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Theory of (Vector Valued) Fourier Hyperfunctions. Their Realization as Boundary Values of (Vector Valued) Slowly Increasing Holomorphic Functions, (II)

By

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Introduction

This paper is the second part of this series of papers, which includes Chapters 5 to 8. For the outline of this paper, see "Contents" in the first part of this series of papers [37]. For References we refer to the lists of references at the ends of the paper [37] and this paper.

Chapter 5. Cases of sheaves $\mathcal{O}^{\#}$, $\mathcal{A}^{\#}$, $\mathcal{O}_{\#}$ and $\mathcal{A}_{\#}$

5.1. The Oka-Cartan-Kawai Theorem B

In this section we will prove the Oka-Cartan-Kawai Theorem B for the sheaves $\mathcal{O}^{\#}$ and $\mathcal{O}_{\#}$.

For a 2-tuple $n=(n_1, n_2)$ of nonnegative integers with $|n|=n_1+n_2 \neq 0$, we denote by $\boldsymbol{F}^n$ the product space $\tilde{\boldsymbol{C}}^{n_1} \times \boldsymbol{E}^{n_2}$ and by $\boldsymbol{D}^n$ the product space $\boldsymbol{D}^{n_1} \times \boldsymbol{D}^{n_2}$ and by $\boldsymbol{C}^{|n|}$ the space $\boldsymbol{C}^{n_1+n_2}=\boldsymbol{C}^{n_1} \times \boldsymbol{C}^{n_2}$. We denote $z=(z', z'') \in \boldsymbol{C}^{|n|}$ so that $z'=(z_1,\dots, z_{n_1})$, $z''=(z_{n_1+1},\dots, z_{|n|})$.

Definition 5.1.1 (The sheaf $\mathcal{O}^{\#}$ of germs of slowly increasing holomorphic functions). *We define $\mathcal{O}^{\#}$ to be the sheafification of the presheaf $\{\mathcal{O}^{\#}(\Omega);\ \Omega \subset \boldsymbol{F}^n$ open$\}$, where the section module $\mathcal{O}^{\#}(\Omega)$ on an open set Ω in $\boldsymbol{F}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega \cap \boldsymbol{C}^{|n|}$ such that, for any positive number ε and for any compact set K in Ω, the estimate* $\sup\{|f(z)|e(-\varepsilon|z|);\ z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ *holds.*

Definition 5.1.2 (The sheaf $\mathcal{O}_{\#}$ of germs of rapidly decreasing holomorphic functions). *We define $\mathcal{O}_{\#}$ to be the sheafification of the presheaf $\{\mathcal{O}_{\#}(\Omega);\ \Omega \subset \boldsymbol{F}^n$ open$\}$, where the section module $\mathcal{O}_{\#}(\Omega)$ on an open set Ω in $\boldsymbol{F}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega \cap \boldsymbol{C}^{|n|}$ such that, for any compact set K in Ω, there exists some positive constant δ so that the estimate* $\sup\{|f(z)|e(\delta|z|);\ z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ *holds.*

Definition 5.1.3. *An open set V in $\boldsymbol{F}^n$ is said to be an $\mathcal{O}^\#$-pseudoconvex open set if it satisfies the conditions*:

(1) $\sup\{|\mathrm{Im}\, z'|, |\mathrm{Im}\, z''|-|\mathrm{Re}\, z''|;\ z=(z', z'')\in V\cap \boldsymbol{C}^{|n|}\}<\infty$.

(2) *There exists a C^∞-plurisubharmonic function $\varphi(z)$ on $V\cap \boldsymbol{C}^{|n|}$ having the following two properties*:

(i) *The closure of $V_c=\{z\in V\cap \boldsymbol{C}^{|n|};\ \varphi(z)<c\}$ in $\boldsymbol{F}^n$ is a compact subset of V for any real c.*

(ii) *$\varphi(z)$ is bounded on $L\cap \boldsymbol{C}^{|n|}$ for any compact subset L of V.*

Then we can prove the Oka-Cartan-Kawai Theorem B by a similar method to that in section 1.1.

Theorem 5.1.4 (The Oka-Cartan-Kawai Theorem B). *For any $\mathcal{O}^\#$-pseudoconvex open set V in $\boldsymbol{F}^n$, we have $H^s(V, (\mathcal{O}^\#)^p)=0$, $(p\geqq 0,\ s\geqq 1)$.*

Proof. Since V is paracompact, $H^s(V, (\mathcal{O}^\#)^p)$ coincides with the Čech cohomology group. So we have only to prove $\varinjlim_{\mathfrak{U}} H^s(\mathfrak{U}, (\mathcal{O}^\#)^p)=0$, where $\mathfrak{U}=\{U_j\}_{j\geqq 1}$ is a locally finite open covering of V so that $V_j=U_j\cap \boldsymbol{C}^{|n|}$ is pseudoconvex. We can choose such a covering of V because V is an $\mathcal{O}^\#$-pseudoconvex open set.

Now we define $C^s(Z^{\mathrm{loc},\#}_{(p,q)}(\{V_j\}))$ to be the set of all cochains $c=\{c_J;\ J=(j_0, j_1,\dots, j_s)\in \boldsymbol{N}^{s+1}\}$ of forms of type (p, q) satisfying the two conditions:

(i) $\bar\partial c_J=0$ in $V_J=V_{j_0}\cap V_{j_1}\cap\cdots\cap V_{j_s}$.

(ii) For any positive ε and any finite subset M of $\boldsymbol{N}^{s+1}$, the estimate

$$\sum_{J\in M}\int_{V_J}|c_J|^2 e(-\varepsilon\|z\|)d\lambda<\infty$$

holds, where $d\lambda$ is the Lebesgue measure on $\boldsymbol{C}^{|n|}$ and $\|z\|$ denotes the modification of $\sum_{j=1}^{|n|}|z_j|$ so as to become C^∞ and convex.

Now we will prove the following

Lemma 5.1.5. *If $c\in C^s(Z^{\mathrm{loc},\#}_{(p,q)}(\{V_j\}))$ satisfies the conditions $\delta c=0$, then we can find some $c'\in C^{s-1}(Z^{\mathrm{loc},\#}_{(p,q)}(\{V_j\}))$ such that $\delta c'=$*c. *Here δ means the coboundary operator.*

If this Lemma is proved, the theorem will follow from this Lemma as the special case where $q=0$ because we can use Cauchy's integral formula to change the L^2-norm to the sup-norm for holomorphic functions.

Proof of Lemma 5.1.5. We denote by $\{\chi_j\}$ the partition of unity subordinate to $\{V_j\}$ and define $b_I=\sum_j \chi_j c_{jI}$ for $I\in \boldsymbol{N}^s$. Since $\delta c=0$, we have $\delta b=c$. So $\delta\bar\partial b=0$ because $\bar\partial c=0$. Since $\sum\chi_j=1$ and $\chi_j\geqq 0$, we have

$$\int_{V_I}|b_I|^2 e(-\varepsilon\|z\|)d\lambda\leqq\sum_j\int_{V_I}\chi_j|c_{jI}|^2 e(-\varepsilon\|z\|)d\lambda$$

for any positive number ε by virtue of Cauchy-Schwarz' inequality.

By the assumption of the existence of C^∞ plurisubharmonic function $\varphi(z)$ in Definition 5.1.3, we can find some plurisubharmonic function $\psi(z)$ on $W=V\cap \boldsymbol{C}^{|n|}$ which satisfies the following two conditions:

(1) $\sum|\bar\partial\chi_j|\leqq e(\psi(z))$,

(2) $\sup\{\psi(z);\ z\in K\cap \boldsymbol{C}^{|n|}\}\leqq C_K$ for any $K\Subset W$.

Thus it follows from the condition on c that

$$\sum_{I\in N}\int_{V_I}|\bar\partial b_I|^2 e(-\varepsilon\|z\|-\psi(z))d\lambda<\infty$$

for any positive number ε and any finite subset N of $\boldsymbol{N}^s$.

Now we consider the case $s=1$. By the fact that $\delta(\bar\partial b)=0$, $\bar\partial b$ defines a global section f on $W=V\cap \boldsymbol{C}^{|n|}$. Then, by Hörmander [4], Theorem 4.4.2, p. 94, we can prove the existence of u such that $\bar\partial u=f$ and the estimate

$$\int_{K\cap \boldsymbol{C}^{|n|}}|u|^2 e(-\varepsilon\|z\|)(1+|z|^2)^{-2}d\lambda<\infty$$

holds for any positive number ε and any $K\Subset V$.

If we define $c'_I=b_I-u|V_I$, then $\bar\partial c'_I=0$ and $\delta c'=\delta b=c$. Clearly $c'\in C^{s-1}\cdot(Z^{\mathrm{loc},\#}_{(p,q)}(\{V_j\}))$.

Now we go on to the case $s>1$. In this case we use the induction on s. By the induction hypotheses there exists $b'\in C^{s-2}(Z^{\mathrm{loc},\#}_{(p,q+1)}(\{V_j\}))$ such that $\delta b'=\bar\partial b$. By virtue of Hörmander [4], Theorem 4.4.2, p. 94, we can also find $b''=\{b''_H\}_{H\in \boldsymbol{N}^{s-1}}$ such that $b'_H=\bar\partial b''_H$ and the estimate

$$\sum_{H\in L}\int_{V_H}|b''_H|^2 e(-\varepsilon\|z\|-\psi(z))(1+|z|^2)^{-2}d\lambda<\infty$$

holds for any positive number ε and any finite subset L of $\boldsymbol{N}^{s-1}$. Therefore $c'=b-\delta b''$ satisfies all the required conditions. Q.E.D.

This completes the proof of the theorem. Q.E.D.

Now we will prove the Malgrange theorem for the sheaf $\mathscr{A}^\#$ of germs of slowly increasing real analytic functions. Here we define the sheaf $\mathscr{A}^\#$ to be the restriction of $\mathscr{O}^\#$ to $\boldsymbol{D}^n$: $\mathscr{A}^\#=\mathscr{O}^\#|\boldsymbol{D}^n$. Then we have the following

Theorem 5.1.6 (Malgrange). *For an arbitrary set Ω in $\boldsymbol{D}^n$, we have* $H^s(\Omega,(\mathscr{A}^\#)^p)=0$, $(p\geqq 0,\ s\geqq 1)$.

Proof. We know, by virtue of Ito [11], Theorem 2.1.13, that Ω has a fundamental system $\{\tilde\Omega\}$ of $\mathscr{O}^\#$-pseudoconvex open neighborhoods. Then, it follows from the Oka-Cartan-Kawai Theorem B and Schapira [34], Theorem B 42, that, for $p\geqq 0$ and $s>0$, we have

$$H^s(\Omega, (\mathscr{A}^\#)^p) = \lim_{\tilde{\Omega} \cap \boldsymbol{D}^n = \Omega} H^s(\tilde{\Omega}, (\mathcal{O}^\#)^p) = 0. \qquad \text{Q.E.D.}$$

Next we will prove the Oka-Cartan-Kawai Theorem B for the sheaf $\mathcal{O}_\#$. This can be proved by a similar method to Theorem 5.1.4. Thus we have the following

Theorem 5.1.7 (The Oka-Cartan-Kawai Theorem B). *For any $\mathcal{O}^\#$-pseudoconvex open set V in $\boldsymbol{F}^n$, we have $H^s(V, \mathcal{O}^p_\#) = 0$ for $p \geqq 0$ and $s \geqq 1$.*

Proof. Since V is paracompact, $H^s(V, \mathcal{O}^p_\#)$ coincides with the Čech cohomology group. So we have only to prove $\lim_{\mathfrak{U}} H^s(\mathfrak{U}, \mathcal{O}^p_\#) = 0$, where $\mathfrak{U} = \{U_j\}_{j \geqq 1}$ is a locally finite open covering of V so that $V_j = U_j \cap \boldsymbol{C}^{|n|}$ is pseudoconvex. We can choose such a covering of V because V is an $\mathcal{O}^\#$-pseudoconvex open set.

Here we use the notations in the proof of Theorem 5.1.4.

For any cocycle $d = \{d_J\}$ representing an element in $H^s(\mathfrak{U}, \mathcal{O}^p_\#)$, we can define an element $c = \{c_J\}$ in $C^s(Z^{\text{loc},\#}_{(p,0)}(\{V_j\}))$ such as $\delta c = 0$ by putting $c_J = d_J \cdot h_\varepsilon(z)$, $h_\varepsilon(z) = (\prod_{j=1}^{n_1} \cosh(\varepsilon z_j)) \cdot \cosh(\varepsilon \sqrt{(z'')^2}/2)$ for some positive ε, where δ denotes the coboundary operator. Then we can find some $c' \in C^{s-1}(Z^{\text{loc},\#}_{(p,0)}(\{V_j\}))$ such that $\delta c' = c$. If we put $d'_I = c'_I \cdot (h_\varepsilon(z))^{-1}$, then $d' = \{d'_I\}$ is a cochain with values in $\mathcal{O}_\#$ such that $\delta d' = d$. Thus the element in $H^s(\mathfrak{U}, \mathcal{O}^p_\#)$ represented by d is zero. Since a class $[d]$ with a representative d is an arbitrary element in $H^s(\mathfrak{U}, \mathcal{O}^p_\#)$, we have $H^s(\mathfrak{U}, \mathcal{O}^p_\#) = 0$. This completes the proof. Q.E.D.

At last we will prove the Malgrange theorem for the sheaf $\mathscr{A}_\#$ of germs of rapidly decreasing real analytic functions. Here we define the sheaf $\mathscr{A}_\#$ to be the restriction of $\mathcal{O}_\#$ to $\boldsymbol{D}^n$: $\mathscr{A}_\# = \mathcal{O}_\# | \boldsymbol{D}^n$. Then we have the following

Theorem 5.1.8 (Malgrange). *For an arbitrary set Ω in $\boldsymbol{D}^n$, we have $H^s(\Omega, \mathscr{A}^p_\#) = 0$ for $p \geqq 0$ and $s \geqq 1$.*

Proof. We can prove this by a method similar to that of Theorem 5.1.6. Q.E.D.

5.2. The Dolbeault-Grothendieck resolutions of $\mathcal{O}^\#$ and $\mathcal{O}_\#$

In this section we will construct soft resolutions of $\mathcal{O}^\#$ and $\mathcal{O}_\#$ and prove some of their consequences.

At first we will recall the definition of the sheaf $L^\# = L^\#_{2,\text{loc}}$ of germs of slowly increasing locally L_2-functions over $\boldsymbol{F}^n$.

Definition 5.2.1. *We define the sheaf $L^\#$ to be the sheafification of the presheaf $\{L^\#(\Omega); \Omega \subset \boldsymbol{F}^n \text{ open}\}$, where, for an open set Ω in $\boldsymbol{F}^n$, the section module $L^\#(\Omega)$ is the space of all $f \in L_{2,\text{loc}}(\Omega \cap \boldsymbol{C}^{|n|})$ such as, for any $\varepsilon > 0$ and any relatively compact open subset ω of Ω, $e(-\varepsilon\|z\|)f(z)|\omega$ belongs to $L_2(\omega \cap \boldsymbol{C}^{|n|})$. Here $e(-\varepsilon\|z\|)f(z)|\omega$ denotes the restriction of $e(-\varepsilon\|z\|)f(z)$ to ω and $\|z\|$ denotes the modification of*

$\sum_{j=1}^{|n|} |z_j|$ *so as to become* C^∞ *and convex.*

Then it is easy to see that $L^\#$ is a soft FS* sheaf. Then we give

Definition 5.2.2. *We define the sheaf* $(\mathscr{L}^\#)^{p,q} = (\mathscr{L}^\#_{2,\mathrm{loc}})^{p,q}$ *to be the sheafification of the presheaf* $\{\mathscr{L}^{\#,p,q}(\Omega);\ \Omega \subset \boldsymbol{F}^n \text{ open}\}$, *where, for an open set* Ω *in* $\boldsymbol{F}^n$, *the section module* $\mathscr{L}^{\#,p,q}(\Omega)$ *is the space of all* $f \in L^{\#,p,q}(\Omega) = L^{\#,p,q}_{2,\mathrm{loc}}(\Omega)$ *such that* $\bar\partial f \in L^{\#,p,q+1}(\Omega) = L^{\#,p,q+1}_{2,\mathrm{loc}}(\Omega)$. *We put* $\mathscr{L}^\# = (\mathscr{L}^\#)^{0,0}$.

Then $(\mathscr{L}^\#)^{p,q}$ is a soft FS* sheaf. Then we have the following

Theorem 5.2.3 (The Dolbeault-Grothendieck resolution). *For some* $d>0$, *put* $U = \mathrm{int}\{z \in \boldsymbol{C}^{|n|};\ |\mathrm{Im}\, z''| - |\mathrm{Re}\, z''| < d\}^a$, *where* $\mathrm{int}\{\ \}^a$ *denotes the interior of the closure in* $\boldsymbol{F}^n$ *of a set* $\{\ \}$. *Then the sequence of sheaves over* U

$$0 \longrightarrow \mathscr{O}^{\#,p}|U \longrightarrow \mathscr{L}^{\#,p,0}|U \xrightarrow{\bar\partial} \mathscr{L}^{\#,p,1}|U \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{L}^{\#,p,|n|}|U \longrightarrow 0$$

is exact.

Proof. The exactness of the sequence

$$0 \longrightarrow \mathscr{O}^{\#,p}|U \longrightarrow \mathscr{L}^{\#,p,0}|U \xrightarrow{\bar\partial} \mathscr{L}^{\#,p,1}|U$$

is evident. In fact, let Ω be a relatively compact open set in U. Let $u \in \mathscr{L}^{\#,p,0}(\Omega)$ such that $\bar\partial u = 0$. Then, if we write u in the form

$$u = \sum_{|I|=p} u_I dz^I,$$

we have

$$\partial u_I / \partial \bar z_j = 0, \quad j = 1, 2, \dots, |n|,$$

from which we obtain

$$\sum_{j=1}^{|n|} \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar z_j} u_I = 0.$$

Since the operator (on $\Omega \cap \boldsymbol{R}^{2|n|}$)

$$\sum_{j=1}^{|n|} \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar z_j}$$

is elliptic, it follows from Weyl's Lemma that u_I's are analytic on Ω. So that we can conclude that u_I's are holomorphic. The fact that $u_I \in \mathscr{O}^\#(\Omega)$ follows from the exchangeability of L_2-norm and sup-norm for holomorphic functions. Thus the exactness of the above sequence was proved.

Next we have to prove the exactness of the sequence

$$\mathscr{L}^{\#,p,0}|U \xrightarrow{\bar\partial} \mathscr{L}^{\#,p,1}|U \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{L}^{\#,p,|n|}|U \longrightarrow 0.$$

For this purpose, we have only to prove the exactness of the sequence of stalks

$$\mathscr{L}_z^{\#,p,0} \xrightarrow{\bar{\partial}} \mathscr{L}_z^{\#,p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{L}_z^{\#,p,|n|} \longrightarrow 0$$

for every $z \in U$. But this is an easy consequence of Hörmander [4], Theorem 4.4.2 because every $z \in U$ has a fundamental system of $\mathcal{O}^{\#}$-pseudoconvex open neighborhoods. Q. E. D.

Corollary 1. *Let U be as in Theorem 5.2.3. For an open set Ω in U, we have the following isomorphism:*

$$H^q(\Omega, \mathcal{O}^{\#,p}) \cong \{f \in \mathscr{L}_{2,\mathrm{loc}}^{\#,p,q}(\Omega);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g \in \mathscr{L}_{2,\mathrm{loc}}^{\#,p,q-1}(\Omega)\}, \quad (p \geqq 0,\ q \geqq 1).$$

Corollary 2. *Let Ω be an $\mathcal{O}^{\#}$-pseudoconvex open set in $\boldsymbol{F}^n$. Then the equation $\bar{\partial}u=f$ has a solution $u \in \mathscr{L}_{2,\mathrm{loc}}^{\#,p,q}(\Omega)$ for every $f \in \mathscr{L}_{2,\mathrm{loc}}^{\#,p,q+1}(\Omega)$ such that $\bar{\partial}f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 5.1.4 and Corollary 1 to Theorem 5.2.3. Q. E. D.

We will now define the sheaf $L_{\#}=L_{\#,2,\mathrm{loc}}$ of germs of rapidly decreasing locally L_2-functions.

Definition 5.2.4. *We define the sheaf $L_{\#}$ to be the sheafification of the presheaf $\{L_{\#}(\Omega);\ \Omega \subset \boldsymbol{F}^n \text{ open}\}$, where, for an open set Ω in $\boldsymbol{F}^n$, the section module $L_{\#}(\Omega)$ is the space of all $f \in L_{\#,2,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^{|n|})$ such as, for any relatively compact open subset ω of Ω, there exists some positive δ such that $e(\delta\|z\|)f(z)|\omega \cap \boldsymbol{C}^{|n|} \in L_2(\omega \cap \boldsymbol{C}^{|n|})$.*

Then it is easy to see that $L_{\#}$ is a soft FS* sheaf.

Definition 5.2.5 (The sheaf $\mathscr{L}_{\#}^{p,q}$). *We define the sheaf $\mathscr{L}_{\#}^{p,q}=\mathscr{L}_{\#,2,\mathrm{loc}}^{p,q}$ to be the sheafification of the presheaf $\{\mathscr{L}_{\#}^{p,q}(\Omega);\ \Omega \subset \boldsymbol{F}^n \text{ open}\}$, where, for an open set Ω in $\boldsymbol{F}^n$, the section module $\mathscr{L}_{\#}^{p,q}(\Omega)$ is the space of all $f \in L_{\#}^{p,q}(\Omega)=L_{\#,2,\mathrm{loc}}^{p,q}(\Omega)$ such that $\bar{\partial}f \in L_{\#}^{p,q+1}(\Omega)=L_{\#,2,\mathrm{loc}}^{p,q+1}(\Omega)$. We put $\mathscr{L}_{\#}=\mathscr{L}_{\#}^{0,0}$.*

Then $\mathscr{L}_{\#}^{p,q}$ is a soft FS* sheaf. Then we have the following

Theorem 5.2.6 (The Dolbeault-Grothendieck resolution). *For some $d>0$, put $U=\mathrm{int}\{z \in \boldsymbol{C}^{|n|};\ |\mathrm{Im}\, z''|-|\mathrm{Re}\, z''|<d\}^a$. Then the sequence of sheaves over U*

$$0 \longrightarrow \mathcal{O}_{\#}^p|U \longrightarrow \mathscr{L}_{\#}^{p,0}|U \xrightarrow{\bar{\partial}} \mathscr{L}_{\#}^{p,1}|U \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{L}_{\#}^{p,|n|}|U \longrightarrow 0$$

is exact.

Proof. The exactness of the sequence

$$0 \longrightarrow \mathcal{O}_{\#}^p|U \longrightarrow \mathscr{L}_{\#}^{p,0}|U \xrightarrow{\bar{\partial}} \mathscr{L}_{\#}^{p,1}|U$$

can be proved by asimilar way to that of Theorem 5.2.3.

Next we have to prove the exactness of the sequence

$$\mathscr{L}_{\#}^{p,0}|U \xrightarrow{\bar\partial} \mathscr{L}_{\#}^{p,1}|U \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{L}_{\#}^{p,|n|}|U \longrightarrow 0.$$

Let $z=(z', z'')\in U$ and Ω an open neighborhood of z of the form $\Omega'\times\Omega''$, where Ω' is an open neighborhood of z' in $\tilde{\boldsymbol{C}}^{n_1}$ and Ω'' is an open neighborhood of z'' in $\boldsymbol{E}^{n_2}$ of the form $V_{\delta,A}$ in Lemma 3.2.7 for some δ and A such as $0<\delta<1$ and $A>0$. Let f be an element in $\mathscr{L}_{\#}^{p,q+1}(\Omega)$ such that $\bar\partial f=0$. Then, for some $\varepsilon>0$, we have $f\cdot h_\varepsilon(z)\in\mathscr{L}^{\#,p,q+1}(\Omega)$, where we put $h_\varepsilon(z)=(\prod_{j=1}^{n_1}\cosh(\varepsilon z_j))\cdot\cosh(\varepsilon\sqrt{(z'')^2}/2)$. Since $\bar\partial(f\cdot h_\varepsilon(z))=0$, we can find some $v\in\mathscr{L}^{\#,p,q}(\omega)$ for some open neighborhood $\omega=\omega'\times\omega''$ of z with $z'\in\omega'\subset\Omega'$ and $z''\in\omega''\subset\Omega''$ such that $\bar\partial v=f\cdot h_\varepsilon(z)$. Here we may assume that $h_\varepsilon(z)\neq 0$ on $\omega\cap\boldsymbol{C}^{|n|}$. Then $u=v/h_\varepsilon(z)$ belongs to $\mathscr{L}_{\#}^{p,q}(\omega)$ and $\bar\partial u=f$ holds. This completes the proof. Q.E.D.

Corollary 1. *Let U be as in Theorem 5.2.6. For an open set Ω in U, we have the following isomorphism:*

$$H^q(\Omega,\mathcal{O}_{\#}^p)\cong\{f\in\mathscr{L}_{\#;2,\mathrm{loc}}^{p;q}(\Omega);\ \bar\partial f=0\}/\{\bar\partial g;\ g\in\mathscr{L}_{\#;2,\mathrm{loc}}^{p;q-1}(\Omega)\},\quad (p\geqq 0,\ q\geqq 1).$$

Corollary 2. *Let Ω be an $\mathcal{O}^{\#}$-pseudoconvex open set in $\boldsymbol{F}^n$. Then the equation $\bar\partial u=f$ has a solution $u\in\mathscr{L}_{\#;2,\mathrm{loc}}^{p;q}(\Omega)$ for every $f\in\mathscr{L}_{\#;2,\mathrm{loc}}^{p;q+1}(\Omega)$ such that $\bar\partial f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 5.1.7 and Corollary 1 to Theorem 5.2.6. Q.E.D.

Now, for later applications, we will construct another soft resolutions of $\mathcal{O}^{\#}$ and $\mathcal{O}_{\#}$.

At first, we will give some preliminary facts.

For an integer $s\geqq 0$ and for an open set Ω in $\boldsymbol{F}^n$, we put

$W_s^{\#}(\Omega)=\{f\in W_{s,\mathrm{loc}}(\Omega\cap\boldsymbol{C}^{|n|})$; *for any positive* ε *and for every relatively compact open subset* ω *of* Ω *and for every* $\alpha\in\bar{\boldsymbol{N}}^{2|n|}$ *such that* $|\alpha|\leqq s$, $(e(-\varepsilon\|z\|)f^{(\alpha)}(z))|\omega\cap\boldsymbol{C}^{|n|}\in L_2(\omega\cap\boldsymbol{C}^{|n|})$ *holds*$\}$,

and denote by $W_s^{\#,p,q}(\Omega)$ the space of all differential forms of type (p, q) whose coefficients in $W_s^{\#}(\Omega)$. Then we have the following

Theorem 5.2.7. *Let Ω be an $\mathcal{O}^{\#}$-pseudoconvex open set in $\boldsymbol{F}^n$ and s an integer such as $0\leqq s\leqq\infty$. Then, for every $f\in W_s^{\#,p,q+1}(\Omega)$ such as $\bar\partial f=0$, we can find a solution $u\in W_{s+1}^{\#;p,q}(\Omega)$ of the equation $\bar\partial u=f$. Every solution of the equation $\bar\partial u=f$ has this property when $q=0$.*

Proof. (a) First assume that $q=0$. We know, from Corollary 2 to Theorem

5.2.3, that the equation $\bar{\partial}u=f$ has a solution $u=\sum' u_I dz^I \in \mathscr{L}^{\sharp;p,0}_{2,\mathrm{loc}}(\Omega)$ because $f\in \mathscr{L}^{\sharp;p,1}_{2,\mathrm{loc}}(\Omega)$ and $\bar{\partial}f=0$. The equation $\bar{\partial}u=f$ means that

$$\partial(u_I|\Omega \cap \boldsymbol{C}^{|n|})/\partial \bar{z}_j = f_{I,j}|\Omega \cap \boldsymbol{C}^{|n|} \in W_{s,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^{|n|})$$

for all I and j. Thus, by Hörmander [4], Theorem 4.2.5, we have $u_I \in W_{s+1,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^{|n|})$. Then, by Nagamachi [25], Lemma 4.3, we can conclude that $u_I \in W^{\sharp}_{s+1}(\Omega)$.

(b) Next we assume that $q>0$. Then, by Hörmander [4], Theorem 4.2.5, we can find $u \in W^{p,q}_{s+1,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^{|n|})$ such that $\bar{\partial}u=f$. Then, by Nagamachi [25], Lemma 4.2, we can conclude that $u \in W^{\sharp;p,q}_{s+1}(\Omega)$. Q. E. D.

Now we will define the sheaf $\mathscr{E}^{\sharp}$ of germs of slowly increasing C^{∞}-functions over $\boldsymbol{F}^n$.

Definition 5.2.8. *We define the sheaf $\mathscr{E}^{\sharp}$ to be the sheafification of the presheaf $\{\mathscr{E}^{\sharp}(\Omega);\ \Omega\subset \boldsymbol{F}^n$ open$\}$, where, for an open set Ω in $\boldsymbol{F}^n$, the section module $\mathscr{E}^{\sharp}(\Omega)$ is defined as follows*:

$\mathscr{E}^{\sharp}(\Omega)=\{f\in \mathscr{E}(\Omega \cap \boldsymbol{C}^{|n|})$; *for any positive ε and any compact set K in Ω and any $\alpha \in \bar{\boldsymbol{N}}^{2|n|}$ the estimate* $\sup\{|f^{(\alpha)}(z)|e(-\varepsilon|z|);\ z\in K\cap \boldsymbol{C}^{|n|}\}<\infty$ *holds*$\}$.

Then it is easy to see that $\mathscr{E}^{\sharp}$ is a soft nuclear Fréchet sheaf. Then we have the following

Theorem 5.2.9. *Let Ω be an $\mathscr{O}^{\sharp}$-pseudoconvex open set in $\boldsymbol{F}^n$. Then the equation $\bar{\partial}u=f$ has a solution $u\in \mathscr{E}^{\sharp,p,q}(\Omega)$ for every $f\in \mathscr{E}^{\sharp,p,q+1}(\Omega)$ such that $\bar{\partial}f=0$. Every solution of the equation $\bar{\partial}u=f$ has this property when $q=0$.*

Proof. Since $f\in W^{\sharp,p,q+1}_{s}(\Omega)$ for every integer $s\geqq 0$, we can find $u\in W^{\sharp;p,q}_{s+1}(\Omega)$ for every s. But, by the well-known Sobolev lemma, we have

$$W^{\sharp;p,q}_{s+2|n|}(\Omega)\subset C^{\sharp,p,q}_{s}(\Omega),$$

where we put

$C^{\sharp}_{s}(\Omega)=\{f\in C^{s}(\Omega \cap \boldsymbol{C}^{|n|})$; *for any positive ε and any compact set K in Ω and any $\alpha\in \bar{\boldsymbol{N}}^{2|n|}$ such that $|\alpha|\leqq s$, the estimate* $\sup\{|f^{(\alpha)}(z)|e(-\varepsilon|z|);$ $z\in K\cap \boldsymbol{C}^{|n|}\}<\infty$ *holds*$\}$.

Thus we have $u\in \mathscr{E}^{\sharp,p,q}(\Omega)$. Q. E. D.

Then we have the following

Theorem 5.2.10 (The Dolbeault-Grothendieck resolution). *For some $d>0$, put $U=\mathrm{int}\,\{z\in \boldsymbol{C}^{|n|};\ |\mathrm{Im}\, z''|-|\mathrm{Re}\, z''|<d\}^a$. Then the sequence of sheaves over U*

$$0 \longrightarrow \mathscr{O}^{\sharp,p}|U \longrightarrow \mathscr{E}^{\sharp,p,0}|U \xrightarrow{\bar{\partial}} \mathscr{E}^{\sharp,p,1}|U \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{E}^{\sharp,p,|n|}|U \longrightarrow 0$$

is exact

Proof. It follows immediately from Theorem 5.2.9. Q.E.D.

Corollary. *We use notations in Theorem* 5.2.10. *For an open set Ω in U, we have the following isomorphism*:

$$H^q(\Omega, \mathcal{O}^{\#,p}) \cong \{f \in \mathcal{E}^{\#,p,q}(\Omega);\ \bar{\partial} f=0\}/\{\bar{\partial} g;\ g \in \mathcal{E}^{\#,p,q-1}(\Omega)\}, \quad (p \geqq 0,\ q \geqq 1).$$

Now we will define the sheaf $\mathcal{E}_\#$ of germs of rapidly decreasing C^∞-functions over $\boldsymbol{F}^n$.

Definition 5.2.11. *We define the sheaf $\mathcal{E}_\#$ to be the sheafification of the presheaf $\{\mathcal{E}_\#(\Omega);\ \Omega \subset \boldsymbol{F}^n$ open$\}$, where the section module $\mathcal{E}_\#(\Omega)$ on an open set Ω in $\boldsymbol{F}^n$ is the space of all C^∞-functions on $\Omega \cap \boldsymbol{C}^{|n|}$ such that, for any compact set K in Ω and any $\alpha \in \bar{\boldsymbol{N}}^{2|n|}$, there exists some positive constant δ so that the estimate* $\sup\{|f^{(\alpha)}(z)|e(\delta|z|);\ z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ *holds.*

Then $\mathcal{E}_\#$ becomes a soft nuclear Fréchet sheaf. Then we have the following

Theorem 5.2.12 (The Dolbeault-Grothendieck resolution). *Put* $U = \operatorname{int}\{z \in \boldsymbol{C}^{|n|};\ |\operatorname{Im} z''| - |\operatorname{Re} z''| < d\}^a$ *for some $d>0$. Then the sequence of sheaves over U*

$$0 \longrightarrow \mathcal{O}^p_\# | U \longrightarrow \mathcal{E}^{p,0}_\# | U \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1}_\# | U \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{p,|n|}_\# | U \longrightarrow 0$$

is exact.

Proof. Let $z=(z', z'') \in U$ and Ω an open neighborhood of z of the form $\Omega' \times \Omega''$, where Ω' is an open neighborhood of z' in $\tilde{\boldsymbol{C}}^{n_1}$ and Ω'' is an open neighborhood of z'' in $\boldsymbol{E}^{n_2}$ of the form $V_{\delta,A}$ in Lemma 3.2.7 for some δ and A such as $0<\delta<1$ and $A>0$. Let f be an element in $\mathcal{E}^{p,q+1}_\#(\Omega)$ such that $\bar{\partial} f=0$. Then, for some $\varepsilon>0$, we have $f \cdot h_\varepsilon(z) \in \mathcal{E}^{\#,p,q+1}(\Omega)$, where we put $h_\varepsilon(z)=(\prod_{j=1}^{n_1} (\cosh(\varepsilon z_j)) \cdot \cosh(\varepsilon\sqrt{(z'')^2}/2)$. Since $\bar{\partial}(f \cdot h_\varepsilon(z))=0$, we can find some $v \in \mathcal{E}^{\#,p,q}(\omega)$ for some open neighborhood $\omega=\omega' \times \omega''$ of z with $z' \in \omega' \subset \Omega'$ and $z'' \in \omega'' \subset \Omega''$ such that $\bar{\partial} v = f \cdot h_\varepsilon(z)$. Here we may assume that $h_\varepsilon(z) \neq 0$ on $\omega \cap \boldsymbol{C}^{|n|}$. Then $u=v/h_\varepsilon(z)$ belongs to $\mathcal{E}^{p,q}_\#(\omega)$ and $\bar{\partial} u = f$ holds. This completes the proof. Q.E.D.

Corollary 1. *Let U be as in Theorem* 5.2.12. *For an open set Ω in U, we have the following isomorphism*:

$$H^q(\Omega, \mathcal{O}^p_\#) \cong \{f \in \mathcal{E}^{p,q}_\#(\Omega);\ \bar{\partial} f=0\}/\{\bar{\partial} g;\ g \in \mathcal{E}^{p,q-1}_\#(\Omega)\}, \quad (p \geqq 0,\ q \geqq 1).$$

Corollary 2. *Let Ω be an $\mathcal{O}^\#$-pseudoconvex open set in $\boldsymbol{F}^n$. Then the equation $\bar{\partial} u = f$ has a solution $u \in \mathcal{E}^{p,q}_\#(\Omega)$ for every $f \in \mathcal{E}^{p,q+1}_\#(\Omega)$ such that $\bar{\partial} f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 5.1.7 and Corollary 1 to Theorem 5.2.12.

Q. E. D.

5.3. The Serre duality theorem

In this section we will prove the Serre duality theorem.

Theorem 5.3.1. *Let Ω be such an open set as in Theorem 5.3.1 and assume that* $\dim H^p(\Omega, \mathcal{O}^\#) < \infty$ *holds for* $p \geqq 1$. *Then we have the isomorphism* $[H^p(\Omega, \mathcal{O}^\#)]' \cong H_c^{|n|-p}(\Omega, \mathcal{O}_\#)$, $(0 \leqq p \leqq |n|)$.

Proof. By virtue of Corollary 1 to Theorem 5.2.3 and Corollary 1 to Theorem 5.2.6, cohomology groups $H^p(\Omega, \mathcal{O}^\#)$ and $H_c^{|n|-p}(\Omega, \mathcal{O}_\#)$ are cohomology groups respectively of the complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathscr{L}^{\#,0,0}(\Omega) & \xrightarrow{\bar\partial} & \mathscr{L}^{\#,0,1}(\Omega) & \xrightarrow{\bar\partial} & \cdots & \xrightarrow{\bar\partial} & \mathscr{L}^{\#,0,|n|}(\Omega) & \longrightarrow & 0 \\ & & \updownarrow & & \updownarrow & & & & \updownarrow & & \\ 0 & \longleftarrow & \mathscr{L}^{0,|n|}_{\#,c}(\Omega) & \xleftarrow{-\bar\partial} & \mathscr{L}^{0,|n|-1}_{\#,c}(\Omega) & \xleftarrow{-\bar\partial} & \cdots & \xleftarrow{-\bar\partial} & \mathscr{L}^{0,0}_{\#,c}(\Omega) & \longleftarrow & 0. \end{array}$$

Here the upper complex is composed of FS* spaces and the lower complex is composed of DFS* spaces. Since the ranges of operators $\bar\partial$ in the upper complex are all closed by virtue of Schwartz' Lemma (cf. Komatsu [20]), the ranges of operators $-\bar\partial = (\bar\partial)'$ in the lower complex are also all closed. Hence we have the isomorphism

$$[H^p(\Omega, \mathcal{O}^\#)]' \cong H_c^{|n|-p}(\Omega, \mathcal{O}_\#)$$

by virtue of Serre's Lemma (cf. Komatsu [20]). Q. E. D.

5.4. The Martineau-Harvey Theorem

In this section we will prove the Martineau-Harvey Theorem.

Theorem 5.4.1. *Let K be a compact set in $\boldsymbol{F}^n$ such that it has an $\mathcal{O}^\#$-pseudoconvex open neighborhood Ω and satisfies the conditions $H^p(K, \mathcal{O}_\#) = 0$ $(p \geqq 1)$. Then we have $H_K^p(\Omega, \mathcal{O}^\#) = 0$ for $p \neq |n|$ and isomorphisms $H_K^{|n|}(\Omega, \mathcal{O}^\#) \cong H^{|n|-1}(\Omega \backslash K, \mathcal{O}^\#) \cong \mathcal{O}_\#(K)'$.*

Remark. If a compact set K in $\boldsymbol{F}^n$ has a fundamental system of $\mathcal{O}^\#$-pseudoconvex open neighborhoods, it satisfies the assumptions in Theorem 5.4.1.

Proof. It goes in a similar way to that of Theorem 1.5.1. Q. E. D.

5.5. The Sato Theorem

In this section we will prove the pure-codimensionality of $\boldsymbol{D}^n$ with respect to $\mathcal{O}^\#$. Then we will realize mixed Fourier hyperfunctions as "boundary values" of slowly increasing holomorphic functions or as (relative) cohomology classes of slowly increasing holomorphic functions.

Theorem 5.5.1 (The Sato Theorem). *Let Ω be an open set in $\boldsymbol{D}^n$ and V an open set in $\boldsymbol{F}^n$ which contains Ω as its closed subsets. Then we have the following*

(1) *The relative cohomology groups $H^p_\Omega(V, \mathcal{O}^\sharp)$ are zero for $p \neq |n|$.*

(2) *The presheaf over $\boldsymbol{D}^n$*

$$\Omega \longrightarrow H^{|n|}_\Omega(V, \mathcal{O}^\sharp)$$

is a sheaf.

(3) *This sheaf (2) is isomorphic to the sheaf $\mathscr{P}$ of mixed Fourier hyperfunctions.*

Proof. (1) It goes in a similar way to that of Kawai [19], p. 482.

(2) By (1) and by the theorem II.3.18 of Komatsu [21], we have the conclusion.

(3) Consider the following exact sequence of relative cohomology groups

$$\begin{aligned}0 &\longrightarrow H^0_{\partial\Omega}(V, \mathcal{O}^\sharp) \longrightarrow H^0_{\Omega^a}(V, \mathcal{O}^\sharp) \longrightarrow H^0_\Omega(V, \mathcal{O}^\sharp)\\ &\longrightarrow H^1_{\partial\Omega}(V, \mathcal{O}^\sharp) \longrightarrow \quad \cdots \quad \longrightarrow H^{|n|-1}_\Omega(V, \mathcal{O}^\sharp)\\ &\longrightarrow H^{|n|}_{\partial\Omega}(V, \mathcal{O}^\sharp) \longrightarrow H^{|n|}_{\Omega^a}(V, \mathcal{O}^\sharp) \longrightarrow H^{|n|}_\Omega(V, \mathcal{O}^\sharp)\\ &\longrightarrow H^{|n|+1}_{\partial\Omega}(V, \mathcal{O}^\sharp) \longrightarrow \cdots.\end{aligned}$$

Then, by (1) and by the Martineau-Harvey Theorem, we have $H^{|n|-1}(V, \mathcal{O}^\sharp)=0$, $H^{|n|+1}_{\partial\Omega}(V, \mathcal{O}^\sharp)=0$. Thus we have the exact sequence

$$0 \longrightarrow H^{|n|}_{\partial\Omega}(V, \mathcal{O}^\sharp) \longrightarrow H^{|n|}_{\Omega^a}(V, \mathcal{O}^\sharp) \longrightarrow H^{|n|}_\Omega(V, \mathcal{O}^\sharp) \longrightarrow 0.$$

Since, by the Martineau-Harvey Theorem, we have isomorphisms

$$H^{|n|}_{\partial\Omega}(V, \mathcal{O}^\sharp) \cong \mathscr{A}_\sharp(\partial\Omega)', \; H^{|n|}_{\Omega^a}(V, \mathcal{O}^\sharp) \cong \mathscr{A}_\sharp(\Omega^a)',$$

we obtain the isomorphism

$$H^{|n|}_\Omega(V, \mathcal{O}^\sharp) \cong \mathscr{A}_\sharp(\Omega^a)'/\mathscr{A}_\sharp(\partial\Omega)' = \mathscr{P}(\Omega).$$

Thus the sheaf $\Omega \to H^{|n|}_\Omega(V, \mathcal{O}^\sharp)$ is isomorphic to the sheaf $\mathscr{P}$ of mixed Fourier hyperfunctions over $\boldsymbol{D}^n$. Q.E.D.

Let Ω be an open set in $\boldsymbol{D}^n$. Then there exists an $\mathcal{O}^\sharp$-pseudoconvex open neighborhood V of Ω such that $V \cap \boldsymbol{D}^n = \Omega$ (cf. Ito [11], Theorem 8.1.9). We put $V_0 = V$ and $V_j = V \backslash \{z \in V; \operatorname{Im} z_j = 0\}^a$, $j = 1, 2, \ldots, |n|$. Then $\mathfrak{U} = \{V_0, V_1, \ldots, V_{|n|}\}$ and $\mathfrak{U}' = \{V_1, \ldots, V_{|n|}\}$ cover V and $V \backslash \Omega$ respectively. Since V_j and their intersections are also $\mathcal{O}^\sharp$-pseudoconvex open sets, the covering $(\mathfrak{U}, \mathfrak{U}')$ satisfies the conditions of Leray's Theorem (cf. Komatsu [21]). Thus, by Leray's Theorem, we obtain the isomorphism $H^{|n|}_\Omega(V, \mathcal{O}^\sharp) = H^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^\sharp)$. Since the covering $\mathfrak{U}$ is composed of only $|n|+1$ open sets V_j $(j=0, 1, \ldots, |n|)$, we easily obtain the isomorphisms

$$Z^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^\sharp) \cong \mathcal{O}^\sharp(\cap_j V_j),$$

$$C^{|n|-1}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\#}) \cong \bigoplus_{j=1}^{|n|} \mathcal{O}^{\#}(\bigcap_{i \neq j} V_i).$$

Hence we have

$$\delta C^{|n|-1}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\#}) \cong \sum_{j=1}^{|n|} \mathcal{O}^{\#}(\bigcap_{i \neq j} V_i) | V_1 \cap \cdots \cap V_{|n|}.$$

Thus we have the isomorphisms

$$\begin{aligned} H_{\Omega}^{|n|}(V, \mathcal{O}^{\#}) &\cong H^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\#}) \\ &= Z^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\#}) / \delta C^{|n|-1}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\#}) \\ &\cong \mathcal{O}^{\#}(\cap_j V_j) / \sum_{j=1}^{|n|} \mathcal{O}^{\#}(\bigcap_{i \neq j} V_i). \end{aligned}$$

Thus we have the following

Theorem 5.5.2. *We use notations as above. Then we have the isomorphisms*

$$H_{\Omega}^{|n|}(V, \mathcal{O}^{\#}) \cong H^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\#}) \cong \mathcal{O}^{\#}(\cap_j V_j) / \sum_{j=1}^{|n|} \mathcal{O}^{\#}(\bigcap_{i \neq j} V_i).$$

At last we will realize mixed Fourier analytic functionals with certain compact carrier as (relative) cohomology classes with coefficients in $\mathcal{O}^{\#}$.

Let K be a compact set in $\boldsymbol{F}^n$ of the from $K = K_1 \times \cdots \times K_{|n|}$ with compact sets K_j in $\tilde{\boldsymbol{C}}$ for $j = 1, 2, \ldots, n_1$ and in $\boldsymbol{E}$ for $j = n_1 + 1, \ldots, |n|$. Assume that K admits a fundamental system of $\mathcal{O}^{\#}$-pseudoconvex open neighborhoods. Then we have

$$H^p(K, \mathcal{O}_{\#}) = 0 \quad \text{for} \quad p > 0.$$

By virtue of the Martineau-Harvey Theorem, there exists the isomorphism

$$\mathcal{O}_{\#}(K)' \cong H_K^{|n|}(\Omega, \mathcal{O}^{\#}).$$

Here Ω denotes an open neighborhood of K. Further assume that there exists an $\mathcal{O}^{\#}$-pseudoconvex open neighborhood Ω of K such that

$$\Omega_j = \Omega \backslash \{z \in \boldsymbol{C}^{|n|};\, z_j \in K_j \cap \boldsymbol{C}\}^a$$

is also an $\mathcal{O}^{\#}$-pseudoconvex open set for $j = 1, 2, \ldots, |n|$. Put $\Omega_0 = \Omega$. Then $\mathfrak{U} = \{\Omega_0, \Omega_1, \ldots, \Omega_{|n|}\}$ and $\mathfrak{U}' = \{\Omega_1, \Omega_2, \ldots, \Omega_{|n|}\}$ form acyclic coverings of Ω and $\Omega \backslash K$. Set

$$\Omega \# K = \bigcap_{j=1}^{|n|} \Omega_j, \quad \Omega^j = \bigcap_{i \neq j} \Omega_i.$$

Let $\sum_j \mathcal{O}^{\#}(\Omega^j)$ be the image in $\mathcal{O}^{\#}(\Omega \# K)$ of $\prod_{j=1}^{|n|} \mathcal{O}^{\#}(\Omega^j)$ by the mapping

$$(f_j)_{j=1}^{|n|} \longrightarrow \sum_{j=1}^{|n|} (-1)^{j+1} f'_j,$$

where f'_j denotes the restriction of f_j to $\Omega \# K$.

Then, by a similar method to that of Theorem 5.5.2, we have the following

Theorem 5.5.3. *We use the notations as above. Then we have the isomorphisms*

$$\mathcal{O}_\#(K)' \cong H_K^{|n|}(\Omega, \mathcal{O}^\#) \cong H^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^\#) \cong \mathcal{O}^\#(\Omega \# K)/\sum_j \mathcal{O}^\#(\Omega^j).$$

Chapter 6. Case of the sheaf ${}^E\mathcal{O}^\#$

6.1. The Dolbeault-Grothendieck resolution of ${}^E\mathcal{O}^\#$

In this section we will construct a soft resolution of ${}^E\mathcal{O}^\#$. In this chapter we always assume that E is a Fréchet space whose topology is defined by a family $\mathscr{T} = \mathscr{T}_E$ of continuous seminorms of E.

At first we will define sheaves ${}^E\mathcal{O}^\#$ and ${}^E\mathscr{E}^\#$.

Definition 6.1.1 (The sheaf ${}^E\mathcal{O}^\#$ of germs of slowly increasing E-valued holomorphic functions over F^n). *We define the sheaf ${}^E\mathcal{O}^\#$ to be the sheafification of the presheaf $\{\mathcal{O}^\#(\Omega; E)\}$, where, for an open set Ω in F^n, the module $\mathcal{O}^\#(\Omega; E)$ is defined as follows:*

$\mathcal{O}^\#(\Omega; E) = \{f \in \mathcal{O}(\Omega \cap C^{|n|}; E)$; *for any positive ε and any compact set K in Ω and any $q \in \mathscr{T}$,* $\sup\{q(f(z))e(-\varepsilon|z|); z \in K \cap C^{|n|}\} < \infty$ *holds*$\}$.

We call this sheaf ${}^E\mathcal{O}^\#$ the sheaf of germs of slowly increasing E-valued holomorphic functions.

Definition 6.1.2 (The sheaf ${}^E\mathscr{E}^\#$ of germs of slowly increasing E-valued C^∞-functions). *We define ${}^E\mathscr{E}^\#$ to be the sheafification of the presheaf $\{\mathscr{E}^\#(\Omega; E)\}$, where, for an open set Ω in F^n, the module $\mathscr{E}^\#(\Omega; E)$ is defined as follows:*

$\mathscr{E}^\#(\Omega; E) = \{f \in \mathscr{E}^\#(\Omega \cap C^{|n|}; E)$; *for any positive ε and any compact set K in Ω and any $\alpha \in \overline{N}^{2|n|}$ and any $q \in \mathscr{T}$,* $\sup\{q(f^{(\alpha)}(z))e(-\varepsilon|z|); z \in K \cap C^{|n|}\} < \infty$ *holds*$\}$.

Then the sheaf ${}^E\mathscr{E}^\#$ is a soft Fréchet sheaf and we have the following

Theorem 6.1.3 (The Dolbeault-Grothendieck resolution of ${}^E\mathcal{O}^{\#,p}$). *The sequence of sheaves*

$$0 \longrightarrow {}^E\mathcal{O}^{\#,p}|U \longrightarrow {}^E\mathscr{E}^{\#,p,0}|U \xrightarrow{\bar\partial} {}^E\mathscr{E}^{\#,p,1}|U \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {}^E\mathscr{E}^{\#,p,|n|}|U \longrightarrow 0$$

is exact, where $U=\mathrm{int}\,\{z\in C^{|n|};\ |\mathrm{Im}\,z''|-|\mathrm{Re}\,z''|<d\}^a$ *for some* $d>0$.

Proof. The exactness of the sequence

$$0\longrightarrow {}^E\mathcal{O}^{\#,p}|U\longrightarrow {}^E\mathcal{E}^{\#,p,0}|U\xrightarrow{\bar\partial}{}^E\mathcal{E}^{\#,p,1}|U$$

is evident.

Next the exactness of the sequence

$${}^E\mathcal{E}^{\#,p,0}|U\xrightarrow{\bar\partial}{}^E\mathcal{E}^{\#,p,1}|U\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}{}^E\mathcal{E}^{\#,p,|n|}|U\longrightarrow 0$$

follows from the following

Lemma 6.1.4. *Let* Ω *be an* $\mathcal{O}^{\#}$*-pseudoconvex open set in* F^n. *Then the equation* $\bar\partial u=f$ *has a solution* $u\in\mathcal{E}^{\#,p,q}(\Omega;E)$ *for every* $f\in\mathcal{E}^{\#,p,q+1}(\Omega;E)$ *such that* $\bar\partial f=0$. *Here* $p,q\geqq 0$.

Proof of Lemma 6.1.4. If we put $Z^{\#,p,q+1}(\Omega)=\{f\in\mathcal{E}^{\#,p,q+1}(\Omega);\ \bar\partial f=0\}$ and $Z^{\#,p,q+1}(\Omega;E)=\{f\in\mathcal{E}^{\#,p,q+1}(\Omega;E);\ \bar\partial f=0\}$, then $Z^{\#,p,q+1}(\Omega)$ is a nuclear Fréchet space and

$$Z^{\#,p,q+1}(\Omega;E)\cong Z^{\#,p,q+1}(\Omega)\hat{\otimes}E$$

holds. By virtue of Theorem 5.2.9, we have an exact sequence

$$\mathcal{E}^{\#,p,q}(\Omega)\xrightarrow{\bar\partial}Z^{\#,p,q+1}(\Omega)\longrightarrow 0$$

for the $\mathcal{O}^{\#}$-pseudoconvex open set Ω. Then, since we have also

$$\mathcal{E}^{\#,p,q}(\Omega;E)\cong\mathcal{E}^{\#,p,q}(\Omega)\hat{\otimes}E,$$

we have an exact sequence

$$\mathcal{E}^{\#,p,q}(\Omega;E)\xrightarrow{\bar\partial}Z^{\#,p,q+1}(\Omega;E)\longrightarrow 0$$

by virtue of Trèves [36], Proposition 4.3.9. Q.E.D.

This completes the proof of Theorem 6.1.3. Q.E.D.

Corollary. *We use notations in Theorem* 6.1.3. *For an open set* Ω *in* U, *we have the following isomorphism*:

$$H^q(\Omega,{}^E\mathcal{O}^{\#,p})\cong\{f\in\mathcal{E}^{\#,p,q}(\Omega;E);\ \bar\partial f=0\}/\{\bar\partial g;\ g\in\mathcal{E}^{\#,p,q-1}(\Omega;E)\},\ (p\geqq 0,\ q\geqq 1).$$

Proof. It follows from Theorem 6.1.3 and Komatsu [21], Theorems II.2.9 and II.2.19. Q.E.D.

6.2. The Oka-Cartan-Kawai Theorem B

We will prove the Oka-Cartan-Kawai Theorem B for the sheaf ${}^E\mathcal{O}^{\#}$.

Theorem 6.2.1 (The Oka-Cartan-Kawai Theorem B). *For any $\mathcal{O}^{\sharp}$-pseudoconvex open set Ω in $\boldsymbol{F}^n$, we have $H^q(\Omega, {}^E\mathcal{O}^{\sharp,p})=0$ for $p\geqq 0$ and $q\geqq 1$.*

Proof. Since we have, by the Oka-Cartan-Kawai Theorem B for $\mathcal{O}^{\sharp}$,

$$H^q(\Omega, \mathcal{O}^{\sharp,p})=0 \qquad (p\geqq 0 \text{ and } q\geqq 1),$$

the complex obtained from Theorem 5.2.10:

$$\mathcal{E}^{\sharp,p,0}(\Omega) \xrightarrow{\bar\partial} \mathcal{E}^{\sharp,p,1}(\Omega) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}^{\sharp,p,|n|}(\Omega) \longrightarrow 0$$

is exact. Since $\mathcal{E}^{\sharp,p,q}(\Omega)$'s are nuclear Fréchet spaces and E is a Fréchet space, the complex

$$\mathcal{E}^{\sharp,p,0}(\Omega; E) \xrightarrow{\bar\partial} \mathcal{E}^{\sharp,p,1}(\Omega; E) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}^{\sharp,p,|n|}(\Omega; E) \longrightarrow 0$$

is also exact by virtue of the isomorphism

$$\mathcal{E}^{\sharp,p,q}(\Omega; E)\cong \mathcal{E}^{\sharp,p,q}(\Omega)\hat{\otimes} E$$

and Ion and Kawai [5], Theorem 1.10. Hence we obtain

$$H^q(\Omega, {}^E\mathcal{O}^{\sharp,p})=0, \qquad (p\geqq 0, q\geqq 1).$$

This completes the proof. Q.E.D.

Corollary. *Let Ω be an $\mathcal{O}^{\sharp}$-pseudoconvex open set in $\boldsymbol{F}^n$. Then the equation $\bar\partial u=f$ has a solution $u\in\mathcal{E}^{\sharp,p,q}(\Omega; E)$ for every $f\in\mathcal{E}^{\sharp,p,q+1}(\Omega; E)$ such that $\bar\partial f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 6.2.1 and Corollary to Theorem 6.1.3. Q.E.D.

6.3. The Serre duality theorem

Theorem 6.3.1. *Let Ω be an open set in $\boldsymbol{F}^n$ such that, for any $z\in\Omega\cap\boldsymbol{C}^{|n|}$, $|\mathrm{Im}\, z''|-|\mathrm{Re}\, z''|<d$ holds for some constant $d>0$ independent of $z\in\Omega\cap\boldsymbol{C}^{|n|}$ and such that* dim $H^p(\Omega, \mathcal{O}^{\sharp})<\infty$ *holds* $(p\geqq 1)$. *Then we have the isomorphism $H^p(\Omega, {}^E\mathcal{O}^{\sharp})\cong L(H_c^{|n|-p}(\Omega, \mathcal{O}_{\sharp}); E)$, $0\leqq p\leqq |n|$.*

Proof. By a similar method to Junker [15], Lemma 3.5, we can obtain the isomorphism $H^p(\Omega, {}^E\mathcal{O}^{\sharp})\cong H^p(\Omega, \mathcal{O}^{\sharp})\hat{\otimes}_\pi E$. Then, by Theorem 5.3.1, we have the following isomorphisms

$$\begin{aligned} H^p(\Omega, {}^E\mathcal{O}^{\sharp}) &\cong H^p(\Omega, \mathcal{O}^{\sharp})\hat{\otimes}_\pi E \cong [H_c^{|n|-p}(\Omega, \mathcal{O}_{\sharp})]'\hat{\otimes}_\pi E \\ &\cong L(H_c^{|n|-p}(\Omega, \mathcal{O}_{\sharp}); E). \end{aligned}$$

Q.E.D.

6.4. The Martineau-Harvey Theorem

Theorem 6.4.1. *Let K be a compact set in $\boldsymbol{F}^n$ such that it has an $\mathcal{O}^{\sharp}$-*

pseudoconvex open neighborhood Ω and satisfies the conditions $H^p(K, \mathcal{O}_\sharp)=0$ $(p \geqq 1)$. Then we have $H_K^p(\Omega, {}^E\mathcal{O}^\sharp)=0$ for $p \neq |n|$ and isomorphisms $H_K^{|n|}(\Omega, {}^E\mathcal{O}^\sharp) \cong H^{|n|-1}(\Omega \backslash K, {}^E\mathcal{O}^\sharp) \cong L(\mathcal{O}_\sharp(K); E)$.

Proof. We can assume that Ω is an $\mathcal{O}^\sharp$-pseudoconvex open neighborhood of K. Then, in the long exact sequence of cohomology groups (cf. Komatsu [21], Theorem II.3.2):

$$\begin{aligned} 0 &\longrightarrow H_K^0(\Omega, {}^E\mathcal{O}^\sharp) \longrightarrow H^0(\Omega, {}^E\mathcal{O}^\sharp) \longrightarrow H^0(\Omega \backslash K, {}^E\mathcal{O}^\sharp) \\ &\longrightarrow H_K^1(\Omega, {}^E\mathcal{O}^\sharp) \longrightarrow H^1(\Omega, {}^E\mathcal{O}^\sharp) \longrightarrow H^1(\Omega \backslash K, {}^E\mathcal{O}^\sharp) \\ &\longrightarrow \cdots \\ &\longrightarrow H_K^{|n|}(\Omega, {}^E\mathcal{O}^\sharp) \longrightarrow H^{|n|}(\Omega, {}^E\mathcal{O}^\sharp) \longrightarrow H^{|n|}(\Omega \backslash K, {}^E\mathcal{O}^\sharp) \longrightarrow \cdots, \end{aligned}$$

we have $H^p(\Omega, {}^E\mathcal{O}^\sharp)=0$ for $p \geqq 1$ and $H_K^0(\Omega, {}^E\mathcal{O}^\sharp)=0$ by the unique continuation theorem. Hence we have isomorphisms

$$H_K^1(\Omega, {}^E\mathcal{O}^\sharp) \cong \mathcal{O}^\sharp(\Omega \backslash K; E)/\mathcal{O}^\sharp(\Omega; E),$$

$$H_K^p(\Omega, {}^E\mathcal{O}^\sharp) \cong H^{p-1}(\Omega \backslash K, {}^E\mathcal{O}^\sharp), \qquad p \geqq 2.$$

But, by a similar method to Junker [15], Lemma 3.5, we have isomorphisms $H^p(V, {}^E\mathcal{O}^\sharp) \cong H^p(V, \mathcal{O}^\sharp) \hat{\otimes}_\pi E$, $0 \leqq p \leqq |n|$, where V is an open set in $\boldsymbol{F}^n$ such that, for any $z \in V \cap \boldsymbol{C}^{|n|}$, $|\operatorname{Im} z''| - |\operatorname{Re} z''| < d$ holds for some constant $d > 0$. So that, by Theorem 5.5.1, we have isomorphisms

$$H_K^p(\Omega, {}^E\mathcal{O}^\sharp) \cong H_K^p(\Omega, \mathcal{O}^\sharp) \hat{\otimes}_\pi E = 0 \quad \text{for } p \neq |n|,$$

and

$$\begin{aligned} H_K^{|n|}(\Omega, {}^E\mathcal{O}^\sharp) &\cong H^{|n|-1}(\Omega \backslash K, {}^E\mathcal{O}^\sharp) \cong H^{|n|-1}(\Omega \backslash K, \mathcal{O}^\sharp) \hat{\otimes}_\pi E \\ &\cong H_K^{|n|}(\Omega, \mathcal{O}^\sharp) \hat{\otimes}_\pi E \cong \mathcal{O}_\sharp(K)' \hat{\otimes}_\pi E \cong L(\mathcal{O}_\sharp(K); E). \end{aligned} \qquad \text{Q.E.D.}$$

6.5. The Sato Theorem

In this section we will prove the pure-codimensionality of $\boldsymbol{D}^n$ with respect to ${}^E\mathcal{O}^\sharp$. Then we will realize E-valued mixed Fourier hyperfunctions as "boundary values" of E-valued slowly increasing holomorphic functions or as (relative) cohomology classes of E-valued slowly increasing holomorphic functions.

Theorem 6.5.1 (The Sato Theorem). *Let Ω be an open set in $\boldsymbol{D}^n$ and V an open set in $\boldsymbol{F}^n$ which contains Ω as its closed subset. Then we have the following*

(1) *The relative cohomology groups $H_\Omega^p(V, {}^E\mathcal{O}^\sharp)$ are zero for $p \neq |n|$.*

(2) *The presheaf over $\boldsymbol{D}^n$*

$$\Omega \longrightarrow H_\Omega^{|n|}(V, {}^E\mathcal{O}^\sharp)$$

is a sheaf.

(3) *This sheaf* (2) *is isomorphic to the sheaf* ${}^E\mathscr{P}$ *of E-valued mixed Fourier hyperfunctions.*

Proof. (1) By the excision theorem, we may assume that V is an $\mathcal{O}^\sharp$-pseudoconvex open set in $\boldsymbol{F}^n$. Consider the following exact sequence of relative cohomology groups

$$\begin{aligned}0 &\longrightarrow H^0_{\partial\Omega}(V, {}^E\mathcal{O}^\sharp) \longrightarrow H^0_{\Omega^a}(V, {}^E\mathcal{O}^\sharp) \longrightarrow H^0_{\Omega}(V, {}^E\mathcal{O}^\sharp) \\ &\longrightarrow H^1_{\partial\Omega}(V, {}^E\mathcal{O}^\sharp) \longrightarrow \cdots \longrightarrow H^{|n|-1}_{\Omega}(V, {}^E\mathcal{O}^\sharp) \\ &\longrightarrow H^{|n|}_{\partial\Omega}(V, {}^E\mathcal{O}^\sharp) \longrightarrow H^{|n|}_{\Omega^a}(V, {}^E\mathcal{O}^\sharp) \longrightarrow H^{|n|}_{\Omega}(V, {}^E\mathcal{O}^\sharp) \\ &\longrightarrow H^{|n|+1}_{\partial\Omega}(V, {}^E\mathcal{O}^\sharp) \longrightarrow \cdots .\end{aligned}$$

By Theorems 5.1.8 and 6.4.1, we may conclude that $H^p_{\partial\Omega}(V, {}^E\mathcal{O}^\sharp)=H^p_{\Omega^a}(V, {}^E\mathcal{O}^\sharp)=0$ for $p\neq|n|$. So that, we have $H^p_{\Omega}(V, {}^E\mathcal{O}^\sharp)=0$ for $p\neq|n|-1, |n|$. On the other hand, by Theorems 5.1.8 and 6.4.1, we also have the exact sequence

$$0 \longrightarrow H^{|n|-1}_{\Omega}(V, {}^E\mathcal{O}^\sharp) \longrightarrow L(\mathscr{A}_\sharp(\partial\Omega); E) \xrightarrow{j} L(\mathscr{A}_\sharp(\Omega^a); E).$$

Since j is injective, we have $H^{|n|-1}_{\Omega}(V, {}^E\mathcal{O}^\sharp)=0$.

(2) By (1) and by the theorem II.3.18 of Komatsu [21], we have the conclusion.

(3) By the proof of (1), we have the exact sequence

$$0 \longrightarrow H^{|n|}_{\partial\Omega}(V, {}^E\mathcal{O}^\sharp) \longrightarrow H^{|n|}_{\Omega^a}(V, {}^E\mathcal{O}^\sharp) \longrightarrow H^{|n|}_{\Omega}(V, {}^E\mathcal{O}^\sharp) \longrightarrow 0.$$

Since, by the Martineau-Harvey Theorem, we have isomorphisms

$$H^{|n|}_{\partial\Omega}(V, {}^E\mathcal{O}^\sharp)\cong L(\mathscr{A}_\sharp(\partial\Omega); E),$$

$$H^{|n|}_{\Omega^a}(V, {}^E\mathcal{O}^\sharp)\cong L(\mathscr{A}_\sharp(\Omega^a); E),$$

we obtain the isomorphism

$$H^{|n|}_{\Omega}(V, {}^E\mathcal{O}^\sharp)\cong L(\mathscr{A}_\sharp(\Omega^a); E)/L(\mathscr{A}_\sharp(\partial\Omega); E)=\mathscr{P}(\Omega; E).$$

Thus the sheaf $\Omega\to H^{|n|}_{\Omega}(V, {}^E\mathcal{O}^\sharp)$ is isomorphic to the sheaf ${}^E\mathscr{P}$ of E-valued mixed Fourier hyperfunctions over $\boldsymbol{D}^n$. Q. E. D.

In the same notations as in Theorem 5.5.2, we have the following

Theorem 6.5.2. $H^{|n|}_{\Omega}(V, {}^E\mathcal{O}^\sharp)\cong H^{|n|}(\mathfrak{U}, \mathfrak{U}', {}^E\mathcal{O}^\sharp)\cong\mathcal{O}^\sharp(\cap_j V_j;\ E)/\sum_{j=1}^{|n|}\mathcal{O}^\sharp(\bigcap_{i\neq j} V_i;$ *E) hold.*

At last we will realize mixed Fourier analytic linear mappings with certain compact carrier as (relative) cohomology classes with coefficients in ${}^E\mathcal{O}^\sharp$.

Let K be a compact set in $\boldsymbol{F}^n$ of the form $K=K_1\times\cdots\times K_{|n|}$ with compact sets

K_j in $\tilde{\boldsymbol{C}}$ for $j=1, 2,\dots, n_1$ and in $\boldsymbol{E}$ for $j=n_1+1,\dots, |n|$. Assume that K admits a fundamental system of $\mathcal{O}^\sharp$-pseudoconvex open neighborhoods. Then we have

$$H^p(K, \mathcal{O}_\sharp)=0 \qquad \text{for} \quad p>0.$$

By virtue of the Martineau-Harvey Theorem, there exists the isomorphism

$$\mathcal{O}'_\sharp(K; E)\cong H_K^{|n|}(\Omega, {}^E\mathcal{O}^\sharp).$$

Here Ω denotes an open neighborhood of K. Further assume that there exists an $\mathcal{O}^\sharp$-pseudoconvex open neighborhood Ω of K such that

$$\Omega_j=\Omega\backslash\{z\in \boldsymbol{C}^{|n|};\ z_j\in K_j\cap \boldsymbol{C}\}^a$$

is also an $\mathcal{O}^\sharp$-pseudoconvex open set for $j=1, 2,\dots, |n|$. Put $\Omega_0=\Omega$. Then $\mathfrak{U}=\{\Omega_0, \Omega_1,\dots, \Omega_{|n|}\}$ and $\mathfrak{U}'=\{\Omega_1, \Omega_2,\dots, \Omega_{|n|}\}$ form acyclic coverings of Ω and $\Omega\backslash K$. Set

$$\Omega\sharp K=\bigcap_{j=1}^{|n|}\Omega_j,$$

$$\Omega^j=\bigcap_{i\neq j}\Omega_i.$$

Let $\sum_j \mathcal{O}^\sharp(\Omega^j; E)$ be the image in $\mathcal{O}^\sharp(\Omega\sharp K; E)$ of $\prod_{j=1}^{|n|} \mathcal{O}^\sharp(\Omega^j; E)$ by the mapping

$$(f_j)_{j=1}^{|n|} \longrightarrow \sum_{j=1}^{|n|}(-1)^{j+1}f'_j,$$

where f'_j denotes the restriction of f_j to $\Omega\sharp K$.

Then, by a similar method to that of Theorem 6.6.2, we have the following

Theorem 6.5.3. *We use the notations as above. Then we have the isomorphisms*

$$\begin{aligned}\mathcal{O}'_\sharp(K; E)&\cong H_K^{|n|}(\Omega, {}^E\mathcal{O}^\sharp)\cong H^{|n|}(\mathfrak{U}, \mathfrak{U}', {}^E\mathcal{O}^\sharp)\\ &\cong \mathcal{O}^\sharp(\Omega\sharp K; E)/\textstyle\sum_j \mathcal{O}^\sharp(\Omega^j; E).\end{aligned}$$

Chapter 7. Cases of sheaves $\mathcal{O}^\flat$, $\mathcal{A}^\flat$, $\mathcal{O}_\flat$ anc $\mathcal{A}_\flat$

7.1. The Oka-Cartan-Kawai Theorem B

In this section we will prove the Oka-Cartan-Kawai Theorem B for the sheaves $\mathcal{O}^\flat$ and $\mathcal{O}_\flat$.

For a 2-tuple $n=(n_1, n_2)$ of nonnegative integers with $|n|=n_1+n_2\neq 0$, we denote by $\boldsymbol{G}^n$ the product space $\boldsymbol{C}^{n_1}\times\tilde{\boldsymbol{C}}^{n_2}$ and by $\tilde{\boldsymbol{D}}^n$ the product space $\boldsymbol{R}^{n_1}\times \boldsymbol{D}^{n_2}$ and by $\boldsymbol{C}^{|n|}$ the space $\boldsymbol{C}^{n_1+n_2}=\boldsymbol{C}^{n_1}\times\boldsymbol{C}^{n_2}$.

We denote by $z=(z', z'')\in \boldsymbol{C}^{|n|}$ so that $z'=(z_1,\dots, z_{n_1})$, $z''=(z_{n_1+1},\dots, z_{|n|})$.

Definition 7.1.1 (The sheaf $\mathcal{O}^{\flat}$ of germs of partially slowly increasing holomorphic functions). *We define $\mathcal{O}^{\flat}$ to be the sheafification of the presheaf $\{\mathcal{O}^{\flat}(\Omega);\ \Omega\subset \boldsymbol{G}^n$ open$\}$, where the section module $\mathcal{O}^{\flat}(\Omega)$ on an open set Ω in $\boldsymbol{G}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega\cap \boldsymbol{C}^{|n|}$ such that, for any positive number ε and for any compact set K in Ω, the estimate $\sup\{|f(z)|e(-\varepsilon|z|);\ z\in K\cap \boldsymbol{C}^{|n|}\}<\infty$ holds.*

Definition 7.1.2 (The sheaf $\mathcal{O}_{\flat}$ of germs of partially rapidly decreasing holomorphic functions). *We define $\mathcal{O}_{\flat}$ to be the sheafification of the presheaf $\{\mathcal{O}_{\flat}(\Omega);\ \Omega\subset \boldsymbol{G}^n$ open$\}$, where the section module $\mathcal{O}_{\flat}(\Omega)$ on an open set Ω in $\boldsymbol{G}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega\cap \boldsymbol{C}^{|n|}$ such that, for any compact set K in Ω, there exists some positive constant δ so that the estimate $\sup\{|f(z)|e(\delta|z|);\ z\in K\cap \boldsymbol{C}^{|n|}\}<\infty$ holds.*

Definition 7.1.3. *An open set V in $\boldsymbol{G}^n$ is said to be an $\mathcal{O}^{\flat}$-pseudoconvex open set if it satisfies the conditions:*

(1) $\sup\{|\mathrm{Im}\, z''|;\ z=(z', z'')\in V\cap \boldsymbol{C}^{|n|}\}<\infty$.

(2) *There exists a C^{∞}-plurisubharmonic function $\varphi(z)$ on $V\cap \boldsymbol{C}^{|n|}$ having the following two properties:*

(i) *The closure of $V_c=\{z\in V\cap \boldsymbol{C}^{|n|};\ \varphi(z)<c\}$ in $\boldsymbol{G}^n$ is a compact subset of V for any real c.*

(ii) *$\varphi(z)$ is bounded on $L\cap \boldsymbol{C}^{|n|}$ for any compact subset L of V.*

Then we can prove the Oka-Cartan-Kawai Theorem B by a similar method to that in section 1.1.

Theorem 7.1.4 (The Oka-Cartan-Kawai Theorem B). *For any $\mathcal{O}^{\flat}$-pseudoconvex open set V in $\boldsymbol{G}^n$, we have $H^s(V, \mathcal{O}^{\flat,p})=0$, $(p\geqq 0,\ s\geqq 1)$.*

Proof. Since V is paracompact, $H^s(V, \mathcal{O}^{\flat,p})$ coincides with the Čech cohomology group. So we have only to prove $\varinjlim_{\mathfrak{U}} H^s(\mathfrak{U}, \mathcal{O}^{\flat,p})=0$, where $\mathfrak{U}=\{U\}_{j\geqq 1}$ is a locally finite open covering of V so that $V_j=U_j\cap \boldsymbol{C}^{|n|}$ is pseudoconvex. We can choose such a covering of V because V is an $\mathcal{O}^{\flat}$-pseudoconvex open set.

Now we define $C^s(Z^{\mathrm{loc},\flat}_{(p,q)}(\{V_j\}))$ to be the set of all cochains $c=\{c_J;\ J=(j_0, j_1,\dots, j_s)\in \boldsymbol{N}^{s+1}\}$ of forms of type (p, q) satisfying the two conditions:

(i) $\bar{\partial}c_J=0$ in $V_J=V_{j_0}\cap V_{j_1}\cap\cdots\cap V_{j_s}$.

(ii) For any positive ε and any finite subset M of $\boldsymbol{N}^{s+1}$, the estimate

$$\sum_{J\in M}\int_{V_J}|c_J|^2 e(-\varepsilon\|z\|)d\lambda<\infty$$

holds, where $d\lambda$ is the Lebesgue measure on $\boldsymbol{C}^{|n|}$ and $\|z\|$ denotes the modification of $\sum_{j=1}^{|n|} |z_j|$ so as to become C^∞ and convex.

Now we will prove the following

Lemma 7.1.5. *If* $c \in C^s(Z^{\mathrm{loc},b}_{(p,q)}(\{V_j\}))$ *satisfies the conditions* $\delta c = 0$, *then we can find some* $c' \in C^{s-1}(Z^{\mathrm{loc},b}_{(p,q)}(\{V_j\}))$ *such that* $\delta c' = c$. *Here* δ *means the coboundary operator.*

If this Lemma is proved, the theorem will follow from this Lemma as the special case where $q=0$ because we can use Cauchy's integral formula to change the L^2-norm to the sup-norm for holomorphic functions.

Proof of Lemma 7.1.5. We denote by $\{\chi_j\}$ the partition of unity subordinate to $\{V_j\}$ and define $b_I = \sum_j \chi_j c_{jI}$ for $I \in \boldsymbol{N}^s$. Since $\delta c = 0$, we have $\delta b = c$. So $\delta\bar{\partial} b = 0$ because $\bar{\partial} c = 0$. Since $\sum \chi_j = 1$ and $\chi_j \geqq 0$, we have

$$\int_{V_I} |b_I|^2 e(-\varepsilon\|z\|)d\lambda \leqq \sum_j \int_{V_I} \chi_j |c_{jI}|^2 e(-\varepsilon\|z\|)d\lambda$$

for any positive number ε by virtue of Cauchy-Schwarz' inequality.

By the assumption of the existence of C^∞ plurisubharmonic function $\varphi(z)$ in Definition 7.1.3, we can find some plurisubharmonic function $\psi(z)$ on $W = V \cap \boldsymbol{C}^{|n|}$ which satisfies the following two conditions

(1) $\sum |\bar{\partial}\chi_j| \leqq e(\psi(z))$,

(2) $\sup\{\psi(z);\ z \in K \cap \boldsymbol{C}^{|n|}\} \leqq C_K$ for any $K \Subset W$.

Thus it follows from the condition on c that

$$\sum_{I \in N} \int_{V_I} |\bar{\partial} b_I|^2 e(-\varepsilon\|z\| - \psi(z))d\lambda < \infty$$

for any positive number ε and any finite subset N of $\boldsymbol{N}^s$.

Now we consider the case $s=1$. By the fact that $\delta(\bar{\partial} b) = 0$, $\bar{\partial} b$ defines a global section f on $W = V \cap \boldsymbol{C}^{|n|}$. Then, by Hörmander [4], Theorem 4.4.2, p. 94, we can prove the existence of u such that $\bar{\partial} u = f$ and the estimate

$$\int_{K \cap \boldsymbol{C}^{|n|}} |u|^2 e(-\varepsilon\|z\|)(1+|z|^2)^{-2} d\lambda < \infty$$

holds for any positive number ε and any $K \Subset V$.

If we define $c'_I = b_I - u|V_I$, then $\bar{\partial} c'_I = 0$ and $\delta c' = \delta b = c$. Clearly $c' \in C^{s-1} \cdot (Z^{\mathrm{loc},b}_{(p,q)}(\{V_j\}))$.

Now we go on to the case $s>1$. In this case we use the induction on s. By the induction hypotheses there exists $b' \in C^{s-2}(Z^{\mathrm{loc},b}_{(p,q+1)}(\{V_j\}))$ such that $\delta b' = \bar{\partial} b$. By virtue of Hörmander [4], Theorem 4.4.2, p. 94, we can also find $b'' = \{b''_H\}_{H \in \boldsymbol{N}^{s-1}}$ such that $b'_H = \bar{\partial} b''_H$ and the estimate

$$\sum_{H \in L} \int_{V_H} |b''_H|^2 e(-\varepsilon\|z\| - \psi(z))(1+|z|^2)^{-2} d\lambda < \infty$$

holds for any positive number ε and any finite subset L of N^{s-1}. Therefore $c'=b-\delta b''$ satisfies all the required conditions. Q.E.D.

This completes the proof of the theorem. Q.E.D.

Now we will prove the Malgrange theorem for the sheaf $\mathscr{A}^\flat$ of germs of partially slowly increasing real analytic functions. Here we define the sheaf $\mathscr{A}^\flat$ to be the restriction of $\mathcal{O}^\flat$ to $\tilde{\boldsymbol{D}}^n$: $\mathscr{A}^\flat=\mathcal{O}^\flat|\tilde{\boldsymbol{D}}^n$. Then we have the following

Theorem 7.1.6 (Malgrange). *For an arbitrary set Ω in $\tilde{\boldsymbol{D}}^n$, we have $H^s(\Omega, \mathscr{A}^{\flat,p})=0$, $(p\geqq 0,\ s\geqq 1)$.*

Proof. We know, by virtue of Ito [11], Theorem 2.1.13, that Ω has a fundamental system $\{\tilde{\Omega}\}$ of $\mathcal{O}^\flat$-pseudoconvex open neighborhoods. Then, it follows from the Oka-Cartan-Kawai Theorem B and Schapira [36], Theorem B 42, that, for $p\geqq 0$ and $s>0$, we have

$$H^s(\Omega, \mathscr{A}^{\flat,p})=\varinjlim_{\tilde{\Omega}\cap \boldsymbol{G}^n=\Omega} H^s(\tilde{\Omega}, \mathcal{O}^{\flat,p})=0. \qquad \text{Q.E.D.}$$

Next we will prove the Oka-Cartan-Kawai Theorem B for the sheaf $\mathcal{O}_\flat$. This can be proved by a similar method to Theorem 7.1.4. Thus we have the following

Theorem 7.1.7 (The Oka-Cartan-Kawai Theorem B). *For any $\mathcal{O}^\flat$-pseudoconvex open set V in $\boldsymbol{G}^n$, we have $H^s(V, \mathcal{O}_\flat^p)=0$ for $p\geqq 0$ and $s\geqq 1$.*

Proof Since V is paracompact, $H^s(V, \mathcal{O}_\flat^p)$ coincides with the Čech cohomology group. So we have only to prove $\varinjlim_{\mathfrak{U}} H^s(\mathfrak{U}, \mathcal{O}_\flat^p)=0$, where $\mathfrak{U}=\{U_j\}_{j\geqq 1}$ is a locally finite open covering of V so that $V_j=U_j\cap \boldsymbol{C}^{|n|}$ is pseudoconvex. We can choose such a covering of V because V is an $\mathcal{O}^\flat$-pseudoconvex open set.

Here we use the notation in the proof of Theorem 7.1.4.

For any cocycle $d=\{d_J\}$ representing an element in $H^s(\mathfrak{U}, \mathcal{O}_\flat^p)$, we can define an element $c=\{c_J\}$ in $C^s(Z^{\mathrm{loc},\flat}_{(p,0)}(\{V_j\}))$ such as $\delta c=0$ by putting $c_J=d_J\cdot h_\varepsilon(z)$, $h_\varepsilon(z)=\prod_{j=n_1+1}^{|n|}\cosh(\varepsilon z_j)$ for some positive ε, where δ denotes the coboundary operator. Then we can find some $c'\in C^{s-1}(Z^{\mathrm{loc},\flat}_{(p,0)}(\{V_j\}))$ such that $\delta c'=c$. If we put $d'_I=c'_I\cdot(h_\varepsilon(z))^{-1}$, then $d'=\{d'_I\}$ is a cochain with values in $\mathcal{O}_\flat$ such that $\delta d'=d$. Thus the element in $H^s(\mathfrak{U}, \mathcal{O}_\flat^p)$ represented by d is zero. Since a class $[d]$ with a representative d is an arbitrary element in $H^s(\mathfrak{U}, \mathcal{O}_\flat^p)$, we have $H^s(\mathfrak{U}, \mathcal{O}_\flat^p)=0$. This completes the proof. Q.E.D.

At last we will prove the Malgrange theorem for the sheaf $\mathscr{A}_\flat$ of germs of partially rapidly decreasing real analytic functions. Here we define the sheaf $\mathscr{A}_\flat$ to be the restriction of $\mathcal{O}_\flat$ to $\tilde{\boldsymbol{D}}^n$: $\mathscr{A}_\flat=\mathcal{O}_\flat|\tilde{\boldsymbol{D}}^n$. Then we have the following

Theorem 7.1.8 (Malgrange). *For an arbitrary set Ω in $\tilde{\boldsymbol{D}}^n$, we have $H^s(\Omega, \mathscr{A}_\flat^p)=0$ for $p\geqq 0$ and $s\geqq 1$.*

Proof. We can prove this by a similar method to that of Theorem 7.1.6.
Q. E. D.

7.2. The Dolbeault-Grothendieck resolution of $\mathcal{O}^{\flat}$ and $\mathcal{O}_{\flat}$

In this section we will construct soft resolutions of $\mathcal{O}^{\flat}$ and $\mathcal{O}_{\flat}$ and prove some of their consequences.

At first we will define the sheaf $\mathscr{E}^{\flat}$ of germs of partially slowly increasing C^{∞}-functions over $\boldsymbol{G}^n$.

Definition 7.2.1. *We define the sheaf $\mathscr{E}^{\flat}$ to be the sheafification of the phesheaf $\{\mathscr{E}^{\flat}(\Omega);\ \Omega\subset \boldsymbol{G}^n$ open$\}$, where, for an open set Ω in $\boldsymbol{G}^n$, the module $\mathscr{E}^{\flat}(\Omega)$ is defined as follows:*

$\mathscr{E}^{\flat}(\Omega)=\{f\in\mathscr{E}(\Omega\cap \boldsymbol{C}^{|n|})$; *for any positive ε and any compact set K in Ω and any $\alpha\in\bar{\boldsymbol{N}}^{2|n|}$, the estimate* $\sup\{|f^{(\alpha)}(z)|e(-\varepsilon|z|);\ z\in K\cap \boldsymbol{C}^{|n|}\}<\infty$ *holds*$\}$.

Then it is easy to see that $\mathscr{E}^{\flat}$ is a soft nuclear Fréchet sheaf. Then we have the following

Theorem 7.2.2 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over $\boldsymbol{G}^n$*

$$0\longrightarrow \mathcal{O}^{\flat,p}\longrightarrow \mathscr{E}^{\flat,p,0}\xrightarrow{\bar{\partial}}\mathscr{E}^{\flat,p,1}\xrightarrow{\bar{\partial}}\cdots\xrightarrow{\bar{\partial}}\mathscr{E}^{\flat,p,|n|}\longrightarrow 0$$

is exact.

Proof. It goes in a similar way to that of Ito [10], Theorem 3.1. Q. E. D.

Corollary 1. *For an open set Ω in $\boldsymbol{G}^n$, we have the following isomorphism:*

$$H^q(\Omega,\mathcal{O}^{\flat,p})\cong\{f\in\mathscr{E}^{\flat,p,q}(\Omega);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g\in\mathscr{E}^{\flat,p,q-1}(\Omega)\},\quad (p\geqq 0,\ q\geqq 1).$$

Proof. It follws from Theorem 7.2.2 and Komatsu [21], Theorems II.2.9 and II.2.19.
Q. E. D.

Corollary 2. *Let Ω be an $\mathcal{O}^{\flat}$-pseudoconvex open set. Then the equation $\bar{\partial}u=f$ has a solution $u\in\mathscr{E}^{\flat,p,q}(\Omega)$ for every $f\in\mathscr{E}^{\flat,p,q+1}(\Omega)$ such that $\bar{\partial}f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 7.1.4 and Corollary 1 to Theorem 7.2.2.
Q. E. D.

Now we will define the sheaf $\mathscr{E}_{\flat}$ of germs of partially rapidly decreasing C^{∞}-functions over $\boldsymbol{G}^n$.

Definition 7.2.3. *We define the sheaf $\mathscr{E}_{\flat}$ to be the sheafification of the presheaf $\{\mathscr{E}_{\flat}(\Omega);\ \Omega\subset\boldsymbol{G}^n$ open$\}$, where the section module $\mathscr{E}_{\flat}(\Omega)$ on an open set Ω in $\boldsymbol{G}^n$ is the space of all C^{∞}-functions on $\Omega\cap\boldsymbol{C}^{|n|}$ such that, for any compact set K in Ω*

and any $\alpha \in \bar{N}^{2|n|}$, *there exists some positive constant* δ *so that the estimate*

$$\sup\{|f^{(\alpha)}(z)|e(\delta|z|);\ z \in K \cap C^{|n|}\} < \infty$$

holds.

Then it is easy to see that $\mathscr{E}_\flat$ is a soft nuclear Fréchet sheaf. Then we have the following

Theorem 7.2.4 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over* G^n

$$0 \longrightarrow \mathcal{O}_\flat^p \longrightarrow \mathscr{E}_\flat^{p,0} \xrightarrow{\bar{\partial}} \mathscr{E}_\flat^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{E}_\flat^{p,|n|} \longrightarrow 0$$

is exact.

Proof. It follows in a similar way to Ito [10], Theorem 3.1. Q.E.D.

Corollary 1. *For an open set* Ω *in* G^n, *we have the following isomorphism:*

$$H^q(\Omega, \mathcal{O}_\flat^p) \cong \{f \in \mathscr{E}_\flat^{p,q}(\Omega);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g \in \mathscr{E}_\flat^{p,q-1}(\Omega)\}, \quad (p \geqq 0,\ q \geqq 1).$$

Corollary 2. *Let* Ω *be an* $\mathcal{O}^\flat$*-pseudoconvex open set. Then the equation* $\bar{\partial}u=f$ *has a solution* $u \in \mathscr{E}_\flat^{p,q}(\Omega)$ *for every* $f \in \mathscr{E}_\flat^{p,q+1}(\Omega)$ *such that* $\bar{\partial}f=0$. *Here* p *and* q *are nonnegative integers.*

Proof. It follows from Theorem 7.1.7 and Corollary 1 to Theorem 7.2.4. Q.E.D.

Now, for later applications, we will construct another soft resolutions of $\mathcal{O}^\flat$ and $\mathcal{O}_\flat$.

We will define the sheaf $L^\flat = L^\flat_{2,\mathrm{loc}}$ of germs of partially slowly increasing locally L_2-functions.

Definition 7.2.5. *We define the sheaf* $L^\flat$ *to be the sheafification of the presheaf* $\{L^\flat(\Omega);\ \Omega \subset G^n \text{ open}\}$, *where, for an open set* Ω *in* G^n, *the section module* $L^\flat(\Omega)$ *is the space of all* $f \in L_{2,\mathrm{loc}}(\Omega \cap C^{|n|})$ *such as, for any* $\varepsilon > 0$ *and any relatively compact open subset* ω *of* Ω, $e(-\varepsilon\|z\|)f(z)|\omega$ *belongs to* $L_2(\omega \cap C^{|n|})$. *Here* $e(-\varepsilon\|z\|)f(z)|\omega$ *denotes the restriction of* $e(-\varepsilon\|z\|)f(z)$ *to* ω *and* $\|z\|$ *denotes the modification of* $\sum_{j=1}^{|n|}|z_j|$ *so as to become* C^∞ *and convex.*

Then it is easy to see that $L^\flat$ is a soft FS* sheaf

Definition 7.2.6 (The sheaf $\mathscr{L}^{\flat,p,q}$**).** *We define the sheaf* $\mathscr{L}^{\flat,p,q} = \mathscr{L}^{\flat,p,q}_{2,\mathrm{loc}}$ *to be the sheafification of the presheaf* $\{\mathscr{L}^{\flat,p,q}(\Omega);\ \Omega \subset G^n \text{ open}\}$, *where, for an open set* Ω *in* G^n, *the section module* $\mathscr{L}^{\flat,p,q}(\Omega)$ *is the space of all* $f \in L^{\flat,p,q}(\Omega) = L^{\flat,p,q}_{2,\mathrm{loc}}(\Omega)$ *such that* $\bar{\partial}f \in L^{\flat,p,q+1}(\Omega) = L^{\flat,p,q+1}_{2,\mathrm{loc}}(\Omega)$. *We put* $\mathscr{L}^\flat = \mathscr{L}^{\flat,0,0}$.

Then $\mathscr{L}^{\flat,p,q}$ is a soft FS* sheaf. Then we have the following

Theorem 7.2.7 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over G^n*

$$0 \longrightarrow \mathcal{O}^{\flat,p} \longrightarrow \mathcal{L}^{\flat,p,0} \xrightarrow{\bar{\partial}} \mathcal{L}^{\flat,p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{L}^{\flat,p,|n|} \longrightarrow 0$$

is exact.

Proof. It goes in a similar way to that of Theorem 1.2.7. Q.E.D.

Corollary 1. *For an open set Ω in G^n, we have the following isomorphism:*

$$H^q(\Omega, \mathcal{O}^{\flat,p}) \cong \{f \in \mathcal{L}^{\flat;p,q}_{2,\mathrm{loc}}(\Omega);\ \bar{\partial}f=0\} / \{\bar{\partial}g;\ g \in \mathcal{L}^{\flat;p,q-1}_{2,\mathrm{loc}}(\Omega)\}, \quad (p \geqq 0,\ q \geqq 1).$$

Corollary 2. *Let Ω be an $\mathcal{O}^\flat$-pseudoconvex open set in G^n. Then the equation $\bar{\partial}u=f$ has a solution $u \in \mathcal{L}^{\flat;p,q}_{2,\mathrm{loc}}(\Omega)$ for every $f \in \mathcal{L}^{\flat;p,q+1}_{2,\mathrm{loc}}(\Omega)$ such that $\bar{\partial}f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 7.1.4 and Corollary 1 to Theorem 7.2.7. Q.E.D.

We will now define the sheaf $L_\flat = L_{\flat,2,\mathrm{loc}}$ of germs of partially rapidly decreasing locally L_2-functions.

Definition 7.2.8. *We define the sheaf $L_\flat$ to be the sheafification of the presheaf $\{L_\flat(\Omega);\ \Omega \subset G^n \text{ open}\}$, where, for an open set Ω in G^n, the section module $L_\flat(\Omega)$ is the space of all $f \in L_{2,\mathrm{loc}}(\Omega \cap C^{|n|})$ such as, for any relatively compact open subset ω of Ω, there exists some positive δ such that $e(\delta\|z\|)f(z)|\omega \in L_2(\omega \cap C^{|n|})$.*

Then it is easy to see that $L_\flat$ is a soft FS* sheaf.

Definition 7.2.9 (The sheaf $\mathcal{L}^{p,q}_\flat$). *We define the sheaf $\mathcal{L}^{p,q}_\flat = \mathcal{L}^{p,q}_{\flat;2,\mathrm{loc}}$ to be the sheafification of the presheaf $\{\mathcal{L}^{p,q}_\flat(\Omega);\ \Omega \subset G^n \text{ open}\}$, where, for an open set Ω in G^n, the section module $\mathcal{L}^{p,q}_\flat(\Omega)$ is the space of all $f \in L^{p,q}_\flat(\Omega) = L^{p,q}_{\flat;2,\mathrm{loc}}(\Omega)$ such that $\bar{\partial}f \in L^{p,q+1}_\flat(\Omega) = L^{p,q+1}_{\flat;2,\mathrm{loc}}(\Omega)$. We put $\mathcal{L}_\flat = \mathcal{L}^{0,0}_\flat$.*

Then $\mathcal{L}^{p,q}_\flat$ is a soft FS* sheaf. Then we have the following

Theorem 7.2.10 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over G^n*

$$0 \longrightarrow \mathcal{O}^p_\flat \longrightarrow \mathcal{L}^{p,0}_\flat \xrightarrow{\bar{\partial}} \mathcal{L}^{p,1}_\flat \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{L}^{p,|n|}_\flat \longrightarrow 0$$

is exact.

Proof. It goes in a similar way to that of Theorem 1.2.7. Q.E.D.

Corollary 1. *For an open set Ω in G^n, we have the following isomorphism:*

$$H^q(\Omega, \mathcal{O}^p_\flat) \cong \{f \in \mathcal{L}^{p,q}_{\flat;2,\mathrm{loc}}(\Omega);\ \bar{\partial}f=0\} / \{\bar{\partial}g;\ g \in \mathcal{L}^{p,q-1}_{\flat;2,\mathrm{loc}}(\Omega)\}, \quad (p \geqq 0,\ q \geqq 1).$$

Corollary 2. *Let Ω be an $\mathcal{O}^{\flat}$-pseudoconvex open set in $\boldsymbol{G}^n$. Then the equation $\bar{\partial}u=f$ has a solution $u\in\mathscr{L}^{p,q}_{\flat,2,\mathrm{loc}}(\Omega)$ for every $f\in\mathscr{L}^{p,q}_{\flat,2,\mathrm{loc}}(\Omega)$ such that $\bar{\partial}f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 7.1.7 and Corollary 1 to Theorem 7.2.10.

Q.E.D.

7.3. The Serre duality theorem

In this section we will prove the Serre duality theorem.

Theorem 7.3.1. *Let Ω be an open set in $\boldsymbol{G}^n$ such that* $\dim H^p(\Omega, \mathcal{O}^{\flat})<\infty$ *holds $(p\geqq 1)$. Then we have the isomorphism $[H^p(\Omega, \mathcal{O}^{\flat})]'=H_c^{|n|-p}(\Omega, \mathcal{O}_{\flat})$, $(0\leqq p\leqq |n|)$.*

Proof. By virtue of Corollary 1 to Theorem 7.2.7 and Corollary 1 to Theorem 7.2.10, cohomology groups $H^p(\Omega, \mathcal{O}^{\flat})$ and $H_c^{|n|-p}(\Omega, \mathcal{O}_{\flat})$ are cohomology groups respectively of the complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathscr{L}^{\flat,0,0}(\Omega) & \xrightarrow{\bar{\partial}} & \mathscr{L}^{\flat,0,1}(\Omega) & \xrightarrow{\bar{\partial}} & \cdots & \xrightarrow{\bar{\partial}} & \mathscr{L}^{\flat,0,|n|}(\Omega) & \longrightarrow & 0 \\ & & \updownarrow & & \updownarrow & & & & \updownarrow & & \\ 0 & \longleftarrow & \mathscr{L}^{0,|n|}_{\flat,c}(\Omega) & \xleftarrow{-\bar{\partial}} & \mathscr{L}^{0,|n|-1}_{\flat,c}(\Omega) & \xleftarrow{-\bar{\partial}} & \cdots & \xleftarrow{-\bar{\partial}} & \mathscr{L}^{0,0}_{\flat,c}(\Omega) & \longleftarrow & 0. \end{array}$$

Here the upper complex is composed of FS* spaces and the lower complex is composed of DFS* spaces. Since the ranges of operators $\bar{\partial}$ in the upper complex are all closed by virtue of Schwartz' Lemma (cf. Komatsu [20]), the ranges of operators $-\bar{\partial}=(\bar{\partial})'$ in the lower complex are also all closed. Hence we have the isomorphism

$$[H^p(\Omega, \mathcal{O}^{\flat})]'\cong H_c^{|n|-p}(\Omega, \mathcal{O}_{\flat})$$

by virtue of Serre's Lemma (cf. Komatsu [20]). Q.E.D.

7.4. The Martineau-Harvey Theorem

In this section we will prove the Martineau-Harvey Theorem.

Theorem 7.4.1. *Let K be a compact set in $\boldsymbol{G}^n$ such that it has an $\mathcal{O}^{\flat}$-pseudoconvex open neighborhood Ω and satisfies the conditions $H^p(K, \mathcal{O}_{\flat})=0$ $(p\geqq 1)$. Then we have $H_K^p(\Omega, \mathcal{O}^{\flat})=0$ for $p\neq|n|$ and isomorphisms $H_K^{|n|}(\Omega, \mathcal{O}^{\flat})\cong H^{|n|-1}(\Omega\backslash K, \mathcal{O}^{\flat})\cong\mathcal{O}_{\flat}(K)'$.*

Remark. If a compact set K in $\boldsymbol{G}^n$ has a fundamental system of $\mathcal{O}^{\flat}$-pseudoconvex open neighborhoods, it satisfies the assumptions in Theorem 7.4.1.

Proof. By the excision theorem, $H_K^p(\Omega, \mathcal{O}^{\flat})$ is independent of an open neighborhood Ω of K. So, we may assume that Ω is an $\mathcal{O}^{\flat}$-pseudoconvex open neighborhood in the assumptions in this theorem. Then in the long exact sequence of cohomology groups (cf. Komatsu [21], Theorem II.3.2):

$$
\begin{aligned}
0 &\longrightarrow H_K^0(\Omega, \mathcal{O}^b) \longrightarrow H^0(\Omega, \mathcal{O}^b) \longrightarrow H^0(\Omega\backslash K, \mathcal{O}^b) \\
&\longrightarrow H_K^1(\Omega, \mathcal{O}^b) \longrightarrow H^1(\Omega, \mathcal{O}^b) \longrightarrow H^1(\Omega\backslash K, \mathcal{O}^b) \\
&\longrightarrow \cdots \\
&\longrightarrow H_K^{|n|}(\Omega, \mathcal{O}^b) \longrightarrow H^{|n|}(\Omega, \mathcal{O}^b) \longrightarrow H^{|n|}(\Omega\backslash K, \mathcal{O}^b) \longrightarrow \cdots,
\end{aligned}
$$

we have $H^p(\Omega, \mathcal{O}^b)=0$ for $p\geqq 1$ and $H_K^0(\Omega, \mathcal{O}^b)=0$ by the unique continuation theorem. Hence we have isomorphisms

$$
\begin{aligned}
H_K^1(\Omega, \mathcal{O}^b) &\cong \mathcal{O}^b(\Omega\backslash K)/\mathcal{O}^b(\Omega), \\
H_K^p(\Omega, \mathcal{O}^b) &\cong H^{p-1}(\Omega\backslash K, \mathcal{O}^b), \quad p\geqq 2.
\end{aligned}
$$

We also have the long exact sequence of cohomology groups with compact support (cf. Komatsu [21], Theorem II.3.15):

$$
\begin{aligned}
0 &\longrightarrow H_c^0(\Omega\backslash K, \mathcal{O}_b) \longrightarrow H_c^0(\Omega, \mathcal{O}_b) \longrightarrow H^0(K, \mathcal{O}_b) \\
&\longrightarrow H_c^1(\Omega\backslash K, \mathcal{O}_b) \longrightarrow H_c^1(\Omega, \mathcal{O}_b) \longrightarrow H^1(K, \mathcal{O}_b) \\
&\longrightarrow \cdots \\
&\longrightarrow H_c^p(\Omega\backslash K, \mathcal{O}_b) \longrightarrow K_c^p(\Omega, \mathcal{O}_b) \longrightarrow H^p(K, \mathcal{O}_b) \longrightarrow \cdots.
\end{aligned}
$$

Here $H^p(K, \mathcal{O}_b)=0$ $(p\geqq 1)$ by the assumption on K. Therefore we obtain the isomorphisms

$$
\begin{aligned}
\mathcal{O}_b(K) &\cong H_c^1(\Omega\backslash K, \mathcal{O}_b), \\
H_c^p(\Omega, \mathcal{O}_b) &\cong H_c^p(\Omega\backslash K, \mathcal{O}_b), \quad p\geqq 2.
\end{aligned}
$$

By the theorem 7.3.1, we have $H_c^p(\Omega, \mathcal{O}_b)=0$ $(p\neq|n|)$. Thus we have the following isomorphisms

$$
\begin{aligned}
&H_c^p(\Omega\backslash K, \mathcal{O}_b)=0, \ p\neq 1, |n|, \\
&H_c^{|n|}(\Omega\backslash K, \mathcal{O}_b) \cong \mathcal{O}^b(\Omega)'.
\end{aligned}
$$

Now we consider the following dual complexes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{L}^{b,0,0}(\Omega\backslash K) & \xrightarrow{\bar\partial_0} & \mathscr{L}^{b,0,1}(\Omega\backslash K) & \xrightarrow{\bar\partial_1} & \cdots & \longrightarrow & (*) \\
 & & \updownarrow & & \updownarrow & & & & \\
0 & \longleftarrow & \mathscr{L}_{b,c}^{0,|n|}(\Omega\backslash K) & \xleftarrow{-\bar\partial_{|n|-1}} & \mathscr{L}_{b,c}^{0,|n|-1}(\Omega\backslash K) & \xleftarrow{-\bar\partial_{|n|-2}} & \cdots & \longleftarrow & (**)
\end{array}
$$

$$
\begin{array}{ccccccc}
(*) & \xrightarrow{\bar\partial_{|n|-2}} & \mathscr{L}^{b,0,|n|-1}(\Omega\backslash K) & \xrightarrow{\bar\partial_{|n|-1}} & \mathscr{L}^{b,0,|n|}(\Omega\backslash K) & \longrightarrow & 0 \\
 & & \updownarrow & & \updownarrow & & \\
(**) & \xleftarrow{-\bar\partial_1} & \mathscr{L}_{b,c}^{0,1}(\Omega\backslash K) & \xleftarrow{-\bar\partial_0} & \mathscr{L}_{b,c}^{0,0}(\Omega\backslash K) & \longleftarrow & 0.
\end{array}
$$

Then, since $H^p_c(\Omega\backslash K, \mathcal{O}_\flat)=0$ $(p \neq 1, |n|)$, the range of $-\bar{\partial}_j=(\bar{\partial}_{|n|-j-1})'$ is closed except for $j=0$, $|n|-1$. However $\bar{\partial}_{|n|-1}$ is of closed range by the Malgrange Theorem. Hence, by the closed range theorem, $-\bar{\partial}_0$ is of closed range (cf. Komatsu [20], Theorem 19, p. 381).

In order to prove the closedrangeness of $-\bar{\partial}_{|n|-1}$, we consider the following diagram:

$$\begin{tikzcd}
0 & \mathscr{L}^{0,|n|}_{\flat,c}(\Omega\backslash K) \arrow[l] \arrow[d, hook, "i"] & \mathscr{L}^{0,|n|-1}_{\flat,c}(\Omega\backslash K) \arrow[l, "-\bar{\partial}^{\Omega\backslash K}_{|n|-1}"'] \arrow[d, hook] \\
0 & \mathscr{L}^{0,|n|}_{\flat,c}(\Omega) \arrow[l] & \mathscr{L}^{0,|n|-1}_{\flat,c}(\Omega), \arrow[l, "-\bar{\partial}^{\Omega}_{|n|-1}"']
\end{tikzcd}$$

where the map i is the natural injection. However, in the dual complexes for Ω, $\bar{\partial}^\Omega_0$ is of closed range since $H^1(\Omega, \mathcal{O}^\flat)=0$. Thus, by the closed range theorem, $\mathrm{Im}\,(-\bar{\partial}^{\Omega\backslash K}_{|n|-1})=i^{-1}(\mathrm{Im}\,(-\bar{\partial}^{\Omega}_{|n|-1}))$ is closed. Therefore all $-\bar{\partial}^{\Omega\backslash K}_j$ are of closed range. Hence, by the Serre-Komatsu duality theorem, we have the isomorphisms $[H^p(\Omega\backslash K, \mathcal{O}^\flat)]'\cong H^{|n|-p}_c(\Omega\backslash K, \mathcal{O}_\flat)$, for $0\leqq p\leqq |n|$. Hence we have $\mathcal{O}^\flat(\Omega\backslash K)'\cong H^{|n|}_c(\Omega\backslash K, \mathcal{O}_\flat)\cong H^{|n|}_c(\Omega, \mathcal{O}_\flat)\cong \mathcal{O}^\flat(\Omega)'$. Here $\mathcal{O}^\flat(\Omega\backslash K)$ and $\mathcal{O}^\flat(\Omega)$ are both FS spaces, a posteriori, reflexive. Hence we have the isomorphism $\mathcal{O}^\flat(\Omega)\cong \mathcal{O}^\flat(\Omega\backslash K)$. Thus $H^1_K(\Omega, \mathcal{O}^\flat)\cong \mathcal{O}^\flat(\Omega\backslash K)/\mathcal{O}^\flat(\Omega)=0$. Hence, for $p\geqq 2$, $p\neq |n|$, we have $0=H^{|n|-p+1}_c(\Omega, \mathcal{O}_\flat)\cong H^{|n|-p+1}_c(\Omega\backslash K, \mathcal{O}_\flat)\cong [H^{p-1}(\Omega\backslash K, \mathcal{O}^\flat)]'\cong [H^p_K(\Omega, \mathcal{O}^\flat)]'$. Thus $H^p_K(\Omega, \mathcal{O}^\flat)=0$. In the cace $p=|n|$, we have the isomorphisms $[H^{|n|}_K(\Omega, \mathcal{O}^\flat)]'\cong [H^{|n|-1}(\Omega\backslash K, \mathcal{O}^\flat)]'\cong H^1_c(\Omega\backslash K, \mathcal{O}_\flat)\cong H^0(K, \mathcal{O}_\flat)\cong \mathcal{O}_\flat(K)$. Since $\mathcal{O}_\flat(K)$ is a DFS space, it follows from the Serre-Komatsu duality theorem that the above isomorphisms are topological isomorphisms. Hence we have the isomorphism $H^{|n|}_K(\Omega, \mathcal{O}^\flat)\cong \mathcal{O}_\flat(K)'$. Q.E.D.

7.5. The Sato Theorem

In this section we will prove the pure-codimensionality of $\tilde{\boldsymbol{D}}^n$ with respect to $\mathcal{O}^\flat$. Then we will realize partial Fourier hyperfunctions as "boundary values" of partially slowly increasing holomorphic functions or as (relative) cohomology classes of partially slowly increasing holomorphic functions.

Theorem 7.5.1 (The Sato Theorem). *Let Ω be an open set in $\tilde{\boldsymbol{D}}^n$ and V an open set in $\boldsymbol{G}^n$ which contains Ω as its closed subsets. Then we have the following*

(1) *$\tilde{\boldsymbol{D}}^n$ is purely $|n|$-codimensional with respect to $\mathcal{O}^\flat$.*

(2) *The presheaf over $\tilde{\boldsymbol{D}}^n$*

$$\Omega \longrightarrow H^{|n|}_\Omega(V, \mathcal{O}^\flat)$$

is a sheaf.

(3) *This sheaf* (2) *is isomorphic to the sheaf $\mathscr{BR}$ of partial Fourier hyperfunctions.*

Proof. (1) It is sufficient to prove $H^p_\Omega(V, \mathcal{O}^b)=0\ (p \neq |n|)$ for a relatively compact open set Ω. Thus, it goes in a similar way to Kawai [19], p. 482.

(2) By (1) and the theorem II.3.18 of Komatsu [21], we have the conclusion.

(3) Consider the following exact sequence of relative cohomology groups for a relatively compact open set Ω

$$
\begin{aligned}
0 &\longrightarrow H^0_{\partial\Omega}(V, \mathcal{O}^b) \longrightarrow H^0_{\Omega^a}(V, \mathcal{O}^b) \longrightarrow H^0_\Omega(V, \mathcal{O}^b) \\
&\longrightarrow H^1_{\partial\Omega}(V, \mathcal{O}^b) \longrightarrow \quad \cdots \quad \longrightarrow H^{|n|-1}_\Omega(V, \mathcal{O}^b) \\
&\longrightarrow H^{|n|}_{\partial\Omega}(V, \mathcal{O}^b) \longrightarrow H^{|n|}_{\Omega^a}(V, \mathcal{O}^b) \longrightarrow H^{|n|}_\Omega(V, \mathcal{O}^b) \\
&\longrightarrow H^{|n|+1}_{\partial\Omega}(V, \mathcal{O}^b) \longrightarrow \cdots.
\end{aligned}
$$

Then, by (1) and by the Martineau-Harvey Theorem, we have $H^{|n|-1}_\Omega(V, \mathcal{O}^b)=0$, $H^{|n|+1}_{\partial\Omega}(V, \mathcal{O}^b)=0$. Thus we have the exact sequence

$$0 \longrightarrow H^{|n|}_{\partial\Omega}(V, \mathcal{O}^b) \longrightarrow H^{|n|}_{\Omega^a}(V, \mathcal{O}^b) \longrightarrow H^{|n|}_\Omega(V, \mathcal{O}^b) \longrightarrow 0.$$

Since, by the Martineau-Harvey Theorem, we have isomorphisms

$$H^{|n|}_{\partial\Omega}(V, \mathcal{O}^b) \cong \mathcal{A}_b(\partial\Omega)', \ H^{|n|}_{\Omega^a}(V, \mathcal{O}^b) \cong \mathcal{A}_b(\Omega^a)',$$

we obtain the isomorphism

$$H^{|n|}_\Omega(V, \mathcal{O}^b) \cong \mathcal{A}_b(\Omega^a)'/\mathcal{A}_b(\partial\Omega)' = \mathcal{BR}(\Omega).$$

Thus the sheaf $\Omega \rightarrow H^{|n|}_\Omega(V, \mathcal{O}^b)$ is isomorphic to the sheaf $\mathcal{BR}$ of partial Fourier hyperfunctions over $\tilde{\boldsymbol{D}}^n$. Q.E.D.

Let Ω be an open set in $\tilde{\boldsymbol{D}}^n$. Then there exists an $\mathcal{O}^b$-pseudoconvex open neighborhood V of Ω such that $V \cap \tilde{\boldsymbol{D}}^n = \Omega$ (cf. Ito [11], Theorem 10.1.10). We put $V_0 = V$ and $V_j = V \backslash \{z \in V;\ \mathrm{Im}\, z_j = 0\}^a$, $j=1, 2,\dots, |n|$. Then $\mathfrak{V} = \{V_0, V_1,\dots, V_{|n|}\}$ and $\mathfrak{V}' = \{V_1,\dots, V_{|n|}\}$ cover V and $V \backslash \Omega$ respectively. Since V_j and their intersections are also $\mathcal{O}^b$-pseudoconvex open sets, the covering $(\mathfrak{V}, \mathfrak{V}')$ satisfies the conditions of Leray's Theorem (cf. Komatsu [23]). Thus, by Leray's Theorem, we obtain the isomorphism $H^{|n|}_\Omega(V, \mathcal{O}^b) \cong H^{|n|}(\mathfrak{V}, \mathfrak{V}', \mathcal{O}^b)$. Since the covering $\mathfrak{V}$ is composed of only $|n|+1$ open sets V_j $(j=0, 1,\dots, |n|)$, we easily obtain the isomorphisms

$$Z^{|n|}(\mathfrak{V}, \mathfrak{V}', \mathcal{O}^b) \cong \mathcal{O}^b(\cap_j V_j),$$

$$C^{|n|-1}(\mathfrak{V}, \mathfrak{V}', \mathcal{O}^b) \cong \bigoplus_{j=1}^{|n|} \mathcal{O}^b(\bigcap_{i \neq j} V_i).$$

Hence we have

$$\delta C^{|n|-1}(\mathfrak{V}, \mathfrak{V}', \mathcal{O}^b) \cong \sum_{j=1}^{|n|} \mathcal{O}^b(\bigcap_{i \neq j} V_i) | V_1 \cap \cdots \cap V_{|n|}.$$

Thus we have the isomorphisms

$$H_{\Omega}^{|n|}(V, \mathcal{O}^b) \cong H^{|n|}(\mathfrak{V}, \mathfrak{V}', \mathcal{O}^b)$$
$$\cong Z^{|n|}(\mathfrak{V}, \mathfrak{V}', \mathcal{O}^b)/\delta C^{|n|-1}(\mathfrak{V}, \mathfrak{V}', \mathcal{O}^b)$$
$$\cong \mathcal{O}^b(\cap_j V_j) / \sum_{j=1}^{|n|} \mathcal{O}^b(\bigcap_{i \neq j} V_i).$$

Thus we have the following

Theorem 7.5.2. *We use notations as above. Then we have the isomorphisms*

$$H_{\Omega}^{|n|}(V, \mathcal{O}^b) \cong H^{|n|}(\mathfrak{V}, \mathfrak{V}', \mathcal{O}^b) \cong \mathcal{O}^b(\cap_j V_j) / \sum_{j=1}^{|n|} \mathcal{O}^b(\bigcap_{i \neq j} V_i).$$

At last we will realize partial Fourier analytic functionals with certain compact carrier as (relative) cohomology classes with coefficients in $\mathcal{O}^b$.

Let K be a compact set in $\boldsymbol{G}^n$ of the form $K = K_1 \times \cdots \times K_{|n|}$ with compact sets K_j in $\boldsymbol{C}$ for $j = 1, \ldots, n_1$ and in $\tilde{\boldsymbol{C}}$ for $j = n_1 + 1, \ldots, |n|$. Assume that K admits a fundamental system of $\mathcal{O}^b$-pseudoconvex open neighborhoods. Then we have

$$H^p(K, \mathcal{O}_b) = 0 \qquad \text{for} \quad p > 0.$$

By virtue of the Martineau-Harvey Theorem, there exists the isomorphism

$$\mathcal{O}_b(K)' \cong H_K^{|n|}(\boldsymbol{G}^n, \mathcal{O}^b).$$

Further assume that there exists an $\mathcal{O}_b$-pseudoconvex open neighborhood K such that

$$\Omega_j = \Omega \backslash \{z \in \boldsymbol{C}^{|n|};\ z_j \in K_j \cap \boldsymbol{C}\}^a$$

is also an $\mathcal{O}^b$-pseudoconvex open set for $j = 1, 2, \ldots, |n|$. Put $\Omega_0 = \Omega$. Then $V = \{\Omega_0, \Omega_1, \ldots, \Omega_{|n|}\}$ and $V' = \{\Omega_1, \Omega_2, \ldots, \Omega_{|n|}\}$ form acyclic coverings of Ω and $\Omega \backslash K$. Set

$$\Omega \sharp K = \bigcap_{j=1}^{|n|} \Omega_j,$$
$$\Omega^j = \bigcap_{i \neq j} \Omega_i.$$

Let $\sum_j \mathcal{O}^b(\Omega^j)$ be the image in $\mathcal{O}^b(\Omega \sharp K)$ of $\prod_{j=1}^{|n|} \mathcal{O}^b(\Omega^j)$ by the mapping

$$(f_j)_{j=1}^{|n|} \longrightarrow \sum_{j=1}^{|n|} (-1)^{j+1} f'_j,$$

where f'_j denotes the restriction of f_j to $\Omega \sharp K$.

Then, by a similar method to that of Theorem 7.6.2, we have the following

Theorem 7.5.3. *We use the notations as above. Then we have the isomorphisms*

$$\mathcal{O}_b(K)' \cong H_K^{|n|}(G^n, \mathcal{O}^b) \cong H^{|n|}(\mathfrak{V}, \mathfrak{V}', \mathcal{O}^b) \cong \mathcal{O}^b(\Omega \# K) / \sum_j \mathcal{O}^b(\Omega^j).$$

By the above theorem, we can define the canonical mapping

$$b: \mathcal{O}^b(\Omega \# K) \longrightarrow \mathcal{O}_b(K)'$$

whose kernel is $\sum_j \mathcal{O}^b(\Omega^j)$.

Then we have the following

Theorem 7.5.4. *We use the above notations.*

(i) *Let $u \in \mathcal{O}_b(K)'$ and put*

$$\tilde{u}(z) = (2i\pi)^{-|n|} u_\xi((\xi - z)^{-1} \exp(-(\xi'' - z'')^2)),$$

where we put

$$(\xi - z)^{-1} = \prod_{j=1}^{|n|} (\xi_j - z_j)^{-1} \text{ and } (\xi'' - z'')^2 = \sum_{j=n_1+1}^{|n|} (\xi_j - z_j)^2.$$

Then $\tilde{u} \in \mathcal{O}^b(\Omega \# K)$ and $b(\tilde{u}) = u$ hold.

(ii) *Let $f \in \mathcal{O}^b(\Omega \# K)$ and $g \in \mathcal{O}_b(K)$. Let $\omega = \omega_1 \times \cdots \times \omega_{|n|} \subset \Omega$ with open neighborhoods ω_j of K_j in $\boldsymbol{C}$ or $\tilde{\boldsymbol{C}}$ and $g \in \mathcal{O}_b(\omega)$ where $\tilde{\omega}$ is an open neighborhood of ω with $\tilde{\omega} \subset \Omega$. Let Γ_j $(j = 1, 2, \ldots, |n|)$ be regular contours in $\omega_j \cap \boldsymbol{C}$ enclosing once $K_j \cap \boldsymbol{C}$ and oriented in the usual way. Then we have*

$$b(f)(g) = (-1)^{|n|} \int_{\Gamma_1} \cdots \int_{\Gamma_{|n|}} f(z) g(z) dz_1 \cdots dz_{|n|}.$$

Proof. The integral

$$(-1)^{|n|} \int_{\Gamma_1} \cdots \int_{\Gamma_{|n|}} f(z) g(z) dz_1 \cdots dz_{|n|}$$

does not depend on the chosen contours and defines a linear mapping

$$b': \mathcal{O}^b(\Omega \# K) \longrightarrow \mathcal{O}_b(K)',$$

which is zero on $\sum_j \mathcal{O}^b(\Omega^j)$. Hence, in order to prove (ii), it is sufficient to prove that, if $u \in \mathcal{O}_b(K)'$, we have

$$b'(\tilde{u}) = u.$$

But

$$\begin{aligned} b'(\tilde{u})(g) &= (-1)^{|n|} (2i\pi)^{-|n|} \int_{\Gamma_1} \cdots \int_{\Gamma_{|n|}} u_\xi((\xi - z)^{-1} \exp(-(\xi'' - z'')^2)) g(z) dz \\ &= u_\xi((2i\pi)^{-|n|} \int_{\Gamma_1} \cdots \int_{\Gamma_{|n|}} g(z)(z - \xi)^{-1} \exp(-(z'' - \xi'')^2) dz) \\ &= u(g). \end{aligned}$$

This proves (i) and completes the proof. Q.E.D.

Chapter 8. Case of the sheaf ${}^{E}\mathcal{O}^{b}$

8.1. The Dolbeault-Grothendieck resolution of ${}^{E}\mathcal{O}^{b}$

In this section we will construct a soft resolution of ${}^{E}\mathcal{O}^{b}$. Here E denotes a quasi-complete LCTVS (always assumed to be Hausdorff) unless the contrary is explicitly mentioned and $\mathscr{T}=\mathscr{T}_E$ denotes the family of continuous seminorms of E defining a locally convex topology on E.

At first we will define sheaves ${}^{E}\mathcal{O}^{b}$ and ${}^{E}\mathscr{E}^{b}$.

Definition 8.1.1 (The sheaf ${}^{E}\mathcal{O}^{b}$ of germs of partially slowly increasing E-valued holomorphic functions over G^n). *We define the sheaf ${}^{E}\mathcal{O}^{b}$ to be the sheafification of the presheaf $\{\mathcal{O}^{b}(\Omega; E)\}$, where, for an open set Ω in G^n, the module $\mathcal{O}^{b}(\Omega; E)$ is defined as follows:*

$\mathcal{O}^{b}(\Omega; E)=\{f\in\mathcal{O}(\Omega\cap C^{|n|}; E)$; *for any positive* ε *and any compact set* K *in* Ω *and any* $q\in\mathscr{T}$, $\sup\{q(f(z))e(-\varepsilon|z|);\ z\in K\cap C^{|n|}\}<\infty$ *holds*$\}$.

We call this sheaf ${}^{E}\mathcal{O}^{b}$ the sheaf of germs of partially slowly increasing E-valued holomorphic functions.

Definition 8.1.2 (The sheaf ${}^{E}\mathscr{E}^{b}$ of germs of partially slowly increasing E-valued C^∞-functions). *We define ${}^{E}\mathscr{E}^{b}$ to be the sheafification of the presheaf $\{\mathscr{E}^{b}(\Omega; E)\}$, where, for an open set Ω in G^n, the module $\mathscr{E}^{b}(\Omega; E)$ is defined as follows:*

$\mathscr{E}^{b}(\Omega; E)=\{f\in\mathscr{E}(\Omega\cap C^{|n|}; E)$; *for any positive* ε *and any compact set* K *in* Ω *and any* $\alpha\in\overline{N}^{2|n|}$ *and any* $q\in\mathscr{T}$, $\sup\{q(f^{(\alpha)}(z))e(-\varepsilon|z|);\ z\in K\cap C^{|n|}\}<\infty$ *holds*$\}$.

Then the sheaf ${}^{E}\mathscr{E}^{b}$ is soft, and we have the following

Theorem 8.1.3 (The Dolbeault-Grothendiek resolution of ${}^{E}\mathcal{O}^{b,p}$). *Let* E be *a quasi-complete* LCTVS. *Then the sequence of sheaves over* G^n

$$0\longrightarrow {}^{E}\mathcal{O}^{b,p}\longrightarrow {}^{E}\mathscr{E}^{b,p,0}\xrightarrow{\bar\partial} {}^{E}\mathscr{E}^{b,p,1}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial} {}^{E}\mathscr{E}^{b,p,|n|}\longrightarrow 0$$

is exact.

Proof. It goes in a similar way to that of Ito [10], Theorem 3.1, p. 989. Q.E.D.

Corollary. *For an open set Ω in G^n, we have the following isomorphism:*

$$H^q(\Omega, {}^{E}\mathcal{O}^{b,p})\cong\{f\in\mathscr{E}^{b,p,q}(\Omega; E);\ \bar\partial f=0\}/\{\bar\partial g;\ g\in\mathscr{E}^{b,p,q-1}(\Omega; E)\},\quad (p\geqq 0,\ q\geqq 1).$$

Proof. It follows from Theorem 8.1.3 and Komatsu [21], Theorems II.2.9 and II.2.19. Q. E. D.

8.2. The Oka-Cartan-Kawai Theorem B

In this section we will prove the Oka-Cartan-Kawai Theorem B for the sheaf ${}^E\mathcal{O}^\flat$. In the sequel of this chapter, E is always assumed to be a Fréchet space.

Theorem 8.2.1 (The Oka-Cartan-Kawai Theorem B). *For any $\mathcal{O}^\flat$-pseudoconvex open set Ω in $\boldsymbol{G}^n$, we have $H^q(\Omega, {}^E\mathcal{O}^{\flat,p})=0$ for $p\geqq 0$ and $q\geqq 1$.*

Proof. Since we have, by the Oka-Cartan-Kawai Theorem B for $\mathcal{O}^\flat$,

$$H^q(\Omega, \mathcal{O}^{\flat,p})=0 \qquad (p\geqq 0,\ q\geqq 1),$$

the complex obtained from Theorem 7.2.2:

$$\mathscr{E}^{\flat,p,0}(\Omega) \xrightarrow{\bar\partial} \mathscr{E}^{\flat,p,1}(\Omega) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{E}^{\flat,p,|n|}(\Omega) \longrightarrow 0$$

is exact. Since $\mathscr{E}^{\flat,p,q}(\Omega)$'s are nuclear Fréchet spaces and E is a Fréchet space, the complex

$$\mathscr{E}^{\flat,p,0}(\Omega; E) \xrightarrow{\bar\partial} \mathscr{E}^{\flat,p,1}(\Omega; E) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{E}^{\flat,p,|n|}(\Omega; E) \longrightarrow 0$$

is also exact by virtue of the isomorphism

$$\mathscr{E}^{\flat,p,q}(\Omega; E)\cong \mathscr{E}^{\flat,p,q}(\Omega)\hat{\otimes} E$$

and Ion and Kawai [5], Theorem 1.10. Hence we obtain

$$H^q(\Omega, {}^E\mathcal{O}^{\flat,p})=0 \qquad (p\geqq 0,\ q\geqq 1).$$

This completes the proof. Q. E. D.

Corollary. *Let Ω be an $\mathcal{O}^\flat$-pseudoconvex open set in $\boldsymbol{G}^n$. Then the equation $\bar\partial u=f$ has a solution $u\in\mathscr{E}^{\flat,p,q}(\Omega; E)$ for every $f\in\mathscr{E}^{\flat,p,q+1}(\Omega; E)$ such that $\bar\partial f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 8.2.1 and Corollary to Theorem 8.1.3. Q. E. D.

8.3. The Serre duality theorem

Theorem 8.3.1. *Let Ω be an open set in $\boldsymbol{G}^n$ such that $\dim H^p(\Omega, \mathcal{O}^\flat)<\infty$ holds $(p\geqq 1)$. Then we have the isomorphism $H^p(\Omega, {}^E\mathcal{O}^\flat)\cong L(H_c^{|n|-p}(\Omega, \mathcal{O}_\flat); E)$, $0\leqq p\leqq |n|$.*

Proof. By a similar method to Junker [15], Lemma 3.5, we can obtain the isomorphism $H^p(\Omega, {}^E\mathcal{O}^\flat)\cong H^p(\Omega, \mathcal{O}^\flat)\hat{\otimes}_\pi E$. Then, by Theorem 7.3.1, we have the following isomrophisms

$$H^p(\Omega, {}^E\mathcal{O}^b) \cong H^p(\Omega, \mathcal{O}^b)\hat{\otimes}_\pi E \cong [H_c^{|n|-p}(\Omega, \mathcal{O}_b)]'\hat{\otimes}_\pi E$$
$$\cong L(H_c^{|n|-p}(\Omega, \mathcal{O}_b); E). \qquad \text{Q.E. D.}$$

8.4. The Martineau-Harvey Theorem

Theorem 8.4.1. *Let K be a compact set in $\boldsymbol{G}^n$ such that it has an $\mathcal{O}^b$-pseudoconvex open neighborhood Ω and satisfies the conditions $H^p(K, \mathcal{O}_b)=0$ $(p\geqq 1)$. Then we have $H_K^p(\Omega, {}^E\mathcal{O}^b)=0$ for $p\neq|n|$ and isomorphisms $H_K^{|n|}(\Omega, {}^E\mathcal{O}^b)\cong H^{|n|-1}(\Omega\backslash K, {}^E\mathcal{O}^b)\cong L(\mathcal{O}_b(K); E)$.*

Proof. We can assume that Ω is an $\mathcal{O}^b$-pseudoconvex open neighborhood of K. Then, in the long exact sequence of cohomology groups (cf. Komatsu [21], Theorem II.3.2):

$$0 \longrightarrow H_K^0(\Omega, {}^E\mathcal{O}^b) \longrightarrow H^0(\Omega, {}^E\mathcal{O}^b) \longrightarrow H^0(\Omega\backslash K, {}^E\mathcal{O}^b)$$
$$\longrightarrow H_K^1(\Omega, {}^E\mathcal{O}^b) \longrightarrow H^1(\Omega, {}^E\mathcal{O}^b) \longrightarrow H^1(\Omega\backslash K, {}^E\mathcal{O}^b)$$
$$\longrightarrow \cdots$$
$$\longrightarrow H_K^{|n|}(\Omega, {}^E\mathcal{O}^b) \longrightarrow H^{|n|}(\Omega, {}^E\mathcal{O}^b) \longrightarrow H^{|n|}(\Omega\backslash K, {}^E\mathcal{O}^b) \longrightarrow \cdots,$$

we have $H^p(\Omega, {}^E\mathcal{O}^b)=0$ for $p\geqq 1$ and $H_K^0(\Omega, {}^E\mathcal{O}^b)=0$ by the unique continuation theorem. Hence we have isomorphisms

$$H_K^1(\Omega, {}^E\mathcal{O}^b)\cong\mathcal{O}^b(\Omega\backslash K; E)/\mathcal{O}^b(\Omega; E),$$
$$H_K^p(\Omega, {}^E\mathcal{O}^b)\cong H^{p-1}(\Omega\backslash K, {}^E\mathcal{O}^b), \quad p\geqq 2.$$

But, by a similar way to that of Junker [15], Lemma 3.5, we have isomorphisms $H^p(V, {}^E\mathcal{O}^b)\cong H^p(V, \mathcal{O}^b)\hat{\otimes}E$, $0\leqq p\leqq|n|$, for any open set V in $\boldsymbol{G}^n$. So that, by Theorem 7.4.1, we have isomorphisms

$$H_K^p(\Omega, {}^E\mathcal{O}^b)\cong H_K^p(\Omega, \mathcal{O}^b)\hat{\otimes}_\pi E=0 \qquad \text{for} \quad p\neq|n|,$$

and

$$H_K^{|n|}(\Omega, {}^E\mathcal{O}^b)\cong H^{|n|-1}(\Omega\backslash K, {}^E\mathcal{O}^b)\cong H^{|n|-1}(\Omega\backslash K, \mathcal{O}^b)\hat{\otimes}_\pi E$$
$$\cong H_K^{|n|}(\Omega, \mathcal{O}^b)\hat{\otimes}_\pi E\cong\mathcal{O}_b(K)'\hat{\otimes}_\pi E\cong L(\mathcal{O}_b(K); E). \qquad \text{Q. E. D.}$$

8.5. The Sato Theorem

In this section we will prove the pure-codimensionality of $\tilde{\boldsymbol{D}}^n$ with respect to ${}^E\mathcal{O}^b$. Then we realize E-valued partial Fourier hyperfunctions as "boundary values" of E-valued partially slowly increasing holomorphic functions or as (relative) cohomology classes of E-valued partially slowly increasing holomorphic functions.

Theorem 8.5.1 (The Sato Theorem). *Let Ω be an open set in $\tilde{\boldsymbol{D}}^n$ and V an open* *set in $\boldsymbol{G}^n$ which contains Ω as its closed subset. Then we have the following*

(1) *$\tilde{\boldsymbol{D}}^n$ is purely $|n|$-codimensional with respect to ${}^E\mathcal{O}^b$.*

(2) *The presheaf over $\tilde{\boldsymbol{D}}^n$*

$$\Omega \longrightarrow H^{|n|}_{\Omega}(V, {}^E\mathcal{O}^b)$$

is a sheaf.

(3) *This sheaf (2) is isomorphic to the sheaf ${}^E(\mathscr{BR})$ of E-valued partial Fourier hyperfunctions.*

Proof. (1) It is sufficient to prove $H^p_{\Omega}(V, {}^E\mathcal{O}^b)=0$ $(p \neq |n|)$ for a relatively compact open set Ω. Thus, by the excision theorem, we may assume that V is an $\mathcal{O}^b$-pseudoconvex open set in $\boldsymbol{G}^n$. Consider the following exact sequence of relative cohomology groups

$$\begin{aligned} 0 &\longrightarrow H^0_{\partial\Omega}(V, {}^E\mathcal{O}^b) \longrightarrow H^0_{\Omega^a}(V, {}^E\mathcal{O}^b) \longrightarrow H^0_{\Omega}(V, {}^E\mathcal{O}^b) \\ &\longrightarrow H^1_{\partial\Omega}(V, {}^E\mathcal{O}^b) \longrightarrow \cdots \longrightarrow H^{|n|-1}_{\Omega}(V, {}^E\mathcal{O}^b) \\ &\longrightarrow H^{|n|}_{\partial\Omega}(V, {}^E\mathcal{O}^b) \longrightarrow H^{|n|}_{\Omega^a}(V, {}^E\mathcal{O}^b) \longrightarrow H^{|n|}_{\Omega}(V, {}^E\mathcal{O}^b) \\ &\longrightarrow H^{|n|+1}_{\partial\Omega}(V, {}^E\mathcal{O}^b) \longrightarrow \cdots. \end{aligned}$$

By Theorems 7.1.8 and 8.4.1, we may conclude that $H^p_{\partial\Omega}(V, {}^E\mathcal{O}^b)=H^p_{\Omega^a}(V, {}^E\mathcal{O}^b)=0$ for $p \neq |n|$. So that, we have $H^p_{\Omega}(V, {}^E\mathcal{O}^b)=0$ for $p \neq |n|-1, |n|$. On the other hand, by Theorems 7.1.8 and 8.4.1, we also have the exact sequence

$$0 \longrightarrow H^{|n|-1}_{\Omega}(V, {}^E\mathcal{O}^b) \longrightarrow L(\mathscr{A}_b(\partial\Omega); E) \xrightarrow{j} L(\mathscr{A}_b(\Omega^a); E).$$

Since j is injective, we have $H^{|n|-1}_{\Omega}(V, {}^E\mathcal{O}^b)=0$.

(2) By (1) and the theorem II.3.18 of Komatsu [21], we have the conclusion.

(3) By the proof of (1), we have the exact sequence for a relatively compact open set Ω

$$0 \longrightarrow H^{|n|}_{\partial\Omega}(V, {}^E\mathcal{O}^b) \longrightarrow H^{|n|}_{\Omega^a}(V, {}^E\mathcal{O}^b) \longrightarrow H^{|n|}_{\Omega}(V, {}^E\mathcal{O}^b) \longrightarrow 0.$$

Since, by the Martineau-Harvey Theorem, we have isomorphisms

$$H^{|n|}_{\partial\Omega}(V, {}^E\mathcal{O}^b) \cong L(\mathscr{A}_b(\partial\Omega); E),$$

$$H^{|n|}_{\Omega^a}(V, {}^E\mathcal{O}^b) \cong L(\mathscr{A}_b(\Omega^a); E),$$

we obtain the isomorphism

$$H^{|n|}_{\Omega}(V, {}^E\mathcal{O}^b) \cong L(\mathscr{A}_b(\Omega^a); E)/L(\mathscr{A}_b(\partial\Omega); E) \cong \mathscr{BR})\Omega; E).$$

Thus the sheaf $\Omega \rightarrow H^{|n|}_{\Omega}(V, {}^E\mathcal{O}^b)$ is isomorphic to the sheaf ${}^E(\mathscr{BR})$ of E-valued partial Fourier hyperfunctions over $\tilde{\boldsymbol{D}}^n$. Q. E. D.

In a similar notations to in Theorem 7.5.2, we have the following

Theorem 8.5.2. $H^{|n|}_{\Omega}(V, {}^{E}\mathcal{O}^{b}) \cong H^{|n|}(\mathfrak{V}, \mathfrak{V}', {}^{E}\mathcal{O}^{b}) \cong \mathcal{O}^{b}(\cap_j V_j; E) / \sum_{j=1}^{|n|} \mathcal{O}^{b}(\bigcap_{i \neq j} V_i; E)$ *hold.*

At last we will realize partial Fourier analytic linear mappings with certain compact carrier as (relative) cohomology classes with coefficients in ${}^{E}\mathcal{O}^{b}$.

Let K be a compact set in $\boldsymbol{G}^n$ of the form $K = K_1 \times \cdots \times K_{|n|}$ with compact sets K_j in $\boldsymbol{C}$ for $j = 1, \ldots, n_1$ and in $\tilde{\boldsymbol{C}}$ for $j = n_1 + 1, \ldots, |n|$. Assume that K admits a fundamental system of $\mathcal{O}^{b}$-pseudoconvex open neighborhoods. Then we have

$$H^p(K, \mathcal{O}_b) = 0 \qquad \text{for} \quad p > 0.$$

By virtue of the Martineau-Harvey Theorem, there exists the isomorphism

$$\mathcal{O}'_b(K; E) \cong H^{|n|}_K(\boldsymbol{G}^n, {}^{E}\mathcal{O}^{b}).$$

Further assume that there exists an $\mathcal{O}^{b}$-pseudoconvex open neighborhood Ω such that

$$\Omega_j = \Omega \backslash \{z \in \boldsymbol{G}^n;\ z_j \in K_j \cap \boldsymbol{C}\}^a$$

is also an $\mathcal{O}^{b}$-pseudoconvex open set for $j = 1, 2, \ldots, |n|$. Put $\Omega_0 = \Omega$. Then $\mathfrak{V} = \{\Omega_0, \Omega_1, \ldots, \Omega_{|n|}\}$ and $\mathfrak{V}' = \{\Omega_1, \Omega_2, \ldots, \Omega_{|n|}\}$ form acyclic coverings of Ω and $\Omega \backslash K$. Set

$$\Omega \# K = \bigcap_{j=1}^{|n|} \Omega_j, \quad \Omega^j = \cap_{i \neq j} \Omega_i.$$

Let $\sum_j \mathcal{O}^{b}(\Omega^j; E)$ be the image in $\mathcal{O}^{b}(\Omega \# K; E)$ of $\prod_{j=1}^{|n|} \mathcal{O}^{b}(\Omega^j; E)$ by the mapping

$$(f_j)_{j=1}^{|n|} \longrightarrow \sum_{j=1}^{|n|} (-1)^{j+1} f'_j,$$

where f'_j denotes the restriction of f_j to $\Omega \# K$.

Then, by a similar way to that of Theorem 8.5.2, we have the following

Theorem 8.5.3. *We use the notations as above. Then we have the isomorphisms*

$$\mathcal{O}'_b(K; E) \cong H^{|n|}_K(\boldsymbol{G}^n, {}^{E}\mathcal{O}^{b}) \cong H^{|n|}(\mathfrak{V}, \mathfrak{V}', {}^{E}\mathcal{O}^{b})$$
$$\cong \mathcal{O}^{b}(\Omega \# K; E) / \sum_j \mathcal{O}^{b}(\Omega^j; E).$$

By the above theorem, we can define the canonical mapping

$$b\colon \mathcal{O}^{b}(\Omega \# K; E) \longrightarrow \mathcal{O}'_b(K; E)$$

whose kernel is $\sum_j \mathcal{O}^{b}(\Omega^j; E)$.

Then we have the following

Theorem 8.5.4. *We use the above notations.*

(i) *Let* $u\in \mathcal{O}_b'(K;E)$ *and put*

$$\tilde{u}(z)=(2i\pi)^{-|n|}u_\xi((\xi-z)^{-1}\exp(-(\xi''-z'')^2)),$$

Then $\tilde{u}\in \mathcal{O}^b(\Omega\# K;E)$ *and* $b(\tilde{u})=u$ *hold.*

(ii) *Let* $f\in \mathcal{O}^b(\Omega\# K;E)$ *and* $g\in \mathcal{O}_b(K)$. *Let* $\omega=\omega_1\times\cdots\times\omega_{|n|}\subset\Omega$ *with open neighborhoods* ω_j *of* K_j *in* $\boldsymbol{C}$ *or* $\tilde{\boldsymbol{C}}$ *and* $g\in\mathcal{O}_b(\tilde{\omega})$ *where* $\tilde{\omega}$ *is an open neighborhood of* ω *with* $\tilde{\omega}\subset\Omega$. *Let* Γ_j $(j=1,2,\dots,|n|)$ *be regular contours in* $\omega_j\cap\boldsymbol{C}$ *enclosing once* $K_j\cap\boldsymbol{C}$ *and oriented in the usual way. Then we have*

$$b(f)(g)=(-1)^{|n|}\int_{\Gamma_1}\cdots\int_{\Gamma_{|n|}}f(z)g(z)dz_1\cdots dz_{|n|}.$$

Proof. The integral

$$(-1)^{|n|}\int_{\Gamma_1}\cdots\int_{\Gamma_{|n|}}f(z)g(z)dz_1\cdots dz_{|n|}$$

does not depend on the chosen contours and defines a linear mapping

$$b'\colon \mathcal{O}^b(\Omega\# K;E)\longrightarrow \mathcal{O}_b'(K;E),$$

which is zero on $\sum_j \mathcal{O}^b(\Omega^j;E)$. Hence, in order to prove (ii), it is sufficient to prove that, if $u\in\mathcal{O}_b'(K;E)$, we have

$$b'(\tilde{u})=u.$$

But

$$\begin{aligned}
&b'(\tilde{u})(g)\\
&=(-1)^{|n|}(2i\pi)^{-|n|}\int_{\Gamma_1}\cdots\int_{\Gamma_{|n|}}u_\xi((\xi-z)^{-1}\exp(-(\xi''-z'')^2))g(z)dz\\
&=u_\xi((2i\pi)^{-|n|}\int_{\Gamma_1}\cdots\int_{\Gamma_{|n|}}g(z)(z-\xi)^{-1}\exp(-(z''-\xi'')^2)dz)\\
&=u(g).
\end{aligned}$$

This proves (i) and completes the proof. Q. E. D.

Note. Recently, I found out that Malgrange Theorems are no longer of any use for proving Sato Theorems which are the main theorems in each chapter. The reason is this. At first we used the Malgrange Theorem for proving the flabbiness of the sheaf $\Omega\to H_\Omega^{|n|}(V,\mathcal{O}^\sharp)$ in the case of Theorem 5.5.1. But the presheaf $\Omega\to H_\Omega^{|n|}(V,\mathcal{O}^\sharp)$ is a sheaf only by virtue of the pure-codimensionality of $\boldsymbol{D}^n$ with respect to $\mathcal{O}^\sharp$. Then this sheaf is isomorphic to the sheaf $\mathscr{P}$ of mixed Fourier hyperfunctions which is flabby. Thus, by this isomorphism, we can conclude that the sheaf $\Omega \rightarrow H_{\Omega}^{|n|}(V, \mathcal{O}^{\#})$ is flabby. All the other cases treated in this series of papers are in the same situation.

Therefore we omitted the section concerning the Malgrange Theorem and renumbered the remaining sections.

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Added in Proof. Having submitted this paper to the editorial committee, I again read carefully the manuscript of this paper and remembered that the Malgrange theorems were used for proving the Martineau-Harvey theorems though the Malgrange theorems were not used directly for proving the Sato theorems. But fortunately the Malgrange theorems are the fairly separate ones. Thus, in this paper, we were forced to get on without the explicit mention of the Malgrange theorems even though the selfcontainedness was sacrificed.