J. Math. Tokushima Univ.
Vol. 26 (1992), 1–8

On Certain Real Bicyclic Biquadratic Fields with Class Number One and Two

By

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(Received September 14, 1992)

Abstract

Let $K = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$ be the bicyclic biquadratic fields, where d_i are the integers expressed in the forms $d_i = m^2 + 4$ or $m^2 + 1$ $(m \in \mathrm{N})$. Using Tatuzawa's lower bound of L-function, we shall show there are only finitely many such fields with class number one and two. Assuming the generalized Riemann Hypothesis, there exist exactly 54 real bicyclic biquadratic fields with class number one and 118 fields with class number two.

1991 Mathematics Subject Classification. Primary 11R29; Secondary 11Y40

Introduction

A.Baker and H.M.Stark proved that there are exactly 9 imaginary quadratic fields with class number one and 18 imaginary quadratic fields with class number two. Using these results, E.Brown, C.Parry and K.Uchida independently proved that there are exactly 47 imaginary bicyclic biquadratic number fields with class number one in [2] and [15]. For the real case, using Tatuzawa's lower bound of L-function, H.K.Kim, M.-G.Leu and T.Ono proved that there are at most 13 quadratic fields $\mathbf{Q}(\sqrt{d})$ with class number one in [7] and M.-G.Leu proved that there are at most 18 quadratic fields with class number two in [12], where d are the square-free integers of the form $d = m^2 + 4$ or $m^2 + 1$ $(m \in \mathrm{N})$.
Using these results, we shall show that there exist 54 real bicyclic biquadratic fields $K = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$ with class number one and 118 real bicyclic biquadratic fields with class number two. Here d_i are the square-free integers expressed in the forms $d_i = m^2 + 4$ or $m^2 + 1$ $(m \in \mathrm{N})$. Since the result of Tatuzawa [14] allow the possibility of one more field, our results allow several more fields. However, in [8], H.K.Kim proved

if one assumes the generalized Riemann Hypothesis, the conclusion of Tatuzawa is true without any exception. So, if we assume the generalized Riemann Hypothesis, there are exactly 54 real bicyclic biquadratic fields with class number one and 118 fields with class number two.

§1. Class numbers and the unit groups of real quadratic fields

First, we recall several facts on the class numbers and the unit groups of real quadratic fields. Let d be a positive square-free integer and k be the real quadratic field $\mathbf{Q}(\sqrt{d})$. We denote the fundamental unit of k by $\varepsilon_d = (t_d + u_d\sqrt{d})/2 > 1$ and denote the class number of k by $h(d)$.

Lemma 1 (Tatuzawa). *For any $x \geq 11.2$, let d be a positive integer such that $d \geq e^x$. Let χ be any non-principal primitive real character to modulus d, and $L(s,\chi)$ be the corresponding L-seres. Then $L(1,\chi) > 0.655d^{-1/x}/x$ holds with one possible exception.*

From the class number formula, we have

$$h(d) = \frac{\sqrt{d}L(1,\chi)}{2\log\varepsilon_d} \quad ,$$

where χ be the Kronecker character belonging to k.
From Lemma 1, we have

$$h(d) > g(x,\log u) = \frac{0.655e^{x/2-1/x}}{x(x+2\log u)} \quad ,$$

where $x = \log d \geq 11.2$.

It is easy to see $g(x,\log u)$ is a monotone increasing function of x $(x \geq 11.2)$ for any fixed u and $\lim_{x\to\infty} g(x,\log u) = \infty$. Therefore, for any positive integer h and any real number $M > 0$, there exist a constant c such that $g(c,\log u) > h$ for any $u \leq M$. Hence if $d \geq e^c$, then $h(d) > h$. Therefore we have shown the following proposition.

Proposition 1. *For any integer $h > 0$ and any real number $M > 0$, there exist only finitely many real quadratic fields k with $h(d) = h$ and $u_d \leq M$.*

For the case $d = m^2+4$ or m^2+1 $(m \in \mathrm{N})$ and d is square-free, we see $\varepsilon_d = (m+\sqrt{m^2+4})/2$ or $(2m+2\sqrt{m^2+1})/2$. Hence $u_d = 1$ or 2 for these cases. One can easily verify $g(14.5,\log 1) = 4.09\cdots > 4$, $g(14.7,\log 2) = 4.02\cdots > 4$ and $1409^2+4 = 1985285 > e^{14.5}$ and $1557^2+1 = 2424250 > e^{14.7}$. On the other hand, using the results of Ankeny-Chowla-Hasse [1] and S.-D.Lang [11], we have the following lemma.

Lemma 2. *Let d be a square-free positive integer of the form $d = m^2 + 4$ (resp. $m^2 + 1$). Let q be the least prime such that $\chi(q) = 1$. If $q^h < m$ $(2q^h < m)$, then $h(d) > h$.*

Combining the genus theory and the above lemmas, we have the following.

Proposition 2. *Let $k = \mathbf{Q}(\sqrt{d})$ be a real quadratic fields such as $d = m^2 + 4$ (resp. $m^2 + 1$) and $h(d) \leq 4$. Then we have the following.*

(1) *$d = p_1$, p_1p_2 or $p_1p_2p_3$, where p_i are the primes such that $p_1 < p_2 < p_3$.*

(2) *Let q be the least prime such that $\chi(q) = 1$, then $q^4 \geq m$ (resp. $2q^4 \geq m$).*

(3) *$m \leq 1408$ (resp. $m \leq 1556$), with the possibility of an exceptional d.*

By the help of a computer, we found there are 12 real quadratic fields with class number 1 and 17 real quadratic fields with class number 2 and 12 real quadratic fields with class number 3 and 29 real quadratic fields with class number 4, with one possible exception of d.

Proposition 3. *There exist 70 positive integers d of the form $m^2 + 4$ or $m^2 + 1$ such that $1 \leq h(d) \leq 4$, with one possible exception.*

(1) $h(d) = 1$ $d = 2, 5, 13, 17, 29, 37, 53, 101, 173, 197, 293, 677$ (Theorem 2 of [7]).

(2) $h(d) = 2$ $d = 10, 26, 65, 85, 122, 362, 365, 485, 533, 629, 965, 1157, 1685, 1853, 2117, 2813, 3365$ (Theorem 2 and 2' of [12]).

(3) $h(d) = 3$ $d = 229, 257, 733, 1229, 1373, 2213, 2917, 4493, 5333, 5477, 8837, 9413$.

(4) $h(d) = 4$ $d = 82, 145, 170, 290, 445, 530, 626, 901, 962, 1370, 2405, 2501, 3029, 3485, 3845, 5045, 5933, 6245, 6893, 7397, 9605, 10205, 11237, 11453, 12773, 14885, 16133, 20165, 24653$.

§2. Class numbers and the unit groups of bicyclic biquadratic fields

First, we recall several facts on the class number relations of elementary abelian 2-extensions of $\mathbf{Q}$. K denotes the composite of independent real quadratic fields $k_1, \ldots, k_n$. Since the Galois group $K/\mathbf{Q}$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$, there are exactly $t = 2^n - 1$ different quadratic subfields of K. We denote them $k_1, \ldots, k_t$ and the fundamental unit of $k_i = \mathbf{Q}(\sqrt{d_i})$ by $\varepsilon_i = (t_i + u_i\sqrt{d_i})/2 > 1$. h_i and E_i denote the class number and the unit group of k_i, and H_K and E_K denote the class number and the unit group of K. Then it is known (c.f. [10]) that

$$(1) \quad H_K = \frac{1}{2^v}[E_K : \prod_{i=1}^{t} E_i] \cdot \prod_{i=1}^{t} h_i,$$

where $v = n(2^{n-1} - 1)$. Combining this fact and the above lemmas, one can generalize

Proposition 1 as follows.

Proposition 4. *Let n be a positive integer. For any integer* $H > 0$ *and any real number* $M > 0$*, there exist only finitely many fields* K *with the Galois group* $Gal(K/\mathbf{Q})$ *isomorphic to* $(\mathbf{Z}/2\mathbf{Z})^n$*,* $H_K = H$ *and* $u_i \leq M \quad (1 \leq i \leq n)$.

In the follwing, we consider the case when $n = 2, H \leq 2$ and $M = 2$. Especially, we shall determine all the bicyclic biquadratic fields $K = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$ with class number $H_K = 1$ and 2, where d_i are the integers expressed in the forms $d_i = m^2+4$ or $m^2+1 \ (m \in \mathbf{N})$. Let E be the group $E_1E_2E_3 \subset E_K$. As usual, we call the number $Q_K = [E_K : E]$ the unit index of K. We denote $d_1d_2 = d_3r^2$, where d_3 is a positive square-free integer and r is an integer. h_i and ε_i denote the class number and the fundamental unit of $\mathbf{Q}(\sqrt{d_i})$. Then $E =< \pm 1, \varepsilon_1, \varepsilon_2, \varepsilon_3 >$. Let N be the norm map from k_i to $\mathbf{Q}$. We denote the Galois group $Gal(K/\mathbf{Q})$ by G and the elements of order 2 which correspond to K_1 and K_2 by σ and τ. Let ε be any unit in K. Then it is easy to see

$$\varepsilon^2 = \frac{(\varepsilon\varepsilon^\sigma)(\varepsilon\varepsilon^\tau)}{\varepsilon^\sigma\varepsilon^\tau} \in E \ .$$

Since K is real, we have an isomorphism $E_K/E \cong {E_K}^2/E^2 \subset E/E^2$.
Therefore, to find a system of generators of the unit group E_K, we have to list up the elements of E which are perfect squares in K from among 7 numbers $\varepsilon_1^\alpha\varepsilon_2^\beta\varepsilon_3^\gamma \ (\neq 1)$, where $\alpha, \beta, \gamma = 0$ or 1. We note that $N\varepsilon_1 = N\varepsilon_2 = -1$. In the case $N\varepsilon_3 = -1$, the number $\varepsilon_1^\alpha\varepsilon_2^\beta\varepsilon_3^\gamma$ does not become totally positive for the value of (α, β, γ) other than $(1, 1, 1)$. In the case $N\varepsilon_3 = +1$, the number $\varepsilon_1^\alpha\varepsilon_2^\beta\varepsilon_3^\gamma$ does not become totally positive for the value of (α, β, γ) other than $(0, 0, 1)$. Hence the unit index Q_K of K is 1 or 2. Then it is known (c.f. [9]) that;

Lemma 3. *In the case* $N\varepsilon_3 = -1$*, put* $C = Tr_{K/\mathbf{Q}}(\varepsilon_1\varepsilon_2\varepsilon_3 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$*. Then* $Q_K = 2$ *if and only if* C *or one of* Cd_i *is a perfect square in* $\mathbf{Q}^\times$.
In the case $N\varepsilon_3 = +1$*, put* $C = N(1 + \varepsilon_3)$*. Then* $Q_K = 2$ *if and only if* C *or one of* Cd_i *is a perfect square in* $\mathbf{Q}^\times$.

In the case $H_K = 1$, all the genus fields of $\mathbf{Q}(\sqrt{d_i})$ are contained in K. Therefore only two primes divide the discriminant of K. Hence $(h_1, h_2, h_3) = (1, 1, 2)$ or $(1, 2, 1)$. Hence, using the results of [7] and [12], we have obtained the following theorem.

Theorem 1. *With the above notation, there are* 54 *real bicyclic biquadratic fields* $K = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$ *with class number* $H_K = 1$, *where*
$(d_1, d_2) = (2,5), (2,13), (2,17), (2,29), (2,37), (2,53), (2,101), (2,122), (2,197), (2,293),$
$(2,362), (2,677), (5,13), (5,17), (5,37), (5,53), (5,173), (5,293), (5,365), (5,485),$
$(5,677), (5,965), (5,1685), (5,3365), (13,17), (13,29), (13,37), (13,197), (13,293),$
$(13,533), (13,1157), (17,29), (17,37), (17,101), (17,197), (17,1853), (29,37), (29,53),$
$(29,101), (29,173), (29,293), (29,2117), (29,2813), (37,53), (37,101), (37,293), (53,173),$
$(53,197), (101,173), (101,197), (101,293), (173,293), (173,677), (197,293)$.

Remark 1. As we mentioned in the introduction, the result of Tatuzawa [14] allows the possibility of one more field. So our result allows several more fields. However, using the results of H.K.Kim [8], the conclusion of Tatuzawa is true without any exception under the generalized Riemann Hypothesis. Therefore, under the generalized Riemann Hypothesis, there are exactly 54 real bicyclic biquadratic fields with class number one.

Let $\tilde{K}$ and $\tilde{K}_i$ be the Hilbert class fields of K and K_i. Since $K \cdot \tilde{K}_i \subset \tilde{K}$, we have the following elementary lemma.

Lemma 4. $H_K \geq h_i/2 \quad (1 \leq i \leq 3)$.

From the formula (1) and the above lemmas, $H_K = 2$ implies (h_1, h_2, h_3)
$= (1,1,4), (1,4,1), (1,2,4), (1,4,2), (2,2,2)$ and $(2,4,1)$. In the case (h_1, h_2, h_3)
$= (1,1,4), (1,4,1)$, Lemma 3 and Lemma 4 imply $Q_K = 2$ and therefore $H_K = 2$.
Using the continued fraction expansion, we have $(h_1, h_2, h_3) = (1,1,4)$ or $(1,4,1)$ in the follwing 23 fields $K = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$,
where $(d_1, d_2) = (2,82), (2,626), (5,29), (5,101), (5,445), (5,3845), (5,6245), (13,53),$
$(13,101), (13,173), (13,677), (13,3029), (13,7397), (13,11453), (17,53), (17,293),$
$(17,5933), (17,11237), (37,197), (37,677), (53,12773), (173,197), (293,677)$.
In other cases, one sees $H_K = 2Q_K \geq 2$ and so $H_K = 2$ if and only if $Q_K = 1$. By the help of a computer, we have calculated the class number h_3 and the number C of Lemma 3. Blanks in the following tables stand for the cases $h_1 h_2 h_3 \neq 8$. Moreover, if it is possible , we omitted these cases. For the cases $h_1 h_2 h_3 = 8$, we write (1) or (d_i) when $C \in (\mathbf{Q}^{\times})^2$ or $d_i(\mathbf{Q}^{\times})^2$ and $*$ when $Q_K = 1$. For example, in the case $K = \mathbf{Q}(\sqrt{2}, \sqrt{65})$, $C = 4160 = 8^2 \times 65 \in d_2(\mathbf{Q}^{\times})^2$ and we have written (d_2) in Table 1.

Table 1 (the cases $(h_1, h_2, h_3)=(1, 2, 4)$)

$d_2\backslash d_1$	2	5	13	17	29	37	53	101	173	197	293	677
10			(d_3)	*	(d_2)	*	(d_1)	(d_1)	*	(d_3)	*	
26		(d_1)			*	*	(1)	(1)	(d_3)	*	*	(1)
65	(d_2)			*	(d_1)	(d_2)	(d_2)		(1)	*		(d_1)
85	*		*		(d_2)	(1)	(d_3)	(d_2)	(d_2)	(d_2)	(d_3)	*
122		*	(1)	*	*	(1)	(d_2)		(d_2)	*	(d_3)	*
362		(d_3)		*	(d_2)	*	*	*	(d_1)	*	(d_3)	(d_1)
365			(d_1)	*	(1)	*	*			(d_1)	(d_3)	(d_3)
485	*		*	(d_2)	(1)	*			(d_2)	(d_2)		(1)
533	(d_3)	*		*	*	*	(d_1)	(1)		(d_3)	(d_3)	(d_1)
629	*	(d_3)	*		(d_3)		(d_1)		(d_3)	(1)	(d_2)	(d_3)
965	*			*	(d_3)	*	(d_2)		*		(d_2)	
1157	*	(1)			*		(d_1)	(1)		(d_1)	*	
1685	(d_1)			(d_3)		*	*	(d_2)	(1)	*	(d_1)	
1853		*	*		(d_2)	*	(d_1)		*	(d_1)		(d_1)
2117	*	(d_1)	*	(1)		*	*	*	(d_2)	(d_2)	(d_3)	
2813		(d_3)	*	*		(d_3)		(d_2)	*		(d_2)	
3365	*		*	*		(d_1)		(d_1)	*	(1)		(d_1)

Table 2 (the cases $(h_1, h_2, h_3)=(2, 2, 2)$)

$d_2\backslash d_1$	10	26	65	85	122	365	485	533	629	965	1685	2117
26	(d_2)											
85	*		*									
122	*											
362		*			(1)							
365	*		(d_2)	*								
485	*		*	(1)		*						
533			*									
629				(d_1)								
925	*		*									
1157		*						(d_1)				
1685			*	(d_1)			(d_2)					
1853				*					*			
2813												*
3365	*		*	*						(d_3)	*	

Table 3 (the cases $(h_1, h_2, h_3)=(1, 4, 2)$ or $(2, 4, 1)$)

$d_2 \backslash d_1$	2	5	13	17	37	10	65	85
170		*						
290		(d_2)						
530	(d_2)	(d_3)						
962	*		*					
1370	(1)					(d_2)		
2405		(d_2)	(d_2)					
3485								*
9605		*		*				*
10205		*					*	
11885		(d_1)						
16133			*	*				
20165		*			*			

Theorem 2. *With the above notation, there are* 118 *real bicyclic biquadratic fields* $K = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$ *with class number* $H_K = 2$, *where*
$(d_1, d_2) = (2,85), (2,485), (2,629), (2,965), (2,1157), (2,2117), (2,3365),$
$(2,82), (2,626), (2,962), (5,29), (5,101), (5,122), (5,533), (5,1853), (5,170), (5,455),$
$(5,3845), (5,6245), (5,9605), (5,10205), (5,20165), (13,53), (13,101), (13,173), (13,677),$
$(13,85), (13,485), (13,629), (13,1853), (13,2117), (13,2813), (13,3365), (13,962),$
$(13,3029), (13,7397), (13,11453), (13,16133), (17,53), (17,293), (17,10), (17,65),$
$(17,122), (17,362), (17,365), (17,533), (17,965), (17,2813), (17,3485), (17,5933),$
$(17,9605), (17,11237), (17,16133), (29,26), (29,122), (29,533), (29,1157), (37,197),$
$(37,677), (37,10), (37,26), (37,362), (37,365), (37,485), (37,533), (37,965), (37,1685),$
$(37,1853), (37,2117), (37,20165), (53,362), (53,365), (53,1685), (53,2117), (53,12773),$
$(101,362), (101,2117), (173,197), (173,10), (173,965), (173,1853), (173,2813),$
$(173,3365), (197,26), (197,65), (197,122), (197,362), (197,1685), (293,677),$
$(293,10), (293,26), (293,1157), (677,85), (677,122), (10,85), (10,122), (10,365),$
$(10,485), (10,965), (10,3365), (26,362), (26,1157), (65,85), (65,485), (65,533),$
$(65,965), (65,1685), (65,3365), (65,10205), (85,365), (85,1853), (85,3365), (85,3485),$
$(85,9605), (365,485), (629,1853), (1685,3365), (2117,2813).$

Remark 2. Under the generalized Riemann hypothesis, the number of the real bicyclic biquadratic fields with class number 2 of the above form are exactly 118.

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