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Fourier Hyperfunction Semi-groups

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Introduction

The notion of generalized semi-groups of operators in a Banach space was first introduced by Lions [7] in the class of the (exponential) distribution semi-groups of operators in a Banach space.

Since then many authors have studied the generalized semi-groups of operators in a topological vector space in the several kinds of classes. For this we refer [1], [9] and many others quoted there.

In this paper we will introduce the notion of the (exponential) Fourier hyperfunction semi-groups of operators in a Banach space and characterize its infinitesimal generator. This is a generalization of the Hille-Yosida Theorem and the Feller-Miyadera-Phillips Theorem and the Lions Theorem.

Applying the theory of Fourier hyperfunction semi-groups, we also characterize the infinitesimal generator of an (exponential) Fourier hyperfunction group. This is a generalization of Stone's Theorem.

As for the notions of Fourier hyperfunctions and vector valued Fourier hyperfunctions, we refer [3], [4], [5], [6], [8].

§1. Fourier hyperfunctions with values in a Banach space

First we recall the concept of Fourier hyperfunctions valued in a Banach space E following Junker [3], [4]. Let $\tilde{\boldsymbol{C}}$ be the space $\boldsymbol{D}\times\sqrt{-1}\boldsymbol{R}$ with the product topology where $\boldsymbol{D}=[-\infty, \infty]$ is the radial compactification of the space $\boldsymbol{R}$ in the sense of Kawai [5], [6]. Let Ω be an open set in $\tilde{\boldsymbol{C}}$. The space $\underset{\sim}{\mathcal{O}}(\Omega)$ of rapidly decreasing holomorphic functions on Ω consists of all holomorphic functions on $\Omega\cap\boldsymbol{C}$ such that, for any compact set K in Ω, there exists some positive constant δ so that the estimate

$$|f(z)|\leqq C\exp(-\delta|z|) \qquad z\in K\cap\boldsymbol{C}$$

holds. $\underset{\sim}{\mathcal{O}}(\Omega)$ is a nuclear FS-space. For a compact set K in $\tilde{\boldsymbol{C}}$, we define the space $\underset{\sim}{\mathcal{O}}(K)$ as the inductive limit of $\underset{\sim}{\mathcal{O}}(\Omega)$ for Ω containing K. Then $\underset{\sim}{\mathcal{O}}(K)$ becomes a nuclear

DFS-space. We put $\mathscr{P}_* = \mathscr{A}(\boldsymbol{D})$ and $\mathscr{P}_0 = \mathscr{A}(\boldsymbol{D}_0)$ where $\boldsymbol{D}_0 = [0, \infty]$. Here $\mathscr{A}$ denotes the restriction to $\boldsymbol{D}$ of the sheaf $\underset{\sim}{\mathscr{O}}$ associated with the presheaf $\{\underset{\sim}{\mathscr{O}}(\Omega); \Omega \subset \tilde{\boldsymbol{C}}\}$.

We denote by $\mathscr{P}'_*(E) = L(\mathscr{P}_*, E)$ the space of continuous linear mappings of $\mathscr{P}_*$ into E with the topology of uniform convergence on each bounded set in $\mathscr{P}_*$, whose elements are said to be Fourier hyperfunctions valued in E or E-valued Fourier hyperfunctions. We define $\mathscr{P}'_0(E) = L(\mathscr{P}_0, E)$ analogously. $\mathscr{P}'_0(E)$ is the subspace of $\mathscr{P}'_*(E)$ formed by E-valued Fourier hyperfunctions with support in $\boldsymbol{D}_0$.

Further results for vector valued Fourier hyperfunctions can be found in the articles cited above.

§ 2. Fourier hyperfunction semi-groups and exponential Fourier hyperfunction semi-groups

In this section we introduce the notions of Fourier hyperfunction semi-groups and exponential Fourier hyperfunction semi-groups.

Let E, F be two complex Banach spaces. Then $L(E, F)$ denotes the space of continuous linear mappings of E into F which is a Banach space with the operator norm. We put $L(E) = L(E, E)$.

We consider an $L(E)$-valued Fourier hyperfunction G with support in $\boldsymbol{D}_0$, namely $G \in \mathscr{P}'_0(L(E))$. For $x \in E$, we define $Gx \in \mathscr{P}'_0(E)$ by the formula

$$Gx(\phi) = G(\phi)x, \ \phi \in \mathscr{P}_0,$$

where $G(\phi)x$ is the image of $x \in E$ by the continuous linear mapping $G(\phi)$ of E into itself.

Definition 2.1. *We call a Fourier hyperfunction semi-group (in E) (FHSG for short) an $L(E)$-valued Fourier hyperfunction G with the following properties*:

(1) $G \in \mathscr{P}'_0(L(E))$.

(2) $G(\phi * \psi) = G(\phi)G(\psi)$ *for all ϕ and ψ in $\mathscr{P}_0$.*

(3) *if $\phi \in \mathscr{P}_0$ and $x \in E$, and if $y = G(\phi)x$, the Fourier hyperfunction Gy is almost everywhere equal to a function $u(t)$ which is continuous for $t \geqq 0$ and $u(0) = y$, and $u(0) = 0$ for $t < 0$ and satisfies the estimate*

$$\|u(t)\| \leqq C \exp(\varepsilon t) \qquad \text{for} \quad t \geqq 0 \quad \text{and for any} \quad \varepsilon > 0,$$

where $\|\cdot\|$ denotes the norm in E.

(4) *the set of elements $G(\phi)x$, where ϕ runs through $\mathscr{P}_0$ and x runs through E, is dense in E.*

(5) *if a given $x \in E$ satisfies $G(\phi)x = 0$ for all $\phi \in \mathscr{P}_0$, then $x = 0$.*

Definition 2.2. *We call G an exponential Fourier hyperfunction semi-group* *(in E) (EFHSG for short) if there exists* $\xi_0 \geqq 0$ *such that, for any* $\xi > \xi_0$, $\exp(-\xi t)G$ *becomes FHSG.*

We define the space $\mathscr{P}_{\xi_0}$ by the formula

$$\mathscr{P}_{\xi_0} = \bigcup_{\xi > \xi_0} \exp(-\xi t)\mathscr{P}_0$$

whose topology is defined as the finest locally convex topology such that the multiplication mapping $\exp(-\xi t)$ of $\mathscr{P}_0$ into $\mathscr{P}_{\xi_0}$ is continuous for any $\xi > \xi_0$. Then we define the space $\mathscr{P}'_{\xi_0}(E)$ by the formula

$$\mathscr{P}'_{\xi_0}(E) = \bigcap_{\xi > \xi_0} \exp(\xi t)\mathscr{P}'_0(E).$$

$G \in \mathscr{P}'_{\xi_0}(E)$ is a continuous linear mapping from $\mathscr{P}_{\xi_0}$ into E.

Then an EFHSG G is an element of $\mathscr{P}'_{\xi_0}(L(E))$ for some $\xi_0 \geqq 0$ such that exp$\cdot$ $(-\xi t)G$ is an FHSG for any $\xi > \xi_0$.

For $G \in \mathscr{P}'_{\xi_0}(L(E))$ and $\phi \in \mathscr{P}_{\xi_0}$, we define

$$G(\phi) = (\exp(-\xi t)G)(\exp(\xi t)\phi)$$

for some $\xi > \xi_0$ such that $\exp(\xi t)\phi \in \mathscr{P}_0$.

§3. Infinitesimal generator of Fourier hyperfunction semi-groups

In this section we will introduce the notion of the infinitesimal generator of Fourier hyperfunction semi-groups.

In order to do this we need to define the operator $G(S)$ for an FHSG G and a hyperfunction S with compact support in $[0, \infty)$.

We will go as follows. We will call a regularizing sequence a sequence ρ_n of $\mathscr{P}_0$ such that ρ_n converges to the Dirac measure δ at the origin in the space of measures with compact support endowed with the weak topology. For example we have a regularizing sequence

$$\rho_n(x) = (n/\sqrt{\pi}) \exp(-n^2 x^2).$$

In fact $\rho_n(x) \in \mathscr{P}_* \subset \mathscr{P}_0$ and $\rho_n \to \delta$.

Let S be a hyperfunction with compact support in $[0, \infty)$. Then for any $\rho \in \mathscr{P}_0$ we have $S * \rho \in \mathscr{P}_0$.

Then we consider an element x in E for which there exists a regularizing sequence ρ_n having the following two properties:

(i) $G(\rho_n)x \to x$ when $\rho_n \to \delta$,

(ii) $G(S * \rho_n)x$ converges, say, to y in the space E.

Suppose that there exists another regularizing sequence, say, σ_m with the same properties, i.e.

(i)′ $G(\sigma_m)x \to x$ when $\sigma_m \to \delta$,

(ii)′ $G(S*\sigma_m)x$ converges, say, to z in E.

We will now show that $y=z$. In fact, for every $\phi \in \mathscr{P}_0$, we have

$$G(\phi)G(S*\rho_n)x = G(S*\phi*\rho_n)x,$$

which gives

$$G(\phi)y = G(S*\phi)x$$

tending ρ_n to δ. In the same way we have

$$G(\phi)z = G(S*\phi)x.$$

Hence

$$G(\phi)(y-z)=0 \qquad \text{for all} \quad \phi \text{ in } \mathscr{P}_0.$$

which implies $y-z=0$ by virtue of Definition 2.1, (5). Hence the desired result follows.

This consideration justifies the following

Definition 3.1. *We say that $x \in D(G(S))$ — the domain of $G(S)$ — if there exists a regularizing sequence ρ_n such that $G(\rho_n)x \to x$ and $G(S*\rho_n)x$ converges in E. The limit y of $G(S*\rho_n)x$ is denoted by $G(S)x$. Hence*

$$G(\rho_n)x \longrightarrow x,$$

$$G(S*\rho_n)x \longrightarrow G(S)x.$$

If B is an unbounded operator in E with the domain $D(B)$, we denote by $D(B^2)$ the set of $x \in D(B)$ such that $Bx \in D(B)$, and we put $B(Bx)=B^2x$. We can define $D(B^m)$ and $B^m x$ analogously.

Lemma 3.1. *For all $x \in E$, and for all $\phi \in \mathscr{P}_0$, $G(\phi)x$ belongs to $D(G(S)^m)$ whatever m is, and*

$$G(S)^m G(\phi)x = G(\underbrace{S*S*\cdots*S}_{(m)}*\phi)x$$

holds.

Proof. Let ρ_n be some regularizing sequence. We have

$$G(\rho_n)G(\phi)x = G(\rho_n*\phi)x \longrightarrow G(\phi)x,$$

$$G(S*\rho_n)G(\phi)x = G(S*\phi*\rho_n)x \longrightarrow G(S*\phi)x,$$

so that $G(\phi)x$ is in $D(G(S))$ and

$$G(S)G(\phi)x = G(S*\phi)x.$$

And so on. Q.E.D.

Corollary. *$D(G(S))$ is dense in E.*

Proof. In fact, by virtue of Lemma 3.1, $D(G(S))$ contains the set of elements $G(\phi)x$, $\phi\in\mathscr{P}_0$, $x\in E$. But this latter set is dense in E by virtue of Definition 2.1, (4). Hence $D(G(S))$ is dense in E. Q.E.D.

Lemma 3.2. *If $x\in D(G(S))$, we have*

$$G(S*\phi)x=G(S)G(\phi)x=G(\phi)G(S)x, \qquad \textit{for all} \quad \phi\in\mathscr{P}_0.$$

Proof. Let ρ_n be as

$$G(\rho_n)x \longrightarrow x, \quad G(S*\rho_n)x \longrightarrow G(S)x,$$

and put $y=G(S)x$. Then we have

$$G(\phi)G(S*\rho_n)x \longrightarrow G(\phi)y.$$

Since

$$G(\phi)G(S*\rho_n)x=G(S*\phi*\rho_n)x \longrightarrow G(S*\phi)x,$$

we obtain the lemma. Q.E.D.

Lemma 3.3. *If $x_j\in D(G(S))$, $x_j\to 0$ in E, and $G(S)x_j\to y$ in E, then $y=0$.*

Proof. By virtue of Lemma 3.2, for any $\phi\in\mathscr{P}_0$, we have

$$G(\phi)G(S)x_j=G(S*\phi)x_j.$$

But

$$G(\phi)G(S)x_j \longrightarrow G(\phi)y,$$

and

$$G(S*\phi)x_j \longrightarrow 0.$$

Hence $G(\phi)y=0$ for all $\phi\in\mathscr{P}_0$, so that $y=0$ by virtue of Definition 2.1, (5). Q.E.D.

Lemma 3.3 justifies the following

Definition 3.2. *$\bar{G}(S)$ denotes the smallest closed linear extension of $G(S)$.*

We can summarize some of the obtained results in the following

Theorem 3.1. *Let G be an FHSG and S a hyperfunction with compact support in $[0, \infty)$. We can define a closed linear operator $\bar{G}(S)$ with dense domain. We have*

$$\bar{G}(S)G(\phi)x=\bar{G}(S)G(\phi)x=G(S*\phi)x,\ \phi\in\mathscr{P}_0,\ x\in E.$$

Among the operators $\bar{G}(S)$, it is the operator $\bar{G}(-\delta')=A$ that plays the fundamental role in this paper, where δ' is the derivative with respect to t of the Dirac measure δ at the origin.

Definition 3.3. *The operator A defined above is the infinitesimal generator of an FHSG G.*

This terminology is justified by the fact that, in the case where G is a usual semi-group, the operator A coincides with the classical infinitesimal generator.

§4. Examples

Proposition 4.1. $\bar{G}(\delta)=I(=identity)$.

Proof. By Theorem 3.1 we have

$$\bar{G}(\delta)G(\phi)x=G(\phi)x.$$

Hence the result follows. Q.E.D.

We now consider the function $\Omega\in\mathscr{P}_*$ or $\mathscr{P}_0$ and we denote by Ω_+ the function $Y\Omega$ discontinuous in general at the origin, where Y is the Heaviside function.

Considering Ω_+ as a hyperfunction with support in $[0,\infty)$, we can evidently define $\bar{G}(\Omega_+)$. We will show the following

Proposition 4.2. *$\bar{G}(\Omega_+)=G(\Omega)$, so that $\bar{G}(\Omega_+)$ is a continuous linear operator of E into itself.*

Proof. Let $\phi\in\mathscr{P}_0$, $x\in E$. We will show that

$$G(\Omega_+)G(\phi)x=G(\Omega)G(\phi)x, \tag{4.1}$$

which asserts the proposition. Put $G(\phi)x=y$. By virtue of Definition 2.1, (3), we see that $Gy=u$ is an E-valued function which is null for $t<0$, continuous for $t\geqq 0$ with $u(0)=y$, and satisfies the estimate

$$\|u(t)\|\leqq C\exp(\varepsilon t)\qquad\text{for any}\quad \varepsilon>0\quad\text{and}\quad t\geqq 0.$$

Hence

$$Gy(\Omega)=G(\Omega)y=u(\Omega)=\int_0^\infty u(t)\Omega(t)dt. \tag{4.2}$$

On the other hand, if ρ_n is a regularizing sequence, we have

$$G(\rho_n)G(\Omega_+)y=G(\Omega_+*\rho_n)y=Gy(\Omega_+*\rho_n)=\int_0^\infty u(t)\,(\Omega_+*\rho_n(t))dt$$

(we have here used Lemma 3.2). When $\rho_n \to \delta$, we have

$$G(\rho_n)G(\Omega_+)y \longrightarrow G(\Omega_+)y$$

and

$$\int_0^\infty u(t)(\Omega_+ * \rho_n(t))dt \longrightarrow \int_0^\infty u(t)\Omega(t)dt,$$

which shows (4.1) comparing with (4.2). Q.E.D.

§5. The properties of the infinitesimal generator of an FHSG

We will consider the domain $D(A)$ of the infinitesimal generator A of an FHSG G as a complex Banach space with the graph norm.

We denote the Fourier hyperfunction valued in $L(D(A), E)$

$$-\delta \otimes A + \delta' \otimes I \qquad (I = \text{the identity})$$

by

$$-A + d/dt.$$

We will show the following

Theorem 5.1. *Let G be an FHSG with the infinitesimal generator A. Then we have*

(1) $G \in \mathscr{P}'_0(L(E, D(A)))$.
(2) $(-A + (d/dt)) * G = \delta \otimes I_E$, ($I_E$ = *the identity in* E).
(3) $G * (-A + (d/dt)) = \delta \otimes I_{D(A)}$, ($I_{D(A)}$ = *the identity in* $D(A)$).

Proof. Let $\Omega \in \mathscr{P}_0$ and $\Omega_+ = Y\Omega$. We have

$$\delta' * \Omega_+ = \Omega'_+ + \Omega(0)\delta.$$

If $\phi \in \mathscr{P}_0$, $x \in E$, then we have

$$G(\delta' * \Omega_+ * \phi)x = G(\delta')G(\Omega_+ * \phi)x = G(\Omega_+)G(\delta' * \phi)x$$
$$= G(\Omega'_+ * \phi + \Omega(0)\phi)x.$$

Hence we have

$$-AG(\Omega_+ * \phi)x = G(\Omega'_+ * \phi)x + \Omega(0)G(\phi)x \tag{5.1}$$

and

$$G(\Omega_+)G(\delta' * \phi)x = G(\Omega'_+ * \phi)x + \Omega(0)G(\phi)x. \tag{5.2}$$

But by virtue of Proposition 4.2, we have

$$\bar{G}(\Omega_+)=G(\Omega),\ \bar{G}(\Omega'_+)=G(\Omega').$$

By virtue of Lemma 3.1, we have

$$G(\Omega_+*\phi)=G(\Omega_+)G(\phi)x,\quad G(\Omega'_+*\phi)=G(\Omega'_+)G(\phi)x,$$

so that (5.1) and (5.2) can be written in the forms

$$-AG(\Omega)G(\phi)x=G(\Omega')G(\phi)x+\Omega(0)G(\phi)x, \tag{5.3}$$

$$-G(\Omega)AG(\phi)x=G(\Omega')G(\phi)x+\Omega(0)G(\phi)x. \tag{5.4}$$

Now, take any y in E. By virtue of Definition 2.1, (4), $G(\phi)x$'s are dense in E, so that there exist a sequence $\phi_n\in\mathscr{P}_0$, and a sequence $x_n\in E$ such that $G(\phi_n)x_n\rightarrow y$. Then

$$G(\Omega)G(\phi_n)x_n \longrightarrow G(\Omega)y, \tag{5.5}$$

and by (5.3) (with ϕ_n and x_n)

$$-A(G(\Omega)G(\phi_n)x_n) \longrightarrow G(\Omega')y+\Omega(0)y. \tag{5.6}$$

Since A is closed, it follows from (5.5) and (5.6) that

$$G(\Omega)y\in D(A) \tag{5.7}$$

and

$$-AG(\Omega)y=G(\Omega')y+\Omega(0)y. \tag{5.8}$$

Hence we see that $G(\Omega)$ maps E into $D(A)$ linearly.

On the othe hand, if $y\rightarrow 0$ in E, then $G(\Omega)y\rightarrow 0$ in E, and (5.8) shows that $AG(\Omega)y\rightarrow 0$ in E. Hence $G(\Omega)\in L(E, D(A))$. If $\Omega\rightarrow 0$ in $\mathscr{P}_0$, $G(\Omega)\rightarrow 0$ in $L(E)$ and by (5.8) $AG(\Omega)\rightarrow 0$ in $L(E)$. Hence $G(\Omega)\rightarrow 0$ in $L(E, D(A))$, which proves (1).

We can write (5.8) in the form

$$-AG(\Omega)+((d/dt)G)(\Omega)=\Omega(0)I_E,$$

which proves (2).

We are now going to prove (3). We use (5.4). Take x in $D(G(-\delta'))$. Then there exists $\phi_n\rightarrow\delta$ with $G(\phi_n)x\rightarrow x$ and $G(-\delta'*\phi_n)\rightarrow Ax$, so that (5.4) with $\phi=\phi_n$ gives

$$-G(\Omega)Ax=G(\Omega')x+\Omega(0)x. \tag{5.9}$$

If now x is in $D(A)$, there exists x_n in $D(G(-\delta'))$ with $x_n\rightarrow x$, $Ax_n\rightarrow Ax$, so that (5.9) (written for $x=x_n$) gives in the limit the same relation (5.9) valid for all x in $D(A)$. This proves (3). The theorem is proved. Q.E.D.

Corollary. *If G is an FHSG with the infinitesimal generator A, the equation*

$$-Au+(d/dt)u=T$$

for a given $T\in\mathscr{P}_0'(E)$ *admits a unique solution* $u=G*T$ *in* $\mathscr{P}_0'(D(A))$.

If $\alpha\geqq 0$ *is the lower bound of the support of* T, u *is null for* $t<\alpha$.

§6. The inverse theorem

Definition 6.1. *Let* A *be a closed linear operator with the domain* $D(A)$ *dense in* E. *If there exists a* $G\in\mathscr{P}_0'(L(E,D(A)))$ *which satisfies the equations*

$$(-A+(d/dt))*G=\delta\otimes I_E,\quad G*(-A+(d/dt))=\delta\otimes I_{D(A)},$$

then we call G *a fundamental solution of the abstract Cauchy problem* (*A. C. P. for short*)

$$-Au+(d/dt)u=T,\quad u\in\mathscr{P}_0'(D(A)), \tag{6.1}$$

for a given $T\in\mathscr{P}_0'(E)$.

In this case A *is said to be well-posed.*

Corollary. *If* A *is well-posed, then the fundamental solution* G *of A. C. P. is unique.*

Proof. See Ito [4].

Proposition 6.1. *If* A *is well-posed and* G *is a fundamental solution of A. C. P., then A. C. P. has a unique solution* $u=G*T$ *for a given* $T\in\mathscr{P}_0'(E)$.

If $\alpha\geqq 0$ *is the lower bound of the support of* T, u *is null for* $t<\alpha$.

In the last section we have shown in Theorem 5.1 that the infinitesimal generator A of an FHSG G is well-posed.

In this section we will show the inverse theorem. That is, if a closed linear operator with the dense domain $D(A)$ is well-posed, then it is an infinitesimal generator of a certain FHSG.

Namely we have the following

Theorem 6.1. *Assume that a closed linear operator* A *with the dense domain* $D(A)$ *is well-posed. Let* G *be the unique fundamental solution of A. C. P.* (6.1). *Then* G *is an FHSG and* A *is the infinitesimal generator of* G.

Proof. We have to show that G satisfies the conditions (1)–(5) of Definition 2.1.

(1) is evident from the definition.

(2), (3). Let $\phi\in\mathscr{P}_0$ and $x\in E$.

Put $y=G(\phi)x$. Then $u(t)=Y(t)G_s(\phi(s-t))x$ has the properties of $u(t)$ of Definition 2.1, (3). Further $u(t)$ satisfies the equation

$$-Au(t)+u'(t)=\delta\otimes y.$$

Hence, by virtue of Proposition 6.1, we have

$$u(t)=G*(\delta\otimes y)=Gy.$$

Let $\psi\in\mathscr{D}_0$. Then

$$\begin{aligned}G(\psi)y&=\int_0^\infty \psi(t)Gy\,dt=\int_0^\infty \psi(t)u(t)\,dt=\int_0^\infty (Y\psi)(t)\;G_s(\phi(s-t))x\,dt\\&=\{(Y\psi)*(G*\check{\phi})^{\vee}\}(0)x=\{(Y\psi)*\check{G}*\phi\}(0)x=\{\check{G}*(Y\psi)*\phi\}(0)x\\&=\{G*((Y\psi)*\phi)^{\vee}\}(0)x=G(\psi*\phi)x.\end{aligned}$$

Here we denote $\check{\phi}(t)=\phi(-t)$ and $\check{G}(\phi)=G(\check{\phi})$.

(5) Suppose $G(\phi)x=0$ for all $\phi\in\mathscr{D}_0$. Then $Gx=0$. Then $u=Gx=G*(\delta\otimes x)$ satisfies the equation

$$-Au+u'=\delta\otimes x.$$

Since $u=0$ we can conclude $x=0$.

(4) We denote by E' and $D(A)'$ the duals of E and $D(A)$ respectively. Since $D(A)$ is dense, we can consider E' to be a subspace of $D(A)'$.

For any $\phi\in\mathscr{D}_0$ we put

$$H(\phi)={}^t(G(\phi))\colon D(A)'\longrightarrow E'.$$

Then we have $H\in\mathscr{D}'_0(L(D(A)',E'))$. If A' denotes the transpose of A, A' is in $L(E',D(A)')$. We take the transpose of the equations

$$-AG(\phi)-G(\phi')=\phi(0)I_E,\qquad -G(\phi)A-G(\phi')=\phi(0)I_{D(A)}.$$

Then we have the equations

$$-H(\phi)A'-H(\phi')=\phi(0)I_{E'},\qquad -A'H(\phi)-H(\phi')=\phi(0)I_{D(A)'}$$

Hence we have

$$H*(-A'+(d/dt))=\delta\otimes I_{E'},\qquad (-A'+(d/dt))*H=\delta\otimes I_{D(A)'}.$$

Now we can show that the set of elements $G(\phi)x$, $\phi\in\mathscr{D}_0$, $x\in E$ is dense. In fact, if $x'\in D(A)'$ and

$$\langle G(\phi)x,x'\rangle=0\qquad\text{for all}\quad \phi\in\mathscr{D}_0,\ x\in E,$$

then we have

$$H(\phi)x'=0\qquad\text{for all}\quad \phi\in\mathscr{D}_0.$$

Then by the above proved fact we can conclude $x'=0$.

At last we will show that A is the infinitesimal generator of the FHSG G.

Now assume that A_1 is the infinitesimal generator of the FHSG G. By the assumption that A is well-posed, the mapping $T \to G*T$ is an isomorphism of $\mathscr{P}'_0(E)$ onto $\mathscr{P}'_0(D(A))$. On the other hand, by virtue of Corollary to Theorem 5.1, the mapping $T \to G*T$ is an isomorphism of $\mathscr{P}'_0(E)$ onto $\mathscr{P}'_0(D(A_1))$. Hence we have $D(A) = D(A_1)$.

Since the relations

$$-AG + (d/dt)G = \delta \otimes I_E, \qquad -A_1 G + (d/dt)G = \delta \otimes I_E$$

hold, we have

$$AG = A_1 G.$$

Hence

$$AG(\phi)x = A_1 G(\phi)x \qquad \text{for all} \quad \phi \in \mathscr{P}_0,\ x \in E.$$

Since the set of elements $G(\phi)x$, $\phi \in \mathscr{P}_0$, $x \in E$ is dense in $D(A) = D(A_1)$, we have $A = A_1$. This completes the proof. Q. E. D.

We have just proved that a closed linear operator A with the dense domain $D(A)$ in E is the infinitesimal generator of a certain FHSG if and only if A is well-posed. Thus by Theorem 2.3 of Ito [2] we have the following

Theorem 6.2. *A closed linear operator A with the dense domain $D(A)$ in a Banach space E is the infinitesimal generator of a certain FHSG is if and only if the following conditions hold:*

(i) *For any complex number λ such as $\operatorname{Re}\lambda > 0$, $(\lambda - A)^{-1}$ exists and it is an $L(E)$-valued holomorphic function in the half plane $\operatorname{Re}\lambda > 0$.*

(ii) *For any $\varepsilon > 0$ and any $\delta > 0$ there exists $C_{\varepsilon,\delta} > 0$ such that*

$$\|(\lambda - A)^{-1}\| \leqq C_{\varepsilon,\delta} \exp(\varepsilon|\lambda|)$$

holds for $\lambda \in \sum_\delta$, where $\sum_\delta$ is the set

$$\textstyle\sum_\delta = \{\lambda \in \boldsymbol{C};\ \operatorname{Re}\lambda \geqq \delta\}.$$

§7. Infinitesimal generator of an EFHSG

In this section we characterize the infinitesimal generator of an EFHSG G. G is an element of $\mathscr{P}'_{\xi_0}(L(E))$. In analogy with FHSG we define $\bar{G}(S)$ as a closed linear operator with the dense domain in E for a hyperfunction S with compact support in $[0, \infty)$.

At first we say that $x \in D(G(S))$ if there exists a regularizing sequence ρ_n in $\mathscr{P}_{\xi_0}$ such that $G(\rho_n)x \to x$ and $G(S*\rho_n)x$ converges in E. Here we can easily see that

$G(S*\rho_n)$ has a sense. Since the relation

$$G(\phi*\psi)=G(\phi)G(\psi) \qquad \text{for all} \quad \phi,\ \psi\in\mathscr{P}_{\xi_0}$$

holds, we can verify that the limit of $G(S*\rho_n)$ does not depend on the choice of regularizing sequence in $\mathscr{P}_{\xi_0}$. Let $G(S)x$ be this limit.

For every ϕ in $\mathscr{P}_{\xi_0}$ and every x in E, $G(\phi)x$ is in $D(G(S))$ and

$$G(S)G(\phi)x=G(S*\phi)x.$$

We define $\bar{G}(S)$ to be the smallest closed linear extension of $G(S)$.

Proposition 7.1. $\bar{G}(S)=\overline{(\exp(-\xi t)G)}(\exp(\xi t)S)$. *Here* $\overline{(\exp(-\xi t)G)}$ *is defined in Definition* 3.2.

Proof. $G(\rho_n)x=(\exp(-\xi t)G)(\exp(\xi t)\rho_n)x\to x$

and

$$G(S*\rho_n)x=(\exp(-\xi t)G)((\exp(\xi t)S)*(\exp(\xi t)\rho_n))x$$
$$\longrightarrow(\exp(-\xi t)G)(\exp(\xi t)S)x.$$

Hence

$$G(S)x=(\exp(-\xi t)G)(\exp(\xi t)S)x.$$

Taking the smallest closed linear extension we have the proposition. Q.E.D.

Proposition 7.2. $\bar{G}(\Omega_+)=G(\Omega)$ *for* $\Omega\in\mathscr{P}_{\xi_0}$, *where* $\Omega_+=Y\Omega$.

Proof. $\bar{G}(\Omega_+)=\overline{(\exp(-\xi t)G)}(\exp(\xi t)\Omega_+)=\overline{(\exp(-\xi t)G)}((\exp(\xi t)\Omega)_+)$ $=(\exp(-\xi t)G)(\exp(\xi t)\Omega)=G(\Omega)$. Q.E.D.

Definition 7.1. *The closed linear operator* $A=\overline{G(-\delta')}$ *with the dense domain in* E *is said to be the infinitesimal generator of an EFHSG* G.

We will show the following

Theorem 7.1. *Let* G *be an EFHSG with the infinitesimal generator* A. *Then we have*

(1) $G\in\mathscr{P}'_{\xi_0}(L(E,\ D(A)))$.
(2) $(-A+(d/dt))*G=\delta\otimes I_E$.
(3) $G*(-A+(d/dt))=\delta\otimes I_{D(A)}$.

Proof. $H=\exp(-\xi t)G$ is an FHSG and the infinitesimal generator B is $\overline{H(-\delta')}=\overline{(\exp(-\xi t)G)}(-\delta')=\bar{G}(-\exp(-\xi t)\delta')=\bar{G}(-\xi\delta-\delta')=A-\xi I$. Then, by Theorem 5.1, we have

$$(-B+(d/dt))*H=\delta\otimes I_E, \quad H*(-B+(d/dt))=\delta\otimes I_{D(A)}.$$

Using the relations $H=\exp(-\xi t)G$ and $B=A-\xi I$ we can rewrite the above two equations into the equation of G, so that we can obtain the conclsuion. Q. E. D.

Corollary. *If $G\in\mathscr{P}'_{\xi_0}(L(E))$ is an EFHSG with the infinitesimal generator A, the equation*

$$-Au+(d/dt)u=T$$

for a given $T\in\mathscr{P}'_{\xi_0}(E)$ admits a unique solution $u=G*T$ in $\mathscr{P}'_{\xi_0}(D(A))$.

If $\alpha\geqq 0$ is the lower bound of the support of T, u is null for $t<\alpha$.

Definition 7.2. *Let A be a closed linear operator with the domain $D(A)$ dense in E. If there exists a $G\in\mathscr{P}'_{\xi_0}(L(E, D(A)))$ which satisfies the equations*

$$(-A+(d/dt))*G=\delta\otimes I_E, \qquad G*(-A+(d/dt))=\delta\otimes I_{D(A)},$$

then we call G a fundamental solution in $\mathscr{P}'_{\xi_0}(L(E, D(A)))$ of the abstract Cauchy problem (A. C. P. in $\mathscr{P}'_{\xi_0}(E)$ for short)

$$-Au+(d/dt)u=T, \quad u\in\mathscr{P}'_{\xi_0}(D(A)), \tag{7.1}$$

for a given $T\in\mathscr{P}'_{\xi_0}(E)$.

In this case A is said to be well-posed in $\mathscr{P}'_{\xi_0}(E)$.

Corollary. *If A is well-posed in $\mathscr{P}'_{\xi_0}(E)$, then the fundamental solution G of A. C. P. in $\mathscr{P}'_{\xi_0}(E)$ is unique.*

Proposition 7.3. *If A is well-posed in $\mathscr{P}'_{\xi_0}(E)$ and G is a fundamental solution of A. C. P. in $\mathscr{P}'_{\xi_0}(E)$, then A. C. P. in $\mathscr{P}'_{\xi_0}(E)$ has a unique solution $u=G*T$ for a given $T\in\mathscr{P}'_{\xi_0}(E)$.*

If $\alpha\geqq 0$ is the lower bound of the support of T, u is null for $t<\alpha$.

We have shown in Theorem 7.1 that the infinitesimal generator A of an EFHSG G is well-posed in $\mathscr{P}'_{\xi_0}(E)$.

Now we will show the inverse theorem.

Theorem 7.2. *Assume that a closed linear operator A with the dense domain $D(A)$ is well-posed in $\mathscr{P}'_{\xi_0}(E)$. Let $G\in\mathscr{P}'_{\xi_0}(L(E, D(A)))$ be the unique fundamental solution of A. C. P. (7.1). Then G is an EFHSG and A is the infinitesimal generator of G.*

Proof. We put $\exp(-\xi t)G=H$. Then $H\in\mathscr{P}'_0(L(E, D(A)))$ and satisfies the equations

$$(-B+(d/dt))*H=\delta\otimes I_E, \quad H*(-B+(d/dt))=\delta\otimes I_{D(A)},$$

since $G=\exp(\xi t)H$ is the fundamental solution of A. C. P. (7.1). Here $B=A-\xi I$. Then $D(B)=D(A)$. Hence $H\in\mathscr{P}_0'(L(E, D(B)))$. Thus B is well-posed for A. C. P. (6.1) and H is a fundamental solution of A. C. P. (6.1). Then H is an FHSG by virtue of Theorem 6.1 and B is the infinitesimal generator of H. This holds for any $\xi>\xi_0$, so that G is an EFHSG in the sense of Definition 2.2. Let A_1 be the infinitesimal generator of G. Then $B_1=A_1-\xi I$ is the infinitesimal generator of $H=\exp(-\xi t)G$. Thus $A-\xi I=B=B_1=A_1-\xi I$. Hence $A=A_1$. Q.E.D.

We have just proved that a closed linear operator A with the dense domain $D(A)$ in E is the infinitesimal generator of a certain EFHSG if and only if A is well-posed in $\mathscr{P}_{\xi_0}'(E)$ for some $\xi_0\geqq 0$. Thus by Theorem 3.2 of Ito [2] we have the following

Theorem 7.3. *A closed linear operator A with the dense domain $D(A)$ in a Banach space E is the infinitesimal generator of a certain EFHSG if and only if the following conditions hold*:

(i) *There exists $\xi_0\geqq 0$ such that, for any complex number λ such as* $\mathrm{Re}\,\lambda>\xi_0$, $(\lambda-A)^{-1}$ *exists and it is an $L(E)$-valued holomorphic function in the domain* $\mathrm{Re}\cdot\lambda>\xi_0$.

(ii) *For any $\varepsilon>0$ and any $\delta>\xi_0$ there exists $C_{\varepsilon,\delta}>0$ such that*

$$\|(\lambda-A)^{-1}\|\leqq C_{\varepsilon,\delta}\exp(\varepsilon|\lambda-\xi_0|)$$

holds for $\lambda\in\sum_\delta$, where we put

$$\textstyle\sum_\delta=\{\lambda\in\boldsymbol{C};\ \mathrm{Re}\,\lambda\geqq\delta\}.$$

§8. Fourier hyperfunction groups.

Definition 8.1. *We call a Fourier hyperfunction group (in E) (FHG for short) an $L(E)$-valued Fourier hyperfunction G such that*

(1) $G\in\mathscr{P}_*'(L(E))$.

(2) $G(\phi*\psi)=G(\phi)G(\psi)$ *for all* $\phi,\ \psi\in\mathscr{P}_*$.

(3) $G=G_++G_-$, *where G_+ and $\check{G}_-$ are FHSG's.*

For a real $\xi_0\geqq 0$, we define the space $\mathscr{P}_*(\xi_0)$ by the formula

$$\mathscr{P}_*(\xi_0)=\bigcup_{\xi>\xi_0}(\exp(-\xi t)\mathscr{P}_0)\cap(\exp(\xi t)\check{\mathscr{P}}_0)$$

whose topology is defined as the finest locally convex topology such that the mapping:

$$(\phi_1,\ \phi_2)\in \textit{diagonal of } \mathscr{P}_0\times\check{\mathscr{P}}_0\longrightarrow(\exp(-\xi t)\phi_1,\ \exp(\xi t)\phi_2)\in\mathscr{P}_*(\xi_0)$$

is continuous for any $\xi>\xi_0$. Here we denote

$$\check{\mathscr{P}}_0 = \{\check{\phi};\ \phi \in \mathscr{P}_0\}.$$

Then we define the space $\mathscr{P}'_*(\xi_0, E)$ to be the space of all continuous linear mappings from $\mathscr{P}_*(\xi_0)$ into E.

Definition 8.2. *We call an element G of $\mathscr{P}'_*(\xi_0, L(E))$ an exponential Fourier hyperfunction group (in E) (EFHG for short) if it satisfies the following conditions:*

(1) $G(\phi * \psi) = G(\phi)G(\psi)$ *for all* $\phi, \psi \in \mathscr{P}_*(\xi_0)$.

(2) *If we denote by G_+ (resp. G_-) the extension of G to $\mathscr{P}_{\xi_0}$ (resp. $\check{\mathscr{P}}_{\xi_0}$), then G_+ and $\check{G}_-$ are EFHSG's.*

For any $\xi > \xi_0$,

$$\hat{G} = \exp(-\xi t)G_+ + \exp(\xi t)G_- = \hat{G}_+ + \hat{G}_-$$

is an FHG. Here we put

$$\hat{G}_+ = \exp(-\xi t)G_+, \quad \hat{G}_- = \exp(\xi t)G_-.$$

Then $\hat{G}_+$ and $(\hat{G}_-)^{\vee}$ are FHSG's. G acts on $\phi \in \mathscr{P}_*(\xi_0)$ by the formula

$$G(\phi) = (\exp(-\xi t)G_+)(\exp(\xi t)\phi) + (\exp(\xi t)G_-)(\exp(-\xi t)\phi)$$

for such $\xi > \xi_0$ as $\exp(\xi t)\phi \in \mathscr{P}_0$ and $\exp(-\xi t)\phi \in \check{\mathscr{P}}_0$.

Theorem 8.1. *Let G be an FHG. If A_+* (resp. A_-) *is the infinitesimal generator of the FHSG G_+* (resp. G_-), *we have*

$$A_+ = -A_-. \tag{8.1}$$

Proof. 1) Let ϕ be any one in $\mathscr{P}_*$ and ρ_n a regularizing sequence in $\mathscr{P}_*$. We have $G_+(\rho_n) = G(\rho_n)$. Hence

$$G_+(\rho_n)G(\phi) = G(\rho_n * \phi)x \longrightarrow G(\phi)x \qquad \text{for any } x \text{ in } E,$$

and

$$G_+(-\rho'_n)G(\phi)x = G(-\rho'_n * \phi)x \longrightarrow G(-\phi')x,$$

which proves that $G(\phi)x \in D(A_+)$ and

$$A_+G(\phi)x = G(-\phi')x \qquad \text{for all } \phi \text{ in } \mathscr{P}_* \text{ and } x \text{ in } E. \tag{8.2}$$

Now we note that $\check{G}_-(\rho_n) = G_-(\check{\rho}_n) = G(\check{\rho}_n)$, so that, since $\check{\rho}_n \to \delta$,

$$\check{G}_-(\rho_n)G(\phi)x = G(\check{\rho}_n * \phi)x \longrightarrow G(\phi)x,$$

$$\check{G}_-(-\rho'_n)G(\phi)x = G_-((\check{\rho}_n)')G(\phi)x = G(\check{\rho}_n * \phi')x \longrightarrow G(\phi')x.$$

Hence $G(\phi)$ is in $D(A_-)$ and

$$A_-G(\phi)x=G(\phi')x.$$

Comparing with (8.2), this shows that

$$A_+G(\phi)+A_-G(\phi)x=0 \qquad \text{for any } \phi \text{ in } \mathscr{P}_* \text{ and } x \text{ in } E. \tag{8.3}$$

2) Let now $x\in D(G_+(-\delta'))$. Then there exists a regularizing sequence ρ_n in $\mathscr{P}_*$ such that

$$G_+(\rho_n)x \longrightarrow x \quad \text{and} \quad G_+(-\rho_n')x \longrightarrow A_+x.$$

Using (8.2), this is also written

$$G(\rho_n)x \longrightarrow x, \quad G(-\rho_n')x=A_+G(\rho_n)x \longrightarrow A_+x.$$

And, using (8.3),

$$A_-G(\rho_n)x=-A_+G(\rho_n)x \longrightarrow -A_+x.$$

Since A_- is closed, it follows from this

$$\text{if } x\in D(G_+(-\delta')), \text{ then } x \text{ is in } D(A_-) \text{ and } A_+x+A_-x=0. \tag{8.4}$$

3) If now x is in $D(A_+)$, there exists a sequence x_n in $D(G_+(-\delta'))$ such that $x_n\to x$ and $A_+x_n\to A_+x$. By virtue of (8.4) each x_n is in $D(A_-)$ and $A_-x_n=-A_+x_n\to -A_+x$, so that $-A_-$ is an extension of A_+.

We can analogously see that A_+ is an extension of $-A_-$. This completes the proof. Q. E. D.

We put

$$A=A_+=-A_- \tag{8.5}$$

and call A the infinitesimal generator of the FHG G.

Such an operator is characterized by the following

Theorem 8.2. *A closed linear operator A with the dense domain in a complex Banach space E is the infinitesimal generator of a certain FHG if and only if the following conditions hold:*

(1) *each of the equations*

$$-Au+(d/dt)u=S, \quad u\in\mathscr{P}_0'(D(A)), \quad S \text{ given in } \mathscr{P}_0'(E), \tag{8.6}$$

$$Av+(d/dt)v=T, \quad v\in\mathscr{P}_0'(D(A)), \quad T \text{ given in } \mathscr{P}_0'(E) \tag{8.7}$$

admits the unique solution depending continuously on S(resp. T).

(2) *If $\alpha\geqq 0$ is the lower bound of the support of* S(resp. T), *then* u(resp. v) *is null for $t<\alpha$.*

Proof. The condition is necessary. In fact, by virtue of Theorem 8.1 and (8.5), A and $-A$ are the infinitesimal generator of FHSG's, which proves the necessity by virtue of Corollary of Theorem 5.1.

The condition is sufficient.

1) By virtue of Proposition 6.1 and Theorem 6.1, A and $-A$ are the infinitesimal generators of FHSG's, say G_+ and H respectively. Put

$$\check{H}=G_-.$$

Then the solution of the equation

$$-Au+u'=S, \quad u\in\mathscr{P}_0'(D(A)), \quad S\in\mathscr{P}_0'(E)$$

is

$$u=G_+*S$$

and the solution of

$$-Au+u'=S, \quad \check{u}\in\mathscr{P}_0'(D(A)), \quad \check{S}\in\mathscr{P}_0'(E)$$

is

$$u=-G_-*S.$$

2) If we put

$$G=G_++G_-,$$

it remains only to show that

$$G(\phi*\psi)=G(\phi)G(\psi) \qquad \text{for all} \quad \phi \text{ and } \psi \text{ in } \mathscr{P}_*. \tag{8.8}$$

Denote by u_+(resp. u_-) the solution in $\mathscr{P}_0'(D(A))$ (resp. $\check{u}_-\in\mathscr{P}_0'$ $(D(A))$) of the equation

$$-Au_++u_+'=\check{\psi}\otimes x.$$

(resp.

$$-Au_-+u_-'=-\check{\psi}\otimes x).$$

We have

$$u_+(0)=G_+(\psi)x,\ u_-(0)=G_-(\psi)x.$$

Hence

$$G(\psi)x=u_+(0)+u_-(0).$$

If we denote by v_+(resp. v_-) the solution in $\mathscr{P}_0'(D(A))$ (resp. $\check{v}_-\in\mathscr{P}_0'(D(A))$) of the equation

$$-Av_{+}+v'_{+}=\check{\phi}\otimes(u_{+}(0)+u_{-}(0))$$

(resp.

$$-Av_{-}+v'_{-}=-\check{\phi}\otimes(u_{+}(0)+u_{-}(0))),$$

we have

$$v_{+}(0)=G_{+}(\phi)G(\psi)x, \quad v_{-}(0)=G_{-}(\phi)G(\psi)x,$$

so that

$$G(\phi)G(\psi)x=v_{+}(0)+v_{-}(0). \tag{8.9}$$

Now, if w_{+}(resp. w_{-}) is the solution in $\mathscr{P}'_0(D(A))$ (resp. $\check{w}_{-}\in\mathscr{P}'_0(D(A))$) of the equation

$$-Aw_{+}+w'_{+}=(\check{\phi}*\check{\psi})\otimes x$$

(resp.

$$-Aw_{-}+w'_{-}=-(\check{\phi}*\check{\psi})\otimes x),$$

then

$$G(\phi*\psi)=w_{+}(0)+w_{-}(0). \tag{8.10}$$

But

$$w_{+}=\check{\phi}*u_{+}, \quad w_{-}=\check{\phi}*u_{-},$$

so that

$$w_{+}(0)=u_{+}(\phi), \quad w_{-}(0)=u_{-}(\phi)$$

and (8.10) gives

$$G(\phi*\psi)=u_{+}(\phi)+u_{-}(\phi). \tag{8.11}$$

Now let Y be the Haeviside function and put $Z(t)=Y(-t)$.
We have

$$-A(Yu_{+})+(Yu_{+})'=(Y\check{\psi})\otimes x+\delta\otimes u_{+}(0),$$

$$-A(Yu_{-})+(Yu_{-})'=-(Y\check{\psi})\otimes x+\delta u_{-}(0),$$

so that

$$-A(Yu_{+}+Yu_{-})+(Yu_{+}+Yu_{-})'=\delta\otimes(u_{+}(0)+u_{-}(0)).$$

From this we deduce

$$(Yu_{+}+Yu_{-})*\phi=v_{+}.$$

In the same way we have

$$(Zu_+ + Zu_-)*\check{\phi} = v_-.$$

From these last two equations, we deduce

$$v_+(0) = u_+(\phi_+) + u_-(\phi_+), \quad v_-(0) = u_+(\phi_-) + u_-(\phi_-),$$

where $\phi_+ = Y\phi$, $\phi_- = Z\phi$.

We conclude from this

$$v_+(0) + v_-(0) = u_+(\phi) + u_-(\phi),$$

which, with (8.9) and (8.11), gives (8.8) and completes the proof of the theorem.
Q. E. D.

The following result is now immeiate.

Theorem 8.3. *A closed linear operator A with the dense domain in a complex Banach space E is the infinitesimal generator of an FHG if and only if the following two conditions are satisfied*:

(1) *For any complex number λ such as $|\mathrm{Re}\,\lambda| > 0$, $(\lambda - A)^{-1}$ exists and it is an $L(E)$-valued holomorphic function there.*

(2) *For any $\varepsilon > 0$ and any $\delta > 0$ there exists $C_{\varepsilon,\delta} > 0$ such that*

$$\|(\lambda - A)^{-1}\| \leqq C_{\varepsilon,\delta} \exp(\varepsilon|\lambda|)$$

holds for $\lambda \in \Lambda_\delta$, where Λ_δ is the set

$$\Lambda_\delta = \{\lambda \in \boldsymbol{C};\ |\mathrm{Re}\,\lambda| \geqq \delta\}.$$

Now we will characterize the infinitesimal generator of an EFHG.

Let G be an EFHG. We will use the notations in Definition 8.2 and following after it. Let A_+(resp. A_-) be the infinitesimal generator of the EFHSG G_+(resp. G_-) and $\hat{A}_+$(resp. $\hat{A}_-$) that of the FHSG $\hat{G}_+$(resp. $\hat{G}_-$). Then $\hat{A}_+ = -\hat{A}_-$. Then $\hat{A}_+ = -\hat{A}_- = \hat{A}$ is the infinitesimal generator of the FHG $\hat{G}$ for $\xi > \xi_0$. Then we have

$$A_+ = \hat{A}_+ + \xi I = \hat{A} + \xi I, \quad A_- = \hat{A}_- - \xi I = -\hat{A} - \xi I.$$

Hence

$$A_+ = -A_-.$$

We put

$$A_+ = -A_- = A$$

and call A the infinitesimal generator of the EFHG G. Then we have

$$A = \hat{A} + \xi I.$$

Hence we have the following

Theorem 8.4. *A closed linear operator A with the dense domain in a complex Banach space E is the infinitesimal generator of a certain EFHG if and only if the following two conditions hold*:

(1) *each of the equations*

$$-Au+(d/dt)u=S, \quad u\in\mathscr{P}'_{\xi_0}(D(A)), \quad S \text{ given in } \mathscr{P}'_{\xi_0}(E),$$

$$Av+(d/dt)v=T, \quad v\in\mathscr{P}'_{\xi_0}(D(A)), \quad T \text{ given in } \mathscr{P}'_{\xi_0}(E)$$

admits the unique solution depending continuously on S (resp. T).

(2) *If $\alpha\geqq 0$ is the lower bound of the support of S* (resp. T), *then u* (resp. v) *is null for $t<\alpha$.*

Theorem 8.5. *A closed linear operator A with the dense domain in a complex Banach space E is the infinitesimal generator of an EFHG if and only if the following two conditions are satisfied.*

(1) *There exists $\xi_0\geqq 0$ such that for any complex number λ such as $|\mathrm{Re}\,\lambda|>\xi_0$, $(\lambda-A)^{-1}$ exists and it is an $L(E)$-valued holomorphic function in the open set $|\mathrm{Re}\,\lambda|>\xi_0$.*

(2) *For any $\varepsilon>0$ and any $\delta>\xi_0$ there exists $C_{\varepsilon,\delta}>0$ such that*

$$\|(\lambda-A)^{-1}\|\leqq C_{\varepsilon,\delta}\exp\{\varepsilon(\min(|\lambda-\xi_0|,\ |\lambda+\xi_0|))\}$$

holds for $\lambda\in\Lambda_\delta$, where we put

$$\Lambda_\delta=\{\lambda\in\boldsymbol{C};\ |\mathrm{Re}\,\lambda|\geqq\delta\}.$$

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