Cauchy problem.

Theorem 2.3. *Let X and A be as in Definition 2.1. Then A is well-posed if and only if the following conditions hold:*

(i) *For any λ such as $Re\lambda>0$, $(\lambda-A)^{-1}$ exists and belongs to $L(X)$.*

(ii) *For any $\varepsilon>0$ and any $\delta>0$ there exists $C_{\varepsilon,\delta}>0$ such that*

$$\|(\lambda-A)^{-1}\|\leqq C_{\varepsilon,\delta}\exp(\varepsilon|\lambda|)$$

holds for $\lambda\in\sum_\delta$, where $\sum_\delta$ is the set

$$\textstyle\sum_\delta=\{\lambda\in\boldsymbol{C};\ \mathrm{Re}\,\lambda\geqq\delta\}.$$

In order to prove the Theorem 2.3, we need the following

Lemma. *Let E be a complex Banach space and $f=[\tilde{f}(z)]$ an E-valued Fourier hyperfunction with support in $[0,+\infty]$. That is,*

$$f\in\Gamma_{[0,+\infty]}(\boldsymbol{D},{}^{E}\mathscr{R}).$$

Define the Laplace transform of f by

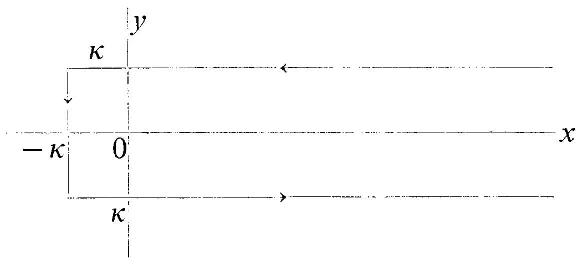
$$\langle f,\exp(-\lambda t)\rangle=-\int_\gamma \tilde{f}(z)\exp(-\lambda z)dz,$$

where γ is a curve encircling the interval $[0,+\infty)$ counter clockwise and $0<a\leqq|\mathrm{Im}\,z|\leqq b$ for $z\in\gamma$ and for any $a>0$ and $b>0$. Then, for any $\varepsilon>0$ and $\delta>0$ there exists $C_{\varepsilon,\delta}>0$ such that

$$\|\langle f,\exp(-\lambda t)\rangle\|_E\leqq C_{\varepsilon,\delta}\exp(\varepsilon|\lambda|)\qquad \textit{for}\quad \lambda\in\textstyle\sum_\delta.$$

Proof. Since $\langle f,\exp(-\lambda t)\rangle$ does not depend on γ by virtue of Cauchy's theorem, we can take γ in the following way:

$$\gamma:\begin{cases}-\kappa\leqq x<\infty,\ y=\kappa>0;\\ x=-\kappa,\ -\kappa\leqq y\leqq\kappa;\\ -\kappa\leqq x<\infty,\ y=-\kappa.\end{cases}$$



Then the estimate

$$\|\tilde{f}(z)\|_E\leqq C_{\kappa,\varepsilon}e^{\varepsilon|z|},\ z\in\gamma$$

holds. If we take $\kappa=\varepsilon/2$, then

$$e^{\varepsilon|z|}|e^{-\lambda z}|=e^{\varepsilon|z|}e^{-\mu x+\nu y}\leqq C_\varepsilon e^{\varepsilon|\lambda|}e^{\varepsilon|x|-\mu x},$$

where $z=x+iy$ and $\lambda=\mu+i\nu$. For any $\delta>0$, if we choose $\varepsilon>0$ so that $\varepsilon<\delta$, then for $\mu=\mathrm{Re}(\lambda)\geqq\delta$ the function $e^{\varepsilon|x|-\mu x}$ is integrable on the curve γ. Hence

$$\|\langle f, \exp(-\lambda t)\rangle\|_E\leqq C_{\varepsilon,\delta}\exp(\varepsilon|\lambda|)$$

for any $\lambda\in\sum_\delta$.

Proof of Theorem 2.3. Necessity. Assume that A is well-posed and let T be the fundamental solution in $\mathscr{R}(\boldsymbol{D}; L(X, [D(A)]))$. Let $\tilde{T}(z)$ be a defining function of T. Then we have

$$\tilde{T}(z)\in\tilde{\mathcal{O}}(\tilde{\boldsymbol{C}}\backslash[0, +\infty]; L(X, [D(A)])).$$

Consider the Laplace transform of T:

$$\langle T, \exp(-\lambda t)\rangle x=-\int_\gamma \tilde{T}(z)\exp(-\lambda z)xdz \qquad \text{for} \quad x\in X,$$

where γ is a curve encircling the interval $[0, \infty)$ in such a way as in the Lemma. Then we have

$$\begin{aligned}
&(\lambda-A)\langle T, \exp(-\lambda t)\rangle x=\langle T, \lambda\exp(-\lambda t)\rangle x-A\langle T, \exp(-\lambda t)\rangle x\\
&\quad=\langle T, -d/dt(\exp(-\lambda t))\rangle x-A\langle T, \exp(-\lambda t)\rangle x\\
&\quad=\langle(\delta'(t)\otimes I-\delta(t)\otimes A)*T, \exp(-\lambda t)\rangle x=\langle\delta(t)\otimes I_X, \exp(-\lambda t)\rangle x=I_Xx=x,\ x\in X\\
&\langle T, \exp(-\lambda t)\rangle(\lambda-A)x=\langle T*(\delta'(t)\otimes I-\delta(t)\otimes A), \exp(-\lambda t)\rangle x\\
&\quad=\langle\delta(t)\otimes I_{[D(A)]}, \exp(-\lambda t)\rangle x=I_{[D(A)]}x=x,\ x\in[D(A)]
\end{aligned}$$

So,

$$(\lambda-A)\langle T, \exp(-\lambda t)\rangle=I_X,\ \langle T, \exp(-\lambda t)\rangle(\lambda-A)=I_{[D(A)]}.$$

Thus we get

$$\langle T, \exp(-\lambda t)\rangle=(\lambda-A)^{-1} \qquad \text{for} \quad \mathrm{Re}\,\lambda>0.$$

Hence the necessity follows from the above Lemma.

Sufficiency. First fix a real $\omega\in\rho(A)$, $\omega>0$ and put $\lambda=\mu+i\nu$. We take two half lines:

$$\Gamma_\omega^+: \mu=\omega,\ \nu\geqq0;\ \Gamma_\omega^-: \mu=\omega,\ \nu\leqq0.$$

We put

$$\tilde{T}^{\omega}_{+}(z)=(1/2\pi i)\int_{\Gamma^{+}_{\omega}} e^{\lambda z}(\lambda-A)^{-1}d\lambda,\ \tilde{T}^{\omega}_{-}(z)=(1/2\pi i)\int_{\Gamma^{-}_{\omega}} e^{\lambda z}(\lambda-A)^{-1}d\lambda.$$

Then we have the estimate

$$\|e^{\lambda z}(\lambda-A)^{-1}\|\leqq C_{\varepsilon,\delta}\exp(\mu x-\nu y+\varepsilon(|\mu|+|\nu|))\quad (z=x+iy,\ \lambda=\mu+i\nu)$$

for any $\varepsilon>0$, and $\lambda\in\sum_{\delta}$ $(0<\delta\leqq\omega)$. For any $y>2\varepsilon$ the integrand of $\tilde{T}^{\omega}_{+}(z)$ decreases more rapidly than the multiple of $e^{-\varepsilon\nu}$ as ν tends to infinity. Hence $\tilde{T}^{\omega}_{+}(z)$ is holomorphic in $\operatorname{Im} z>0$ and the estimate

$$\|\tilde{T}^{\omega}_{+}(z)\|\leqq C_{\omega,\varepsilon}\exp(\omega|z|),\ \operatorname{Im} z>a>0,$$

holds. Analogously we can show that $\tilde{T}^{\omega}_{-}(z)$ is holomorphic in $\operatorname{Im} z<0$ and

$$\|\tilde{T}^{\omega}_{-}(z)\|\leqq C_{\omega,\varepsilon}\exp(\omega|z|),\ 0<a<-\operatorname{Im} z.$$

If $\operatorname{Re} z<0$, by Cauchy's Theorem we can deform the paths Γ^{+}_{ω} and Γ^{-}_{ω} into the same path γ: $\mu\geqq\omega$, $\gamma=0$ without changing the values of $\tilde{T}^{\omega}_{+}(z)$ and $\tilde{T}^{\omega}_{-}(z)$. Hence $\tilde{T}^{\omega}_{+}(z)$ and $\tilde{T}^{\omega}_{-}(z)$ are holomorphic in the half plane $\operatorname{Re} z<0$ and coincide with each other there and

$$\|\tilde{T}^{\omega}_{+}(z)\|=\|\tilde{T}^{\omega}_{-}(z)\|\leqq C_{\omega,\varepsilon}\exp(\omega|z|),\ \operatorname{Re} z\leqq -b<0.$$

Thus we have an $L(X, [D(A)])$-valued holomorphic function $\tilde{T}^{\omega}(z)$ defined on $\tilde{\boldsymbol{C}}\backslash[0,+\infty]$ which is the holomorphic extension of $\tilde{T}^{\omega}_{\pm}(z)$. We can easily see that $\tilde{T}^{\omega}(z)$ and $\tilde{T}^{\omega}_{\pm}(z)$ do not depend on $\omega>0$ and define $\tilde{T}(z)\in\tilde{\mathcal{O}}(\tilde{\boldsymbol{C}}\backslash[0,+\infty];\ L(X, [D(A)]))$ and $\tilde{T}_{\pm}(z)\in\tilde{\mathcal{O}}(\tilde{\boldsymbol{C}}_{\pm};\ L(X, [D(A)]))$. Then $\tilde{T}(z)$ is the holomorphic extension of $\tilde{T}_{\pm}(z)$. Differentiate $\tilde{T}^{\omega}_{\pm}(z)$. Then we have

$$d\tilde{T}^{\omega}_{\pm}/dz=A\tilde{T}^{\omega}_{\pm}(z)+(-1/2\pi i)(e^{\omega z}/z)I_X\qquad\text{on}\quad X,$$

$$d\tilde{T}^{\omega}_{\pm}/dz=\tilde{T}^{\omega}_{\pm}(z)A+(-1/2\pi i)(e^{\omega z}/z)I_{[D(A)]}\qquad\text{on}\quad [D(A)].$$

Since

$$\delta(t)\otimes I_X=[(-1/2\pi iz)I_X]=[(-e^{\omega z}/2\pi iz)I_X],$$

$$\delta(t)\otimes I_{[D(A)]}=[(-1/2\pi iz)I_{[D(A)]}]=[(-e^{\omega z}/2\pi iz)I_{[D(A)]}],$$

we see that $T=[\tilde{T}(z)]=[\tilde{T}^{\omega}_{\pm}(z)]$ is a Fourier hyperfunction fundamental solution.

Q. E. D.

§ 3. Fourier hyperfunction fundamental solution of exponential increase

Definition 3.1. *Let X be a complex Banach space and A a closed linear operator in X.*

Assume that A is well-posed. Then we say that the Fourier hyperfunction *fundamental solution* T *is of exponential increase if there exists a real number* ξ_0 *such that* $\exp(-\xi t)T \in \mathscr{R}(\boldsymbol{D}, L(X, [D(A)]))$ *for* $\xi > \xi_0$. *We write it EFHFS of type* ξ_0 *for short.*

Then we have a criterion for the existence of the EFHFS of type ξ_0 of the abstract Cauchy problem.

Theorem 3.2. *Let* X *and* A *be as in Definition* 3.1. *Then* A *has an EFHFS of type* ξ_0 *if and only if the following conditions are satisfied*:

(i) *For any* λ *such as* $\operatorname{Re}\lambda > \xi_0$, $(\lambda - A)^{-1}$ *exists and belongs to* $L(X)$.

(ii) *For any* $\varepsilon > 0$ *and any* $\delta > \xi_0$ *there exists* $C_{\varepsilon,\delta} > 0$ *such that*

$$\|(\lambda - A)^{-1}\| \leqq C_{\varepsilon,\delta} \exp(\varepsilon|\lambda - \xi_0|)$$

holds for $\lambda \in \sum_\delta$, *where we put*

$$\textstyle\sum_\delta = \{\lambda \in \boldsymbol{C};\ \operatorname{Re}\lambda \geqq \delta\}.$$

Proof. Necessity. Let T be the EFHFS of type ξ_0 and $\tilde{T}(z)$ its defining function. Then

$$e^{-\xi z}\tilde{T}(z) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}} \backslash [0, +\infty]; L(X, [D(A)])) \qquad \text{for} \quad \xi > \xi_0.$$

We have only to consider the Laplace transform of T:

$$\langle T, \exp(-\lambda t)\rangle x = \langle \exp(-\xi t)T, \exp(-(\lambda - \xi)t)\rangle x$$

$$= -\int_\gamma (\exp(-\xi z)\tilde{T}(z)) \exp(-(\lambda - \xi)z) x dz \qquad \text{for} \quad x \in X$$

where γ is a curve such as in the proof of Theorem 2.3. Here $\operatorname{Re}(\lambda - \xi) > 0$. Since $\xi > \xi_0$ is arbitrary, we have

$$\langle T, \exp(-\lambda t)\rangle = (\lambda - A)^{-1} \qquad \text{for} \quad \operatorname{Re}\lambda > \xi_0$$

in the same way as in the proof of Theorem 2.3. The condition (ii) follows from the same arguments for the Laplace transform of T as in Lemma.

Sufficiency. First fix a real $\omega \in \rho(A)$, $\omega > \xi_0$. We take two half lines Γ_ω^+ and Γ_ω^- as before and define $\tilde{T}_+^\omega$ and $\tilde{T}_-^\omega$ in the same way as before. Then $\tilde{T}_\pm^\omega$ can be extended to an $L(X, [D(A)])$-valued holomorphic function $\tilde{T}(z)$ in $\boldsymbol{C} \backslash [0, +\infty)$ which does not depend on ω. Then we have

$$e^{-\xi z}\tilde{T}(z) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}} \backslash [0, +\infty]; L(X, [D(A)])) \qquad \text{for} \quad \xi > \xi_0.$$

Now $T = [\tilde{T}(z)]$ is a desired EFHFS. Q.E.D.

Department of Mathematics

College of General Education
Tokushima University

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