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Boundary Values of (Slowly Increasing) Holomorphic Functions

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Abstract

In this paper, we investigate the conditions that (slowly increasing) holomorphic functions $F_\pm(z)$ on $\boldsymbol{C}_\pm$ have the boundary values $F_\pm(x \pm i0) = \lim_{\varepsilon \to +0} F_\pm(x \pm i\varepsilon)$ in the sense of $\mathscr{A}'([a, b])$ (or $\mathscr{Q}'(\boldsymbol{D})$) and define the Sato (Fourier) hyperfunction $f(x) = F_+(x + i0) - F_-(x - i0)$ on $[a, b]$ (or $\boldsymbol{D}$) as their boundary values.

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§1. Introduction

In this paper we consider the inherent meaning of the boundary values of holomorphic functions. On the occasion of saying that Sato-Fourier hyperfunctions are realized as boundary values of holomorphic functions, we consider them to be (relative) cohomology classes with coefficients in the sheaf $\mathscr{O}$ or $\tilde{\mathscr{O}}$ of germs of holomorphic functions or slowly increasing holomorphic functions. But the boundary values of functions are in its own meaning the values taken by them on the boundary of their domain of definition. So that, the relation between the boundary values of holomorphic functions and (relative) cohomology classes with coefficients in the sheaf $\mathscr{O}$ or $\tilde{\mathscr{O}}$ is not so clear in the direct manner. Thus, the point of this paper is to show in what meaning we can consider the (relative) cohomology classes with coefficients in the sheaf $\mathscr{O}$ or $\tilde{\mathscr{O}}$ as the boundary values of holomorphic functions.

For the sake of simplicity, we consider the 1-dimensional case.

Let $\boldsymbol{C}$ be the complex plane, and $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ the upper half plane and the lower half plane respectively. Then $\boldsymbol{C} = \boldsymbol{C}_+ \cup \boldsymbol{R} \cup \boldsymbol{C}_-$ holds. Here $\boldsymbol{R}$ denotes the real axis.

Now we consider two holomorphic functions $F_+(z)$ and $F_-(z)$ defined on $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ respectively. Then we ask what the boundary value of the holomorphic functions is, which is represented, symbolically, as

$$F_{\pm}(x \pm i0) = \lim_{\varepsilon \to +0} F_{\pm}(x \pm i\varepsilon) \tag{1}$$

and

$$f(x) = F_{+}(x + i0) - F_{-}(x - i0). \tag{2}$$

If $F(z)$ is holomorphic on $\bar{\boldsymbol{C}}_{+}$ or $\bar{\boldsymbol{C}}_{-}$, the right hand side of (1) converges pointwise and the limit

$$f(x) = F(x) = F(x \pm i0)$$

represents an ordinary function. Thus the function $f(x)$ is certainly the boundary value. But, generally speaking, the meaning of the limit (1) is a question. As for the limit of functions, we have the classical concepts of convergence such as pointwise convergence, uniform convergence and extended uniform convergence, and those in the sense of Schwartz' distributions and in the sense of Roumieu-Beurling's ultradistributions. Even if there are no limits in the classical sense, we happen to have the limits in the sense of distributions or ultradistributions. In fact, the holomorphic function $F_{\pm}(x \pm i\varepsilon)$, $(\varepsilon > 0)$, defines a distribution or ultradistribution as a function of x. Then, for certain holomorphic functions $F_{\pm}(z)$, there exist the limits (1) in the sense of distributions or ultradistributions and they become distributions or ultradistributions respectively.

In this time we wish to assert that the limit (1) in the sense of Sato hyperfunctions or Fourier hyperfunctions becomes a Sato hyperfunction or a Fourier hyperfunction respectively. In general, the topology of the space of Sato-Fourier hyperfunctions on an open set is trivial. But we can endow the space of Sato-Fourier hyperfunctions on a compact set the topology of Fréchet space. In the sense of this topology do we consider the limit (1).

The space of Sato hyperfunctions on a bounded closed interval $[a, b]$ is the space $\mathscr{A}'([a, b])$ of real analytic functionals with support in $[a, b]$. Let $F(z) \in \mathcal{O}(\boldsymbol{C} \setminus [a, b])$ and put $F_{\pm}(z) = F(z) | \boldsymbol{C}_{\pm}$. Then $F_{\pm}(x \pm i\varepsilon)$, $(\varepsilon > 0)$, can be considered as elements of $\mathscr{A}'([a, b])$ as functions of x. Then the limits (1) exist in the sense of Sato hyperfunctions and $f(x) = F_{+}(x + i0) - F_{-}(x - i0)$ becomes a Sato hyperfunction on $[a, b]$. This means that a Sato hyperfunction $f(x)$ is the boundary value of the holomorphic function $F(z)$. As for the notations of this paragraph, see section 4.

At last, the space of Fourier hyperfunctions on $\boldsymbol{D} = [-\infty, \infty]$ is the space $\underset{\sim}{\mathscr{A}}'(\boldsymbol{D})$. Let $F(z) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}} \setminus \mathbf{D})$ and put $F_{\pm}(z) = F(z) | \tilde{\boldsymbol{C}}_{\pm}$. Then $F_{\pm}(x \pm i\varepsilon)$, $(\varepsilon > 0)$, can be considered as elements of $\underset{\sim}{\mathscr{A}}'(\boldsymbol{D})$ as functions of x. Then the limits (1) exist in the sense of Fourier hyperfunctions and $f(x) = F_{+}(x + i0) - F_{-}(x - i0)$ becomes a Fourier hyperfunction on $\boldsymbol{D}$. This means that a Fourier hyperfunction $f(x)$ is the boundary value of the slowly increasing holomorphic function $F(z)$. As for the notations of this paragraph, see section 5.

Thereby we can represent Sato-Fourier hyperfunctions as boundary values of (slowly increasing) holomorphic functions in the inherent sense of the word.

§2. Case of distributions

First, we will consider the convergence in the sense of Schwartz' distribution. Here, we denote by $\mathscr{D}$ the space of infinitely differentiable functions with compact support on $\boldsymbol{R}$. Then, if, for each $\varphi \in \mathscr{D}$, there exist

$$\lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} F_+(x+i\varepsilon)\varphi(x)\mathrm{d}x, \tag{3}$$

$$\lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} F_-(x-i\varepsilon)\varphi(x)\mathrm{d}x,$$

there exist Schwartz' distributions $F_+^{\mathscr{D}}(x+i0)$ and $F_-^{\mathscr{D}}(x-i0)$ so that, for $\varphi \in \mathscr{D}$,

$$\langle F_+^{\mathscr{D}}(x+i0), \varphi(x)\rangle = \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} F_+(x+i\varepsilon)\varphi(x)\mathrm{d}x$$

and

$$\langle F_-^{\mathscr{D}}(x-i0), \varphi(x)\rangle = \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} F_-(x-i\varepsilon)\varphi(x)\mathrm{d}x$$

hold. Then $f(x)$ in (2) is a Schwartz' distribution

$$f(x) = F_+^{\mathscr{D}}(x+i0) - F_-^{\mathscr{D}}(x-i0).$$

Namely, if the limits in (3) exist, the convergence in (1) is meaningful in the sense of Schwartz' distribution and the limit (2) can be considered as a Schwartz' distribution. For which pair $F_+(z)$ and $F_-(z)$ of holomorphic functions do the limits (3) exist? As for this, we have the following.

Theorem 1 (Tillmann [12]). *Let $F_+(z)$ and $F_-(z)$ be holomorphic functions on $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ respectively. Then, in order that there exist $F_+^{\mathscr{D}}(x+i0)$ and $F_-^{\mathscr{D}}(x-i0)$, which define a Schwartz' distribution $f(x) = F_+^{\mathscr{D}}(x+i0) - F_-^{\mathscr{D}}(x-i0)$ on $\boldsymbol{R}$, it is necessary and sufficient that, for every $r>0$, there exists a natural number such that*

$$\sup\{|y|^m|F_+(x+iy)|,\ |y|^m|F_-(x-iy)|;\ |x|\leqq r,\ 0<|y|\leqq r\} < \infty$$

holds.

§3. Case of Roumieu-Beuling ultradistributions

Next, we will consider the convergence in the sense of Roumieu-Beuling ultradistributions. In the Komatsu's notations [4], [5], we denote by $\mathscr{D}^{(M_p)}(\boldsymbol{R})$ or $\mathscr{D}^{\{M_p\}}(\boldsymbol{R})$ the space of ultradifferentiable functions with compact support and by $\mathscr{D}^*(\boldsymbol{R})$ either of them. Then, if, for each $\varphi \in \mathscr{D}^*(\boldsymbol{R})$, there exist

$$(4) \qquad \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} F_+(x + i\varepsilon)\varphi(x)\mathrm{d}x,$$

$$\lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} F_-(x - i\varepsilon)\varphi(x)\mathrm{d}x,$$

there exist Roumieu-Beurling ultradistributions $F_+^*(x + i0)$ and $F_-^*(x - i0)$ so that, for $\varphi \in \mathscr{D}^*(\boldsymbol{R})$,

$$\langle F_+^*(x + i0), \varphi(x)\rangle = \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} F_+(x + i\varepsilon)\varphi(x)\mathrm{d}x$$

and

$$\langle F_-^*(x - i0), \varphi(x)\rangle = \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} F_-(x - i\varepsilon)\varphi(x)\mathrm{d}x$$

hold. Then $f(x)$ in (2) is a Roumieu-Beurling ultradistribution

$$f(x) = F_+^*(x + i0) - F_-^*(x - i0).$$

Namely, if the limits in (4) exist, the convergence in (1) is meaningful in the sense of Roumieu-Beurling ultradistribution and the limit can be considered as a Roumieu-Beurling ultradistribution. For which pair $F_+(z)$ and $F_-(z)$ of holomorphic functions do the limits (4) exist? As for this, we have the following.

Theorem 2 (Komatsu [4]). *Let $F_+(z)$ and $F_-(z)$ be holomorphic functions on $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ respectively. Then, in order that there exist $F_+^*(x + i0)$ and $F_-^*(x - i0)$, which define a Roumieu-Beurling ultradistribution $f(x) = F_+^*(x + i0) - F_-^*(x - i0)$ on $\boldsymbol{R}$, it is necessary and sufficient that the function $F(z) = (F_+(z), F_-(z))$ on $\boldsymbol{C} \setminus \boldsymbol{R}$ satisfies the following growth condition: for any compact set K in $\boldsymbol{R}$, there exist constants L and C (for any $L > 0$, there exists a constant C) such that*

$$\sup\{|F(x + iy)|;\, x \in K\} \leqq C \exp M^*(L/|y|)$$

holds.

§4. Case of Sato hyperfunctions

In this respect, how should we consider the convergence in the sense of Sato hyperfunction. The topology of the space of Sato hyperfunctions on an open set in $\boldsymbol{R}$ is trivial. So that we restrict ourselves to consider Sato hyperfunctions with support in a bounded closed interval $[a, b]$. Let $\mathscr{A}([a, b])$ be the space of real analytic functions in some neighborhood of $[a, b]$. Then $\mathscr{A}([a, b])$ is a DFS-space. An element of the topological dual space $\mathscr{A}'([a, b])$ is by definition a Sato hyperfunction with support in $[a, b]$. Then we have

$$\mathscr{A}([a, b]) = \lim_{\eta \to +0} \operatorname{ind} \mathscr{A}([a - \eta, b + \eta])$$

and

$$\mathscr{A}'([a, b]) = \lim_{\eta \to +0} \operatorname{proj} \mathscr{A}'([a - \eta, b + \eta]).$$

Thus every $\varphi \in \mathscr{A}([a, b])$ belongs to $\mathscr{A}([a - \eta, b + \eta])$ for some $\eta > 0$. As for their definitions, we refer the reader to Schapira [10] and Ito [2].

Let $F(z)$ be a holomorphic function on $\boldsymbol{C} \setminus [a, b]$, and $F_+(z)$ and $F_-(z)$ its restrictions to $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ respectively.

For $\varphi \in \mathscr{A}([a - \eta, b + \eta])$, we put

$$\langle F_+^\eta(x + i\varepsilon), \varphi(x) \rangle = \int_{a-\eta}^{b+\eta} F_+(x + i\varepsilon)\varphi(x)\mathrm{d}x$$

for every $\varepsilon > 0$. Then, $F_+^\eta(x + i\varepsilon)$ can be considered as an element of $\mathscr{A}'([a - \eta, b + \eta])$. Since $\varphi(z)$ is holomorphic in some complex neighborhood of $[a - \eta, b + \eta]$, we can consider that it is holomorphic in a neighborhood of $[a - \delta, b + \delta] \times i[-\delta', \delta']$ with some positive numbers δ and δ' with $\eta < \delta$.

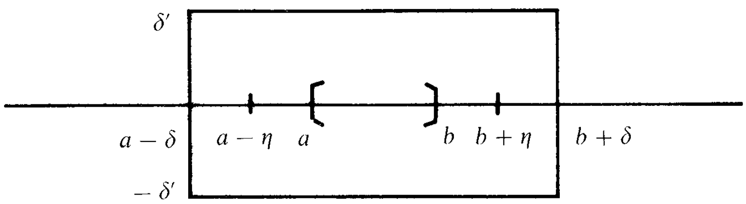

Fig. 1

Now, let γ be the boundary of this rectangle and oriented in the positive sense. Put $\gamma_+ = \gamma \cap \boldsymbol{C}_+$ and $\gamma_- = \gamma \cap \boldsymbol{C}_-$. Put also $\Gamma_+ = [a - \delta, a - \eta] \cup \gamma_+ \cup [b + \eta, b + \delta]$ and $\Gamma_- = [a - \delta, a - \eta] \cup \gamma_- \cup [b + \eta, b + \delta]$. Let Γ_+ be oriented in the clockwise sense and Γ_- oriented in the counterclockwise sense. Then, by changing the path of integration by the way of Cauchy's integral theorem, we have

$$\langle F_+^\eta(x+i\varepsilon), \varphi(x)\rangle = \int_{\Gamma_+} F_+(z+i\varepsilon)\varphi(z)\mathrm{d}z$$

for $\varphi \in \mathscr{A}([a-\eta, b+\eta])$. Here, letting $\varepsilon \to +0$, we have, by virtue of the Lebesgue convergence theorem,

$$\begin{aligned}\lim_{\varepsilon\to+0} &\int_{a-\eta}^{b+\eta} F_+(x+i\varepsilon)\varphi(x)\mathrm{d}x \\ &= \lim_{\varepsilon\to+0}\int_{\Gamma_+} F_+(z+i\varepsilon)\varphi(z)\mathrm{d}z \\ &= \int_{\Gamma_+} F_+(z)\varphi(z)\mathrm{d}z.\end{aligned}$$

Thus, if we put

$$\langle F_+^\eta(x+i0), \varphi(x)\rangle = \int_{\Gamma_+} F_+(z)\varphi(z)\mathrm{d}z,$$

we obtain $F_+^\eta(x+i0) \in \mathscr{A}'([a-\eta, b+\eta])$. For η and η' with $0<\eta'<\eta$, $F_+^\eta(x+i\varepsilon)$ and $F_+^{\eta'}(x+i\varepsilon)$ are defined by the same function $F_+(x+i\varepsilon)$. Thus, for every $\varphi \in \mathscr{A}([a-\eta', b+\eta'])$, we have,

$$\begin{aligned}&\langle F_+^\eta(x+i\varepsilon)\,|\,[a-\eta', b+\eta'], \varphi(x)\rangle \\ &= \int_{a-\eta'}^{b+\eta'} F_+(x+i\varepsilon)\varphi(x)\mathrm{d}x \\ &= \langle F_+^{\eta'}(x+i\varepsilon), \varphi(x)\rangle.\end{aligned}$$

Here $F_+^\eta(x+i\varepsilon)\,|\,[a-\eta', b+\eta']$ means the restriction of $F_+^\eta(x+i\varepsilon)$ to $[a-\eta', b+\eta']$. Thus, for every η and η' with $0<\eta'<\eta$, we have

$$F_+^\eta(x+i\varepsilon)\,|\,[a-\eta', b+\eta'] = F_+^{\eta'}(x+i\varepsilon).$$

Thus, letting $\varepsilon \to +0$, we have

$$F_+^\eta(x+i0)\,|\,[a-\eta', b+\eta'] = F_+^{\eta'}(x+i0).$$

Thus the family $\{F_+^\eta(x+i0); \eta>0\}$ defines an element $F_+(x+i0)$ by the formula

$$F_+(x+i0) = F_+^\eta(x+i0)\,|\,[a, b]$$

for every $\eta > 0$.

Thus, for every $\varphi \in \mathscr{A}([a, b])$, we have

$$\langle F_+(x+i0), \varphi \rangle = \int_{\Gamma_+} F_+(z)\varphi(z)\mathrm{d}z$$

by the above notations for sufficiently small η and δ with $0 < \eta < \delta$.

By a similar way, we have

$$\begin{aligned}\lim_{\varepsilon \to +0} \int_{a-\eta}^{b+\eta} & F_-(x - i\varepsilon)\varphi(x)\mathrm{d}x \\ &= \lim_{\varepsilon \to +0} \int_{\Gamma_-} F_-(z - i\varepsilon)\varphi(z)\mathrm{d}z \\ &= \int_{\Gamma_-} F_-(z)\varphi(z)\mathrm{d}z.\end{aligned}$$

Thus, we obtain $F_-(x - i0) \in \mathscr{A}'([a, b])$ by the formula

$$\langle F_-(x - i0), \varphi(x) \rangle = \int_{\Gamma_-} F_-(z)\varphi(z)\mathrm{d}z.$$

Then

$$f(x) = F_+(x+i0) - F_-(x - i0)$$

becomes a Sato hyperfunction with support in $[a, b]$. The operation of $f(x)$ on $\varphi(x) \in \mathscr{A}([a, b])$ is given by the formula

$$\begin{aligned}\langle f, \varphi \rangle &= \int_{\Gamma_+} F_+(z)\varphi(z)\mathrm{d}z - \int_{\Gamma_-} F_-(z)\varphi(z)\mathrm{d}z \\ &= -\int_{\gamma} F(z)\varphi(z)\mathrm{d}z\end{aligned} \tag{5}$$

by the choice of a convenient path γ. We note that $f(x)$ does not depend on the choice of η.

Conversely, if $f(x) \in \mathscr{A}'([a, b])$, we put

$$F(z) = \langle f(\xi), (2\pi i(\xi - z))^{-1} \rangle.$$

Evidently, we have $F(z) \in \mathscr{O}(\boldsymbol{C} \setminus [a, b])$. Making $F_+(z)$ and $F_-(z)$ as above using this $F(z)$ and calculating the limit in (1) as above, we have

$$\begin{aligned}\langle F_+(x+i0) - F_-(x - i0), \varphi(x) \rangle &= -\int_{\gamma} F(z)\varphi(z)\mathrm{d}z \\ = \left\langle f(\xi), \int_{\gamma} \varphi(z)(2\pi i(z - \xi))^{-1}\mathrm{d}z \right\rangle &= \langle f, \varphi \rangle.\end{aligned}$$

Since this holds for every $\varphi \in \mathscr{A}([a, b])$,

$$f(x) = F_+(x + i0) - F_-(x - i0)$$

holds. Hence, if we define a linear map

$$R: \mathcal{O}(\boldsymbol{C} \setminus [a, b]) \longrightarrow \mathscr{A}'([a, b])$$

by the formula

$$R(F) = F_+(x + i0) - F_-(x - i0) = \lim_{\varepsilon \to +0} (F_+(x + i\varepsilon) - F_-(x - i\varepsilon)),$$

this is a surjection.

Now we will show $R^{-1}(0) = \mathcal{O}(\boldsymbol{C})$. Since $\mathcal{O}(\boldsymbol{C}) \subset R^{-1}(0)$ is evident, we will show the converse. Assume $R(F) = f(x) = 0$. Then $F_+(x + i0) = F_-(x - i0)$ holds. Then, for every $\varphi \in \mathscr{A}([a, b])$, we have

$$\text{(6)} \qquad \begin{aligned} 0 &= \int_{\Gamma_+} F_+(z)\varphi(z)\mathrm{d}z - \int_{\Gamma_+} F_-(z)\varphi(z)\mathrm{d}z \\ &= -\int_{\gamma} F(z)\varphi(z)\mathrm{d}z. \end{aligned}$$

Here γ, Γ_+ and Γ_- may vary as φ does.

Now choose an increasing sequence of open discs $([a, b] \subset) D_1 \subset D_2 \subset \cdots$ such that $\bigcup_{i=1}^{\infty} D_i = \boldsymbol{C}$ and put $\partial D_i = \Gamma_i$. Then if we put

$$G_i(z) = \int_{\Gamma_i} F(\zeta)(2\pi i(\zeta - z))^{-1}\mathrm{d}\xi,$$

$G_i(z)$ is holomorphic in D_i and $G_{i+1}(z) | D_i = G_i(z)$ holds $(i = 1, 2, \cdots)$. Then $G(z) | \boldsymbol{C}_+ = F_+(z)$ and $G(z) | \boldsymbol{C}_- = F_-(z)$ holds. In fact, if we let $z \in \boldsymbol{C}_+$, there exists some D_i so that $z \in D_i$. Then we can choose the curve γ in the above so that $\gamma \subset D_i$ and z is in the outside of γ. Then, since $F(z)$ is holomorphic in the domain enclosed by Γ_i and γ, we have

$$F(z) = \int_{\Gamma_i} F(\zeta)(2\pi i(\zeta - z))^{-1}\mathrm{d}\zeta - \int_{\gamma} F(\zeta)(2\pi i(\zeta - z))^{-1}\mathrm{d}\zeta.$$

Hence we have

$$G(z) = G_i(z) = F(z) + \int_{\gamma} F(\zeta)(2\pi i(\zeta - z))^{-1}\mathrm{d}\zeta.$$

Then, since z is in the outside of γ, $(2\pi i(\xi - z))^{-1}$ belongs to $\mathscr{A}([a, b])$ as a function of ξ. Thus, we have, by virtue of (5) and (6),

$$\int_\gamma F(\zeta)(2\pi i(\zeta - z))^{-1} d\zeta = -\langle f(\xi), (2\pi i(\xi - z))^{-1}\rangle = 0.$$

Hence, $G(z)|\boldsymbol{C}_+ = F_+(z)$ holds.

Similarly $G(z)|\boldsymbol{C}_- = F_-(z)$ holds. Hence $F(z)$ can be continued analytically to a function $G(z)$ holomorphic on $\boldsymbol{C}$. By the above, we have the following.

Theorem 3. *Let $F_+(z)$ and $F_-(z)$ be holomorphic functions on $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ respectively. Then, in order that the boundary value $f(x) = F_+(x + i0) - F_-(x - i0)$ in the topology of $\mathscr{A}'([a, b])$ is a Sato hyperfunction with support in $[a, b]$, it is necessary and sufficient that there exists some $F(z) \in \mathcal{O}(\boldsymbol{C} \setminus [a, b])$ such that $F_+(z) = F(z)|\boldsymbol{C}_+$ and $F_-(z) = F(z)|\boldsymbol{C}_-$ hold. Especially, in order that $f(x) = 0$ holds, it is necessary and sufficient that there exists some $F(z) \in \mathcal{O}(\boldsymbol{C})$ such that $F_+(z) = F(z)|\boldsymbol{C}_+$ and $F_-(z) = F(z)|\boldsymbol{C}_-$ hold.*

By the above correspondence, we have the following.

Corollary 1. *In the notations of Theorem 3, we have the algebraic isomorphism*

$$\mathscr{A}'([a, b]) \cong \mathcal{O}(\boldsymbol{C} \setminus [a, b]) / \mathcal{O}(\boldsymbol{C}).$$

As another Corollary, we have the Edge of the Wedge theorem.

Corollary 2. *For $F(z) \in \mathcal{O}(\boldsymbol{C} \setminus [a, b])$, put $F_+(z) = F(z)|\boldsymbol{C}_+$ and $F_-(z) = F(z)|\boldsymbol{C}_-$. Then, if $F_+(x + i0)$ and $F_-(x - i0)$ exist in the topology of $\mathscr{A}'([a, b])$ and $F_+(x + i0) = F_-(x - i0)$ holds, $F(z)$ can be continued analytically to a holomorphic function on $\boldsymbol{C}$.*

§5. Case of Fourier hyperfunctions

At last, we will consider the convergence (1) in the sense of Fourier hyperfunctions. Put $\boldsymbol{D} = [-\infty, \infty]$ and $\tilde{\boldsymbol{C}} = \boldsymbol{D} \times i\boldsymbol{R}$. Let $\tilde{\mathcal{O}}(\tilde{\boldsymbol{C}} \setminus \boldsymbol{D})$ and $\tilde{\mathcal{O}}(\tilde{\boldsymbol{C}})$ be the spaces of slowly increasing holomorphic functions on $\tilde{\boldsymbol{C}} \setminus \boldsymbol{D}$ and $\tilde{\boldsymbol{C}}$ respectively and $\mathscr{A}(\boldsymbol{D})$ the space of rapidly decreasing real analytic functions on $\boldsymbol{D}$. Then $\mathscr{A}(\boldsymbol{D})$ becomes a DFS-space. An element of the topological dual space $\mathscr{A}'(\boldsymbol{D})$ is by definition a Fourier hyperfunction on $\boldsymbol{C}$. As for their definitions, we refer the reader to Sato [7], Kawai [3] and Ito [2].

If we put $\tilde{\boldsymbol{C}}_+ = \boldsymbol{D} \times i\boldsymbol{R}_+$, $\tilde{\boldsymbol{C}}_- = \boldsymbol{D} \times i\boldsymbol{R}_-$, $\boldsymbol{R}_+ = (0, \infty)$ and $\boldsymbol{R}_- = (-\infty, 0)$, $F(z) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}} \setminus \boldsymbol{D})$ is composed of two functions $F_+(z) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}}_+)$ and $F_-(z) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}}_-)$. Now, if $F(z) = (F_+(z), F_-(z)) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}} \setminus \boldsymbol{D})$ is given, we put, for every $\varphi \in \mathscr{A}(\boldsymbol{D})$,

$$\langle F_+(x + i\varepsilon), \varphi(x)\rangle = \int_{-\infty}^{\infty} F_+(x + i\varepsilon)\varphi(x) dx.$$

Then $F_+(x + i\varepsilon)$ defines a Fourier hyperfunction on $\boldsymbol{D}$. Then, since $F_+(z + i\varepsilon)\varphi(z)$

is a rapidly decreasing holomorphic function in some neighborhood of $\boldsymbol{D}$, we have, for a sufficiently small positive number δ

$$\langle F_+(x+i\varepsilon), \varphi(x)\rangle = \int_{-\infty}^{\infty} F_+(x+i(\delta+\varepsilon))\varphi(x+i\delta)\mathrm{d}x$$

by changing the path of integration by virtue of Cauchy's integral theorem. Here, letting $\varepsilon \to +0$, we have, by the Lebesgue convergence theorem,

$$\begin{aligned}\lim_{\varepsilon\to+0} &\langle F_+(x+i\varepsilon), \varphi(x)\rangle \\ &= \lim_{\varepsilon\to+0} \int_{-\infty}^{\infty} F_+(x+i(\delta+\varepsilon))\varphi(x+i\delta)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} F_+(x+i\delta)\varphi(x+i\delta)\mathrm{d}x.\end{aligned}$$

By a similar way, we have

$$\begin{aligned}\lim_{\varepsilon\to+0} &\langle F_-(x-i\varepsilon), \varphi(x)\rangle \\ &= \lim_{\varepsilon\to+0} \int_{-\infty}^{\infty} F_-(x-i(\delta+\varepsilon))\varphi(x-i\delta)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} F_-(x-i\delta)\varphi(x-i\delta)\mathrm{d}x.\end{aligned}$$

Hence, if we put

$$\langle F_+(x+i0), \varphi(x)\rangle = \int_{-\infty}^{\infty} F_+(x+i\delta)\varphi(x+i\delta)\mathrm{d}x$$

and

$$\langle F_-(x-i0), \varphi(x)\rangle = \int_{-\infty}^{\infty} F_-(x-i\delta)\varphi(x-i\delta)\mathrm{d}x,$$

$F_+(x+i0)$ and $F_-(x-i0)$ become Fourier hyperfunctions on $\boldsymbol{D}$, so that

$$f(x) = F_+(x+i0) - F_-(x-i0)$$

is also a Fourier hyperfunction on $\boldsymbol{D}$. Then, the operation of $f(x)$ on $\varphi(x)\in\mathscr{A}(\boldsymbol{D})$ is given by the formula

(7) $$\langle f, \varphi\rangle = \int_{-\infty}^{\infty} F_+(x+i\delta)\varphi(x+i\delta)\mathrm{d}x$$

$$-\int_{-\infty}^{\infty} F_-(x-i\delta)\varphi(x-i\delta)\mathrm{d}x.$$

By virtue of Cauchy's integral theorem, this right hand side does not depend on the choice of a sufficiently small positive number δ.

Conversely, let $f\in\mathscr{A}'(\boldsymbol{D})$. Put

$$F(z)=\langle f(\xi), \exp(-(\xi-z)^{-2})(2\pi i(\xi-z))^{-1}\rangle.$$

Then, evidently, we have $F(z)\in\tilde{\mathcal{O}}(\tilde{\boldsymbol{C}}\setminus\boldsymbol{D})$. Putting $F_+(z)=F(z)|\tilde{\boldsymbol{C}}_+$, $F_-(z)=F(z)|\tilde{\boldsymbol{C}}_-$ and calculating the limit in a similar way as above, we have

$$\begin{aligned}&\langle F_+(x+i0)-F_-(x-i0), \varphi(x)\rangle\\ &=\int_{-\infty}^{\infty} F_+(x+i\delta)\varphi(x+i\delta)\mathrm{d}x\\ &\quad-\int_{-\infty}^{\infty} F_-(x-i\delta)\varphi(x-i\delta)\mathrm{d}x\\ &=\langle f(\xi), \int_{-\infty}^{\infty}\varphi(x+i\delta)\exp(-(\xi-x-i\delta)^2)(2\pi i(\xi-x-i\delta))^{-1}\mathrm{d}x\\ &\quad-\int_{-\infty}^{\infty}\varphi(x-i\delta)\exp(-(\xi-x+i\delta)^2)(2\pi i(\xi-x+i\delta))^{-1}\mathrm{d}x\\ &=\langle f, \varphi\rangle.\end{aligned}$$

Since this holds for every $\varphi\in\mathscr{A}(\boldsymbol{D})$, we have

$$f(x)=F_+(x+i0)-F_-(x-i0).$$

Hence, if we define a linear map

$$R\colon \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}}\setminus\boldsymbol{D})\longrightarrow\mathscr{A}'(\boldsymbol{D})$$

by the formula

$$R(F)=F_+(x+i0)-F_-(x-i0)=\lim_{\varepsilon\to+0}(F_+(x+i\varepsilon)-F_-(x-i\varepsilon)),$$

this is a surjection.

Now, we will show $R^{-1}(0)=\tilde{\mathcal{O}}(\tilde{\boldsymbol{C}})$. Since $\tilde{\mathcal{O}}(\tilde{\boldsymbol{C}})\subset R^{-1}(0)$ is evident, we will show the converse. Assume $R(F)=f(x)=0$. Then $F_+(x+i0)=F_-(x-i0)$ holds. Then, we have, for every $\varphi\in\mathscr{A}(\boldsymbol{D})$,

$$0=\langle f, \varphi\rangle=\int_{-\infty}^{\infty} F_+(x+i\delta)\varphi(x+i\delta)\mathrm{d}x \tag{8}$$

$$- \int_{-\infty}^{\infty} F_+(x - i\delta)\varphi(x - i\delta)\mathrm{d}x.$$

Here, δ can vary as φ does.

Now, put $D_n = \boldsymbol{D} \times i(-n, n)$. Then, $D_1 \subset D_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} D_n = \tilde{\boldsymbol{C}}$ hold. Put $\partial D_n = \Gamma_n$. Let Γ_n be oriented in the positive sense. Then, if we put

$$G_n(z) = \int_{\Gamma_n} F(\zeta) \exp(-(\zeta - z)^2)(2\pi i(\zeta - z))^{-1}\mathrm{d}\zeta,$$

we have $G_n(z) \in \tilde{\mathcal{O}}(D_n)$ and $G_{n+1}(z) | D_n = G_n(z)$ holds $(n = 1, 2, \cdots)$. Hence we can define $G(z) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}})$ so that $G(z) | D_n = G_n(z)$. Then we have $G(z) | \tilde{\boldsymbol{C}}_+ = F_+(z)$ and $G(z) | \tilde{\boldsymbol{C}}_- = F_-(z)$. In fact, assume $z \in \boldsymbol{C}_+$. Then there exists some D_n such that $z \in D_n$. Then choose δ so that $\operatorname{Im} z > \delta > 0$. Let γ be the boundary of $\boldsymbol{D} \times i(-\delta, \delta)$ and oriented in the positive sense. Then, since $F(z)$ is holomorphic in the region enclosed by Γ_n and γ, we have

$$F(z) = \int_{\Gamma_n} F(\zeta) \exp(-(\zeta - z)^2)(2\pi i(\zeta - z))^{-1}\mathrm{d}\zeta$$
$$- \int_{\gamma} F(\zeta) \exp(-(\zeta - z)^2)(2\pi i(\zeta - z))^{-1}\mathrm{d}\zeta.$$

Hence we have

$$G(z) = G_n(z) = F(z) + \int_{\gamma} F(\zeta) \exp(-(\zeta - z)^2)(2\pi i(\zeta - z))^{-1}\mathrm{d}\zeta.$$

Then, since z is in the outside of the domain enclosed by γ, $\exp(-(\xi - z)^2)(2\pi i(\xi - z))^{-1}$ belongs to $\mathscr{A}(\boldsymbol{D})$ as a function of ξ. Thus, by virtue of (7) and (8), we have

$$\int_{\gamma} F(\zeta) \exp(-(\zeta - z)^2)(2\pi i(\zeta - z))^{-1}\mathrm{d}\zeta$$
$$= -\left(\int_{-\infty}^{\infty} F_+(\xi + i\delta) \exp(-(\xi + i\delta - z)^2)(2\pi i(\xi + i\delta - z))^{-1}\mathrm{d}\xi\right.$$
$$\left.- \int_{-\infty}^{\infty} F_-(\xi - i\delta) \exp(-(\xi - i\delta - z)^2)(2\pi i(\xi - i\delta - z))^{-1}\mathrm{d}\xi\right)$$
$$= -\langle f(\xi), \exp(-(\xi - z)^2)(2\pi i(\xi - z))^{-1}\rangle = 0.$$

Hence $G(z) | \tilde{\boldsymbol{C}}_+ = F_+(z)$ holds.

Similarly, $G(z) | \tilde{\boldsymbol{C}}_+ = F_-(z)$ holds. Hence $F(z)$ can be continued analytically to a slowly increasing holomorphic function $G(z)$ on $\tilde{\boldsymbol{C}}$. By the above, we have the following.

Theorem 4. *Let $F_+(z)$ and $F_-(z)$ be holomorphic functions on $\tilde{\boldsymbol{C}}_+$ and $\tilde{\boldsymbol{C}}_-$ respectively. Then, in order that the boundary value $f(x) = F_+(x+i0) - F_-(x-i0)$ in the topology of $\mathscr{Q}'(\boldsymbol{D})$ defines a Fourier hyperfunction on $\boldsymbol{D}$, it is necessary and sufficient that there exists some $F(z) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}} \setminus \boldsymbol{D})$ such that $F_+(z) = F(z) | \tilde{\boldsymbol{C}}_+$ and $F_-(z) = F(z) | \tilde{\boldsymbol{C}}_-$ hold. Especially, in order that $f(x) = 0$ holds, it is necessary and sufficient that there exists $F(z) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}})$* such that *$F_+(z) = F(z) | \tilde{\boldsymbol{C}}_+$ and $F_-(z) = F(z) | \tilde{\boldsymbol{C}}_-$ holds.*

By the above correspondence, we have the following.

Corollary 1. *In the notations of Theorem* 4, *we have the algebraic isomorphism*

$$\mathscr{Q}'(\boldsymbol{D}) \cong \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}} \setminus \boldsymbol{D}) / \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}}).$$

As another Corollary, we have the Edge of the Wedge theorem as follows.

Corollary 2. *For $F(z) \in \tilde{\mathcal{O}}(\tilde{\boldsymbol{C}} \setminus \boldsymbol{D})$, put $F_+(z) = F(z) | \tilde{\boldsymbol{C}}_+$ and $F_-(z) = F(z) | \tilde{\boldsymbol{C}}_-$. Then, if there exist $F_+(x+i0)$ and $F_-(x-i0)$ in the topology of $\mathscr{Q}'(\boldsymbol{D})$ and $F_+(x+i0) = F_-(x-i0)$ holds, $F(z)$ can be continued analytically to a slowly increasing holomorphic function on $\tilde{\boldsymbol{C}}$.*

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