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On the Borel Graph Theorem and the Open Mapping Theorem

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Abstract

In this paper, we extend the Schwartz' Borel graph theorem and the open mapping theorem to the case of not necessarily locally convex TVS and solve Grothendieck's Conjecture.

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Let E, F be two topological vector spaces (TVS). Then the graph of a linear map $u: E \to F$ means the subset $G(u) = \{(x, u(x));\, x \in E\}$ of the product space $E \times F$. $G(u)$ is said to be a Borel graph when it is a Borel set in $E \times F$, and it is said to be a closed graph when it is a closed set in $E \times F$. A closed graph is, of course, a Borel graph. Then we have the following.

Problem. Assume that E and F are two Hausdorff TVS. Then what kind of TVS is permissible for E and F for which the following two assertions hold simultaneously?

(i) A linear map of E into F with the Borel graph is continuous (Borel Graph Theorem).

(ii) A continuous linear map of F onto E is an open mapping (Open Mapping Theorem).

These theorems are first discovered by Banach [1] in 1932 under rather stronger conditions. Namely, we have the following.

Banach Theorem. *Let E, F be two complete metrizable* TVS. *Then*

(i) *A linear map of E into F with the closed graph is continuous.*

(ii) *A continuous linear map of F onto E is an open mapping.*

We note expressly that the local convexity of TVS E and F is not assumed in this place.

This theorem has many important applications. Many mathematicians extended this theorem in the more general situations.

In the following we will mention several results postponing the explanation of terminologies.

Grothendieck Theorem [2]. *Let E and F be two locally convex Hausdorff* TVS. *Assume that E is of type (β) and F is an* LF-*space. Then*

(i) *A linear map of E into F with the closed graph is continuous.*

(ii) *A continuous linear map of F onto E is an open mapping.*

Here we say E to be of type (β) when it is a locally convex TVS which is an inductive limit of a (not necessarily countable) family of Banach spaces. We say F to be an LF-space when it is a locally convex TVS which is an inductive limit of a sequence of Fréchet spaces.

A quasi-complete bornological space is of type (β). Here a locally convex space E is said to be bornological when each absolutely convex set M which absorbs all bounded sets of E is a neighborhood of zero in E. Especially a Fréchet space is of type (β). An inductive limit of a sequence of spaces of type (β) is of type (β). Especially an LF-space is of type (β).

There Grothendieck presented the following.

Conjecture. In addition to Banach spaces the theorem is also true for all the spaces F obtained by a finite or an infinite numbers of operations of the following types departing from Banach spaces, namely by the operations of making topological direct products or topological direct sums of a countable families of locally convex spaces and by the transition into closed subspaces or Hausdorff quotient spaces.

Then Fréchet spaces are permissible because they are closed subspaces of the topological direct products of sequences of Banach spaces. LF-spaces are also permissible because they are Hausdorff quotient spaces of topological direct sums of sequences of Fréchet spaces.

In order to solve Grothendieck's Conjecture, many mathematicians tried to extend the closed graph theorem and the open mapping theorem. Among many results concerning them, the result of the Borel graph theorem of L. Schwartz [4] in 1966 was surprising.

Schwartz Theorem. *Let E and F be two locally convex Hausdorff* TVS. *Assume that E is an inductive limit of an arbitrary family of Banach spaces and F is a Souslin space. Then*

(i) *A linear map of E into F with the Borel graph is continuous.*

(ii) *A continuous linear map of F onto E is an open mapping.*

While Schwartz' first proof was measure theoretical, *A.* Martineau gave another proof without the measure theory in the same year [3]. Since a closed graph is a Borel graph, this solved Grothendieck's Conjecture by the afterward mentioned properties of Souslin spaces. But, taking into consideration the fact that the Banach's first result is given for not necessarily locally convex, complete metrizable TVS, it is hard yet to say that the Banach's result was extended finally. Therefore, it seems to have an important meaning that one extends the Schwartz' Borel graph theorem to the case of not necessarily locally convex TVS.

Our result will be mentioned in the following.

Theorem. *Let E and F be two (not necessarily locally convex) Hausdorff* TVS. *Assume that E is one of the following* $(\mathrm{E}_1) \sim (\mathrm{E}_4)$:

(E_1) *a complete metrizable* TVS,

(E_2) *a Baire Souslin* TVS,

(E_3) *an inductive limit of an arbitrary family of complete metrizable* TVS,

(E_4) *an inductive limit of an arbitrary family of Baire Hausdorff Suoslin* TVS.

Assume that F is one of the following (F_1), (F_2):

(F_1) *a complete metrizable* TVS,

(F_2) *a Souslin* TVS.

Then

(i) *A linear map of E into F with Borel graph is continuous.*

(ii) *A continuous linear map of F onto E is an open mapping.*

Here we recall the definitions of terminologies used above. A TVS E is said to be an inductive limit of an arbitrary family $\{E_\alpha\}$ of TVS when for each α a linear mapping $\phi_\alpha\colon E_\alpha \to E$ is given so that $E = \bigcup_\alpha \phi_\alpha(E_\alpha)$ holds and E is endowed with the finest topology with respect to which each ϕ_α is continuous.

A topological space P is said to be Polish when it is a separable complete metrizable topological space and a Hausdorff topological space S is said to be a Souslin space when it is an image of some Polish space by a continuous map.

A subset M of a topological space X is said to be meager if M is contained in a countable union of closed sets without any interior point. X is said to be a Baire space if no meager subset of X has interior points.

The class of Souslin spaces is closed with respect to the following operations:

(1) transition to closed or open subspaces.

(2) making countable products and disjoint unions.

(3) making countable intersections or countable unions of Souslin subspaces of a Hausdorff topological space.

(4) making continuous images.

By these properties of Souslin spaces, Schwartz' Borel graph theorem and our Borel graph theorem in the case of $F = (\mathrm{F}_2)$ give solutions of Grothendieck's Conjecture in the extended form.

The case $E=(\mathrm{E}_1)$, $F=(\mathrm{F}_1)$ of our theorem gives a direct generalzation of the classical Banach theorem.

In our theorem, the class of complete metrizable TVS and the class of Souslin TVS are considered. The former is contained in the latter when the former is restricted to the separable case. But it is meaningful to consider the both classes since in general it does not happen that one of them becomes a subclass of the other.

Proof of the theorem. At first we will show (i). In the case where E is (E_3) or (E_4), assume that E is an inductive limit of the family $\{\mathrm{E}_\alpha\}$ with respect to the mapping $\phi_\alpha\colon E_\alpha \to E$. Then for a linear map u of E into F to be continuous is equivalent for a linear map $u_\alpha = u \circ \phi_\alpha$ of E_α into F to be continuous for any α. The graph $G(u_\alpha)$ of u_α is the preimage of the graph $G(u)$ of u with respect to the mapping $(x_\alpha, y) \to (\phi_\alpha(x_\alpha), y)$ of $E_\alpha \times F$ into $E \times F$. Hence these cases are reduced to the cases (E_1) or (E_2). In the case (E_1), it is enough to show that u is sequentially continuous, so that we have only to consider the restriction of u to the smallest closed subspace of E containing an arbitrarily given sequence of E. Hence we have only to consider the case where E is a separable complete metrizable TVS. In this time (E_1) is a Polish space, consequently it becomes a Baire Souslin space and is reduced to (E_2). Hence, we have only to consider the case where $E=(\mathrm{E}_2)$. Then in the case of $F=(\mathrm{F}_1)$, we have only to consider the smallest closed subspace of F containing $u(E)$, but this is a separable complete metrizable TVS, namely, a Polish space since E is separable. Hence we have only to consider the case $F=(\mathrm{F}_2)$.

Hence, in order to prove the theorem, we have only to consider the case where E and F are (E_2) and (F_2) respectively.

In this case, we need the following.

Lemma. *Let E be a baire Hausdorff* TVS. *Let S and S' be two balanced and absorbing subsets such that $S'+S'+S'+S' \subset S$. If S and S' are nonmeager and Souslin spaces, S is a neighborhood of zero in E.*

Proof. Since S' is nonmeager, the largest open set $O(S')$ having the property that every open subset of $O(S')$ contains some nonmeager part of S' is not empty. Let $a \in S' \cap O(S')$. Since the operation $S' \to O(S')$ is translation invariant, we have $O(S'') \supset O(S'-a) = O(S')-a$, where $S''=S'+S'$. But $O(S')-a$ is a neighborhood of zero because $O(S')$ is an open set containing a. Hence $O(S'')$ is also a neighborhood of zero. Hence the proof will be complete if we show $O(S'') \subset S$. Let $x \in O(S'')$. Since $O(S'')$ is a neighborhood of zero, $O(S'') \cap [x - O(S'')]$ is a nonempty open set which differs from $S'' \cap (x - S'')$ by only a meager set. Hence, $S'' \cap (x-S'')$ is nonempty. This means that there is $y \in S''$ such that $x - y \in S''$. Hence $x \in S''+S'' = S'+S'+S'+S' \subset S$. (Q.E.D.)

Now we will return to the proof of the theorem. Let G denote the graph of u. Let V be an arbitrary closed balanced absorbing neighborhood of zero in F. Let W denote the intersection of G and $E \times V$. Of course, W is a Borel set. Since E and F are Souslin spaces, $E \times F$ is a Souslin space [F. Trèves [5], Proposition A.4(b), p. 551]. By the Proposition A.5 of F. Trèves [5], p. 552, W is also a Souslin space. Hence this is also true for its image by the first coordinate projection [F. Trèves [5], Proposition A.4(d), p. 551]. This is nothing else but $U = u^{-1}(V)$. Since $E = u^{-1}(F) = \bigcup_{n=1}^{\infty} nu^{-1}(V) = \bigcup_{n=1}^{\infty} nU$, U is non-meager. Let V' be a closed balanced absorbing neighborhood of zero in F such that $V' + V' + V' + V' \subset V$. Then $U' = u^{-1}(V')$ has also the same properties as U. Hence $S = U$ and $S' = U'$ satisfy the conditions of the Lemma. Thus U is a neighborhood of zero in E.

Next we will show (ii). Let $v: F \to E$ be a surjective continuous linear map. Now let $\bar{v}: F/\mathrm{Ker}\, v \to E$ be an associated injective map. Then $\bar{v}$ is continuous. Hence the graph of $\bar{v}^{-1}$ is closed. But $F/\mathrm{Ker}\, v$ satisfies the conditions (F_1) or (F_2) of the theorem. Hence, by (i), $\bar{v}^{-1}$ is continuous. (Q.E.D).

References

[1] S. Banach, Théorie des Opérations Linéaires. Warsaw, 1932.

[2] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires. Mem. Amer. Math. Soc., **16**, 1955.

[3] A. Martineau, Sur le théorème du graphe fermé. C. R. Acad. Sc., Paris, **263** (1966), Sér. A, 870–871.

[4] L. Schwartz, Sur le theoreme du graphe fermé. C. R. Acad. Sc., Paris, **263** (1966), Sér. A, 602–605.

[5] F. Trèves, Topological Vector Spaces, Distributions and Kernels. Academic Press, New York-London, 1967.