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Stationary Fourier Hyperfields

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Abstract

In this paper, we define stationary Fourier hyperfields and prove the structure theorems of stationary Fourier hyperfields. Thereby we determine the whole class of all stationary Fourier hyperfields.

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§1. Introduction

In this paper, we define stationary Fourier hyperfields and study their properties and prove the structure theorem of stationary Fourier hyperfields. A random Fourier hyperfield is defined to be a vector-valued Fourier hyperfunction on $\boldsymbol{D}^d$ whose values are random variables with mean 0 and finite variance. Then we introduce the concept of stationary Fourier hyperfields. Stationary Fourier hyperfields have similar properties to those of stationary Fourier hyperprocesses. So a similar method can be applicable for studying stationary Fourier hyperfields (See Ito [9]).

At first we show the existence of the covariance Fourier hyperfunction of a stationary Fourier hyperfield. Next this covariance Fourier hyperfunction ρ is the Fourier transform of a nonnegative measure μ with infraexponential increase. This measure μ is called the spectral measure of ρ. Thirdly we prove that any stationary Fourier hyperfield is the Fourier transform of a random hypomeasure with respect to the spectral measure of its covariance Fourier hyperfunction. At last we prove the structure theorem of stationary Fourier hyperfields (Theorem 7.3).

In §2, we first introduce some fundamental notions and prepare the notation.

In §3, we define the covariance Fourier hyperfunctions of stationary random Fourier hyperfields.

In §4, we prove the spectral decomposition theorem of covariance Fourier hyperfunctions.

In § 5, we prove the spectral decomposition theorem of stationary Fourier hyperfields.

In § 6, we mention the derivative of stationary Fourier hyperfields.

In § 7, we mention the structure theorem of stationary Fourier hyperfields.

§ 2. Fundamental notions and notation

In this paper we restrict ourselves to complex-valued random variables with mean 0 and finite variance. Let H be the Hilbert space constituted by all such variables. In H, we define the inner product by the following relation:

$$(X, Y) = E(X \cdot \bar{Y}), \quad \text{for } X \text{ and } Y \text{ in } H,$$

where E denotes the expectation. We consider only the strong topology on H. A continuous random fields $X(x)$, $x \in \boldsymbol{R}^d$, is an H-valued continuous function on $\boldsymbol{R}^d$, where d is a positive integer. The set of all continuous random fields is denoted by $C(H)$.

Now we remember the notions of Fourier hyperfunctions and vector-valued Fourier hyperfunctions following Kawai [17], [18], Ito and Nagamachi [12], [13], Junker [14], [15], Ito [7], [10], [11].

Let $\boldsymbol{D}^d$ be the radial compactification of $\boldsymbol{R}^d$ in the sense of Kawai [18], Definition 1.1.1. Namely $\boldsymbol{D}^d$ is the disjoint union $\boldsymbol{R}^d \sqcup S_\infty^{d-1}$ of $\boldsymbol{R}^d$ and the $(d-1)$-dimensional sphere S_∞^{d-1} at infinity. When x is a vector in $\boldsymbol{R}^d \setminus \{0\}$, we denote by $x\infty$ the point in S_∞^{d-1} whose representative is x in the identification of S_∞^{d-1} with $(\boldsymbol{R}^d \setminus \{0\})/\boldsymbol{R}^+$. Here $\boldsymbol{R}^+$ denotes the set of all positive real numbers. Each element in $\boldsymbol{R}^+$ is considered as a multiplication operator on $\boldsymbol{R}^d \setminus \{0\}$. The space $\boldsymbol{D}^d$ is endowed with the following natural topology. Namely, (i) if a point x of $\boldsymbol{D}^d$ belongs to $\boldsymbol{R}^d$, a fundamental system of neighborhoods of x is given by the family of all open spheres in $\boldsymbol{R}^d$ including x. (ii) If a point x of $\boldsymbol{D}^d$ belongs to S_∞^{d-1}, a fundamental system of neighborhoods of $x(=y\infty)$ is given by the family $\{(C+a) \cup C_\infty ; C_\infty \ni y\infty\}$. Here a runs through all points in $\boldsymbol{R}^d$ and C runs through all open cones in $\boldsymbol{R}^d$ with vertex at the origin which contains $y \in \boldsymbol{R}^d \setminus \{0\}$ and C_∞ denotes the set $\{z\infty ; z \in C\}$.

We denote by $\tilde{\boldsymbol{C}}^d$ the space $\boldsymbol{D}^d \times \sqrt{-1}\boldsymbol{R}^d$ endowed with the direct product topology. $\boldsymbol{D}^d$ and S_∞^{d-1} are identified with the subsets of $\tilde{\boldsymbol{C}}^d$ by the relations $\boldsymbol{D}^d \rightrightarrows \boldsymbol{D}^d \times \sqrt{-1}\{0\} \subsetneq \tilde{\boldsymbol{C}}^d$ and $S_\infty^{d-1} \rightrightarrows S_\infty^{d-1} \times \sqrt{-1}\{0\} \subsetneq \tilde{\boldsymbol{C}}^d$. For a subset U of $\tilde{\boldsymbol{C}}^d$, we denote by $\operatorname{int}(U)$ its interior and by U^{cl} its closure in $\tilde{\boldsymbol{C}}^d$.

We define the sheaf $\underset{\sim}{\mathcal{O}}$ of rapidly decreasing and holomorphic functions over $\tilde{\boldsymbol{C}}^d$ to be the sheaf $\{\underset{\sim}{\mathcal{O}}(\Omega); \Omega \text{ is an open set in } \tilde{\boldsymbol{C}}^d\}$, where the section module $\underset{\sim}{\mathcal{O}}(\Omega)$ on an open set Ω in $\tilde{\boldsymbol{C}}^d$ is the space of all holomorphic functions $f(z)$ on

$\Omega \cap C^d$ such that, for any compact set K in Ω, there exists some positive constant δ so that the estimate $\sup\{|f(z)|e(\delta|z|); z \in K \cap C^d\} < \infty$ holds.

Then the sheaf $\mathscr{A}$ is defined to be the sheaf $\mathscr{A} = \underset{\sim}{\mathcal{O}}|_{D^d}$. Namely $\mathscr{A}$ is the sheaf of rapidly decreasing and real-analytic functions over D^d.

We now define the topology of the space $\underset{\sim}{\mathcal{O}}(K)$, where K is a compact set in $\tilde{C}^d$. Let $\{U_m\}_{m \geq 1}$ be a fundamental system of neighborhoods of K such that $U_{m+1} \subset\subset U_m$ holds. Here $U_{m+1} \subset\subset U_m$ means that U_{m+1} has a compact neighborhood in U_m with respect to the topology of $\tilde{C}^d$. Then we endow the topology of $\underset{\sim}{\mathcal{O}}(K)$ with the inductive limit topology $\lim \operatorname{ind}_m \mathcal{O}_b^{-1/m}(U_m)$. Here the space $\mathcal{O}_b^{-1/m}(U)$ is the Banach space

$$\mathcal{O}_b^{-1/m}(U) = \{f \in C(U^{cl} \cap C^d); f|_{U \cap C^d} \in \mathcal{O}(U \cap C^d), \sup\{f(z)e(|z|/m); z \in U^{cl} \cap C^d\} < \infty\}.$$

Then $\underset{\sim}{\mathcal{O}}(K)$ becomes a DFS-space. If K is a compact subset of D^d, we identify $\mathscr{A}(K) = \underset{\sim}{\mathcal{O}}(K)$. In this sense $\mathscr{A}(K)$ is also a DFS-space.

Let $\underset{\sim}{A} = \mathscr{A}(D^d)$ be the space of all sections of $\mathscr{A}$ on D^d. $\underset{\sim}{A}$ is endowed with the usual DFS topology. A Fourier analytic functional defined on $\underset{\sim}{A}$ is called a Fourier hyperfunction on D^d and a Fourier analytic linear mapping from $\underset{\sim}{A}$ to H is called an H-valued Fourier hyperfunction on D^d. We denote by $\underset{\sim}{A}'$ the space of all Fourier hyperfunctions on D^d and by $\underset{\sim}{A}'(H)$ the space of all H-valued Fourier hyperfunctions on D^d.

Definition 2.1. A random Fourier hyperfield on D^d is defined to be an H-valued Fourier hyperfunction on D^d.

In this paper we only concern random Fourier hyperfields. So we call them Fourier hyperfields for short.

Let $\tilde{C}(H)$ be the set of all H-valued continuous functions on R^d which satisfy the following estimate

$$\sup\{\|X(x)\| e^{-\varepsilon|x|}; x \in K \cap R^d\} < \infty$$

for any $\varepsilon > 0$ and any compact set K in D^d, where $\|\cdot\|$ denotes the norm on H. An element of $\tilde{C}(H)$ is called a slowly increasing and continuous random fields. Then $\tilde{C}(H)$ may be considered as a subsystem of $\underset{\sim}{A}'(H)$, since we can identify a slowly increasing and continuous random field $X(x)$ with the following Fourier hyperfield $X(\phi)$ determined by it:

$$X(\phi) = \int X(x)\phi(x)\,dx \equiv \int_{R^d} X(x)\phi(x)\,dx, \qquad \text{for } \phi \in \underset{\sim}{A}.$$

The following notation is often used in the theory of Fourier hyperfunctions. Let $F \in \underset{\sim}{A}'$ or $F \in \underset{\sim}{A}'(H)$ and $\phi \in \underset{\sim}{A}$.

τ_h(shift transformation): $\tau_h\phi(x) = \phi(x-h)$,

$$\tau_h F(\phi) = F(\tau_{-h}\phi).$$

D_j(partial derivative, $1 \leqq j \leqq d$): $D_j\phi(x) = \partial\phi(x)/\partial x_j$,

$$D_j F(\phi) = -F(D_j\phi).$$

$\check{}$(inversion): $\check{\phi}(x) = \phi(-x)$, $\check{F}(\phi) = F(\check{\phi})$.

$\bar{}$(conjugate): $\bar{\phi}(t) = \overline{\phi(t)}$, $\bar{F}(\phi) = \overline{F(\bar{\phi})}$.

$\tilde{}$($=\bar{\check{}} = \check{\bar{}}$): $\tilde{\phi}(x) = \overline{\phi(-x)}$, $\tilde{F}(\phi) = \overline{F(\tilde{\phi})}$.

$\hat{}$(Fourier transformation): $\hat{\phi}(\lambda) = \int e^{-2\pi i\lambda\cdot x}\phi(x)\,dx$,

$$\hat{F}(\phi) = F(\hat{\phi}),$$

where, for $x = (x_1,\dots,x_d)$ and $\lambda = (\lambda_1,\dots,\lambda_d) \in \boldsymbol{R}^d$, we put $\lambda\cdot x = \lambda_1 x_1 + \cdots + \lambda_d x_d$.

The following relations should be noted:

$$(F*\phi)(0) = F(\check{\phi}) = \check{F}(\phi),\ (\phi*\psi)^{\wedge} = \hat{\phi}\cdot\hat{\psi},\ \hat{\hat{\phi}} = \bar{\bar{\phi}}.$$

Generalizing Khintchine-Itô's notions of (weakly) stationary processes (see Itô [6]), we have the following.

Definition 2.2. We call $X \in \underset{\sim}{A}'(H)$ weakly stationary or merely stationary for short if we have, for any ϕ and ψ in $\underset{\sim}{A}$,

$$(\tau_h X(\phi),\ \tau_h X(\psi)) = (X(\phi),\ X(\psi))$$

and strictly stationary if the joint probability law of

$$(\tau_h X(\phi_1),\dots,\tau_h X(\phi_n))$$

is independent of h for any n and for $\phi_1,\dots,\phi_n \in \underset{\sim}{A}$.

We adopt here the following notation:

S : the totality of stationary (random) Fourier hyperfields,
S^0: the totality of slowly increasing, stationary and continuous random fields,
$\bar{S}$: the totality of strictly stationary (random) Fourier hyperfields,
$\bar{S}^0$: the totality of slowly increasing, strictly stationary and continuous random fields.

Clearly we have

$$S \supset \bar{S} \cup S^0,\ S^0 \supset \bar{S}^0.$$

Definition 2.3. A random Fourier hyperfield X is called a complex-normal (random) Fourier hyperfield if $X(\phi)$, $(\phi \in \underset{\sim}{A})$ constitues a complex-normal system and a real-normal (random) Fourier hyperfield if X is real viz. $X = \bar{X}$ and $X(\phi)$, ϕ running over real functions in $\underset{\sim}{A}$, constitutes a (real-) normal system (see Itô [4], [5] and Hida [3]).

This is a generalization of normal processes or Gaussian processes (Doob [1], II, §3) and complex-(or real-) normal random distributions (Itô [6]). A (complex- as well as real-) normal Fourier hyperfield is strictly stationary, if it is weakly stationary. The corresponding fact is also true regarding stationary continuous random fields.

§3. Covariance Fourier hyperfunctions

At first we consider the covariance function $\rho(x)$ of a stationary continuous field $X(x)$ on $\boldsymbol{R}^d$. By definition we have

$$\rho(x) = (X(x), X(0)) = (X(x+y), X(y)).$$

Then we have, for any ϕ and ψ in $\mathscr{D}$,

$$\begin{aligned}(X(\phi), X(\psi)) &= \left(\int X(x)\phi(x)\mathrm{d}x, \int X(y)\psi(y)\mathrm{d}y\right)\\ &= \iint (X(x), X(y))\,\phi(x)\bar{\psi}(y)\mathrm{d}x\mathrm{d}y\\ &= \iint \rho(x-y)\phi(x)\,\overline{\psi(y)}\,\mathrm{d}x\mathrm{d}y\\ &= \int \rho(x)\int \phi(x-y)\,\overline{\psi(-y)}\,\mathrm{d}y\mathrm{d}x,\end{aligned}$$

where $\mathscr{D}$ denotes the space of all C^∞ functions on $\boldsymbol{R}^d$ with compact support. Thus we have $\rho \in \mathscr{D}'$ and

$$(X(\phi), X(\psi)) = \rho(\phi * \tilde{\psi}).$$

Similarly to Khintchine-Itô's notion of covariance distributions, we here define the notion of the covariance Fourier hyperfunctions.

Theorem 3.1. *Let $X(\phi)$ be any stationary Fourier hyperfield. Then there exists one and only one Fourier hyperfunction $\rho \in \underset{\sim}{A}'$ satisfying the relation*

$$(X(\phi), X(\psi)) = \rho(\phi * \tilde{\psi}), \qquad \textit{for } \phi \textit{ and } \psi \textit{ in } \underset{\sim}{A}.$$

Definition 3.2. The Fourier hyperfunction ρ in Theorem 3.1 is called the covariance Fourier hyperfunction of X.

Proof of Theorem 3.1. If we put

$$T_{\phi}(\psi) = (X(\phi),\ X(\bar{\psi})), \qquad \text{for } \phi \text{ and } \psi \text{ in } \underset{\sim}{A},$$

then we get a Fourier hyperfunction $T_{\phi} \in \underset{\sim}{A}'$ for each $\phi \in \underset{\sim}{A}$. Taking into account the fact that $T_{\phi}(\psi)$ is continuous in $(\phi, \psi) \in \underset{\sim}{A} \times \underset{\sim}{A}$ and by virtue of Kernel Theorem, we easily see that $\phi \to T_{\phi}$ is a continuous linear mapping from $\underset{\sim}{A}$ into $\underset{\sim}{A}'$ (see Grothendieck [2], Chapter II, Théorème 12 and Ito [10]). Furthermore this transformation commutes with the shift transformation:

$$\begin{aligned}(\tau_h T_{\phi})(\psi) &= T_{\phi}(\tau_{-h}\psi) = (X(\phi),\ X(\overline{\tau_{-h}\psi})) \\ &= (X(\phi),\ X(\tau_{-h}\bar{\psi})) = (X(\tau_h\phi),\ X(\bar{\psi})) \\ &= T_{\tau_h\phi}(\psi).\end{aligned}$$

Here we use the following.

Lemma. *A continuous linear mapping $\phi \to T_{\phi}$ from $\underset{\sim}{A}$ to $\underset{\sim}{A}'$ commutes with the shift transformation if and only if there exists a Fourier hyperfunction T such that $T_{\phi} = T * \phi$ holds.*

Proof of the Lemma. This can be proved similarly to that of the Lemma of Ito [9], §2. Q.E.D.

Now we go back to the proof of Theorem 3.1. Thus, by the Lemma above, T_{ϕ} is expressible as the convolution of a Fourier hyperfunction T and $\phi \in \underset{\sim}{A}$:

$$T_{\phi} = T * \phi.$$

Hence it follows that

$$\begin{aligned}(X(\phi),\ X(\psi)) &= T_{\phi}(\bar{\psi}) = (T * \phi)(\bar{\psi}) \\ &= (T * \phi * \tilde{\psi})(0) = \rho(\phi * \tilde{\psi}),\end{aligned}$$

where we put $\rho = \check{T}$.

The uniqueness of ρ follows at once from the fact that the set of all elements of the form $\phi * \psi$, $(\phi, \psi \in \underset{\sim}{A})$ is dense in $\underset{\sim}{A}$. Q.E.D.

Theorem 3.3. *If $X(\phi)$ is a real and stationary Fourier hyperfield, then its covariance Fourier hyperfunction is real, i.e., $\rho = \bar{\rho}$.*

Proof. Let ρ be the covariance Fourier hyperfunction. Then that of $\bar{X}$ is $\bar{\rho}$, since we have

$$(\bar{X}(\phi), \bar{X}(\psi)) = (\overline{X(\bar{\phi})}, \overline{X(\bar{\psi})}) = \overline{(X(\bar{\phi}), X(\bar{\psi}))}$$
$$= \overline{\rho(\bar{\phi} * \tilde{\bar{\psi}})} = \overline{\rho(\overline{\phi * \tilde{\psi}})}$$
$$= \bar{\rho}(\phi * \tilde{\psi}).$$

Thus $X = \bar{X}$ implies $\rho = \bar{\rho}$. Q.E.D.

§ 4. Spectral decomposition of covariance Fourier hyperfunctions

Let $X(x)$ be any stationary and continuous random field which has the covariance function $\rho(x)$. Then, similarly to Khintchine's Theorem [19], $\rho(x)$ can be expressible in the form

$$\rho(x) = \int e^{-2\pi i \lambda \cdot x} \mathrm{d}\mu(\lambda)$$

in one and only one way, where μ is a nonnegative measure on $\boldsymbol{R}^d$ such that $\mu(\boldsymbol{R}^d) < \infty$. The distribution $\rho(\phi)$ induced by $\rho(x)$ may be expressed as

$$\rho(\phi) \equiv \int \rho(x)\phi(x)\mathrm{d}x = \int \hat{\phi}(\lambda)\mathrm{d}\mu(\lambda), \qquad \text{for } \phi \in \mathscr{D}.$$

Let $X(\phi)$ now be any stationary Fourier hyperfield with the covariance Fourier hyperfunction ρ. Then we have

$$\rho(\phi * \tilde{\phi}) = (X(\phi), X(\phi)) \geqq 0,$$

which implies that ρ is a positive semidefinite Fourier hyperfunction. Thus, by virtue of Bochner-Nagamachi-Mugibayashi-Junker's Theorem (see Nagamachi-Mugibayashi [20], Theorem 4.1 and Junker [15], Theorem 5.8), we have the following.

Theorem 4.1. *We use the notation above. Then ρ is expressible in the form*

$$\text{(I)} \quad \rho(\phi) = \int \hat{\phi}(\lambda)\mathrm{d}\mu(\lambda), \qquad \textit{for } \phi \in \underset{\sim}{A}$$

in one and only one way, where μ is a nonnegative measure satisfying

$$\int e^{-\varepsilon|\lambda|}\mathrm{d}\mu(\lambda) < \infty$$

for every $\varepsilon > 0$.

Definition 4.2. We call the expression (I) the spectral decomposition of ρ and μ the spectral measure of ρ.

Conversely we have the following.

Theorem 4.3. *Any Fourier hyperfunction of the form* (I) *above is the covariance Fourier hyperfunction of a stationary Fourier hyperfield which is complex-normal.*

Proof. Let ρ be a Fourier hyperfunction of the form above. Put

$$\Gamma(\phi, \psi) = \rho(\phi * \tilde{\psi}), \qquad \text{for } \phi \text{ and } \psi \text{ in } \underset{\sim}{A}.$$

Then $\Gamma(\phi, \psi)$ is positive semidefinite in (ϕ, ψ), as we have

$$\sum_{i,j=1}^{n} \Gamma(\phi_i, \phi_j)\xi_i\bar{\xi}_j = \rho(\theta * \tilde{\theta}) \geqq 0, \qquad \text{for } \theta = \sum_i \xi_i \phi_i.$$

Therefore we can define a complex-normal system $X(\phi)$, $(\phi \in \underset{\sim}{A})$ such that $EX(\phi) = 0$ and $E(X(\phi) \cdot \overline{X(\psi)}) = \Gamma(\phi, \psi) = \rho(\phi * \tilde{\psi})$ as in Itô [5] and Hida [3]. Then we can easily see that $X(\phi)$ is a continuous linear mapping from $\underset{\sim}{A}$ to H. Thus $X(\phi)$ is a random Fourier hyperfield. Thus our theorem is completely proved. Q.E.D.

Here we remember the Bochner-Nagamachi-Mugibayashi-Junker's Theorem.

Theorem 4.4. *A Fourier hyperfunction T on $\boldsymbol{D}^d$ is positive semidefinite, that is, it satisfies the condition*

$$T(\varphi * \tilde{\varphi}) \geqq 0, \qquad \textit{for every } \varphi \in \underset{\sim}{A}$$

if and only if it is the Fourier transform of some slowly increasing, positive measure μ on $\boldsymbol{R}^d$. Here a positive measure μ on $\boldsymbol{R}^d$ is said to be slowly increasing if it satisfies the condition

$$\int e^{-\varepsilon|\lambda|} \mathrm{d}\mu(\lambda) < \infty, \qquad \textit{for every } \varepsilon > 0.$$

Next we discuss the case of real and stationary Fourier hyperfields. By Theorem 3.3 we see that $\rho = \bar{\rho}$ holds in this case. But we have

$$\bar{\rho}(\phi) = \rho(\bar{\phi}) = \int \bar{\hat{\phi}}(\lambda) \mathrm{d}\mu(\lambda) = \int \hat{\phi}(-\lambda) \mathrm{d}\mu(\lambda)$$
$$= \int \hat{\phi}(\lambda) \mathrm{d}\check{\mu}(\lambda),$$
$$(\check{\mu}(E) = \mu(-E), \; -E = \{x; -x \in E\}).$$

By the uniqueness of the spectral measure, we obtain the following.

Theorem 4.5. *In the case of a real and stationary Fourier hyperfield, the spectral measure μ is symmetric with respect to* 0, *viz.* $\mu(E) = \mu(-E)$.

Conversely we have the following.

Theorem 4.6. *Any Fourier hyperfunction of the form* (I) *with a symmetric measure* μ *is the covariance Fourier hyperfunction of a certain stationary Fourier hyperfield which is real-normal.*

The proof is similar to that of Theorem 4.3. Here we use the existence theorem of a real-normal system instead of that of a complex-normal one.

§5. Spectral decompositon of stationary Fourier hyperfields

We first recall the notion and properties of random hypomeasures. Let μ be a nonnegative measure defined for all Borel sets in $\boldsymbol{R}^d$ and $\boldsymbol{B}^*$ denote the system of all Borel sets with finite μ-measure. Then an H-valued function $M(E)$ defined for $E \in \boldsymbol{B}^*$ is called a random hypomeasure with respect to μ if

$$(M(E_1), M(E_2)) = \mu(E_1 \cap E_2)$$

holds for E_1 and E_2 in $\boldsymbol{B}^*$. Then we have the following (1) $\sim$ (3):

(1) $\|M(E)\|^2 = \mu(E)$, for $E \in \boldsymbol{B}^*$.

(2) $M(E_1) \perp M(E_2)$ if $E_1, E_2 \in \boldsymbol{B}^*$ and $E_1 \cap E_2 = \emptyset$.

(3) If $E_1, E_2, \cdots$ are disjoint to each other and belong to $\boldsymbol{B}^*$ with their sum $E = \sum_{n=1}^{\infty} E_n$, then we have $M(E) = \sum_{n=1}^{\infty} M(E_n)$, (in H).

Then we can easily define the integral with respect to the random hypomeasure (Doob [1], IX, §2):

$$M(f) = \int f(\lambda) \mathrm{d} M(\lambda)$$

for $f \in L^2(\boldsymbol{R}^d, \mu)$. Then we have, for $f_1, f_2 \in L^2(\boldsymbol{R}^d, \mu)$ and for $c_1, c_2 \in \boldsymbol{C}$,

(4) $$(M(f_1), M(f_2)) = (f_1, f_2) \left(\equiv \int f_1(\lambda) \overline{f_2(\lambda)} \, \mathrm{d}\mu(\lambda) \right),$$

(5) $$M(c_1 f_1 + c_2 f_2) = c_1 M(f_1) + c_2 M(f_2).$$

Let $X(x)$ be any stationary and continuous random field which has the spectral measure μ. Then we can see that $X(x)$ may be expressed in one and only one way as

$$X(x) = \int e^{-2\pi i \lambda \cdot x} \mathrm{d} M(\lambda)$$

where M is the random hypomeasure with respect to μ. This can be proved by considering a stationary distribution $X(\phi)$, $(\phi \in \mathscr{D})$, induced by $X(x)$ and by applying a similar method to that in the proof of Theorem 4.1 of Itô [6]. Therefore the stationary distribution $X(\phi)$ induced by $X(x)$ may be expressed as

$$X(\phi) = \int \hat{\phi}(\lambda) \mathrm{d} M(\lambda) = M(\hat{\phi}), \qquad \text{for } \phi \in \mathscr{D}.$$

We now show that this identity holds for any general stationary Fourier hyperfield. Namely we have the following.

Theorem 5.1. *Let X be any stationary Fourier hyperfield with the spectral measure μ. Then $X(\phi)$ is expressible in the form*

$$\text{(II)} \quad X(\phi) = \int \hat{\phi}(\lambda) \mathrm{d} M(\lambda) = M(\hat{\phi}), \qquad \textit{for } \phi \in \underset{\sim}{A}$$

in one and only one way, M being a random hypomeasure with respect to μ. Conversely, any random Fourier hyperfield of such form is a stationary Fourier hyperfield.

Definition 5.2. We call the expression (II) the spectral decomposition of X and M the spectral random hypomeasure of X.

Proof of Theorem 5.1. We first remark that $\underset{\sim}{A}$ is dense in $L^2 \equiv L^2(\boldsymbol{R}^d, \mu)$. Then the Fourier transformation is a topological isomorphism from $\underset{\sim}{A}$ onto itself. Thus the uniqueness of the expression is clear.

In order to prove the possibility of the expression, we put

$$T(\psi) = X(\phi), \qquad \text{for } \psi = \hat{\phi}.$$

Then T is a mapping from $\underset{\sim}{A} (\subset L^2)$ into H, which is clearly linear and isometric on account of the identity:

$$\begin{aligned} \| T(\psi) \|^2 &= (X(\phi), X(\phi)) = \rho(\phi * \tilde{\phi}) \\ &= \int |\psi(\lambda)|^2 \mathrm{d}\mu(\lambda) = \| \psi \|^2, \end{aligned}$$

since $(\phi * \tilde{\phi})\,\hat{} = |\hat{\phi}|^2$. $\underset{\sim}{A}$ being dense in L^2, we can extend $T(\psi)$ to a linear isometric mapping from L^2 into H. As the chracteristic function $\chi_E(\lambda)$ of a set $E \in \boldsymbol{B}^*$ belongs to L^2, we may define $M(E)$ as follows:

$$M(E) = T(\chi_E).$$

Then we have

$$(M(E_1),\ M(E_2)) = \int \chi_{E_1}(\lambda)\overline{\chi_{E_2}(\lambda)}\,\mathrm{d}\mu(\lambda) = \mu(E_1 \cap E_2),$$

since T is isometric. In adition to this, we have

$$\text{(III)} \quad M(f) = T(f), \qquad \text{for } f \in L^2,$$

for this is evidently true for any simple function f in L^2 by the definition and we easily see that it is also true for any $f \in L^2$, by taking into account the fact that both sides of (III) are isometric in f and any $f \in L^2$ is expressed as the L^2-limit of a sequence of simple functions. If we put $f = \hat{\phi}$ in (III), we obtain (II) at once. The last part of the theorem is clear by the definition. Q.E.D.

Making use of this theorem, we can characterize the class of slowly increasing, stationary and continuous random fields.

Theorem 5.3. *A slowly increasing, stationary and continuous random field X is a stationary Fourier hyperfield with the spectral measure μ such that the following holds:*

$$\int \mathrm{d}\mu(\lambda) < \infty.$$

Proof. A slowly increasing, stationary and continuous random field X is a stationary and continuous random field. Then, by the facts mentioned in the beginning of §4 and before Theorem 5.1, its spectral measure μ satisfies the assumption of the theorem and we have the expression

$$X(\phi) = \int \hat{\phi}(\lambda)\mathrm{d}M(\lambda) = M(\hat{\phi}).$$

Thus, by Theorem 5.1, X is a stationary Fourier hyperfield with the spectral measure μ.

Conversely, let X be a stationary Fourier hyperfield with the spectral measure μ which satisfies the assumption of the theorem. Then we have

$$X(\phi) = \int \hat{\phi}(\lambda)\mathrm{d}M(\lambda),\ (M(E_1),\ M(E_2)) = \mu(E_1 \cap E_2),$$

where the following holds:

$$\int \mathrm{d}\mu(\lambda) < \infty.$$

Put

$$Y(x) = \int e^{-2\pi i\lambda\cdot x}\mathrm{d}M(\lambda),$$

which may be defined, since the λ-function $e^{2\pi i\lambda\cdot x}$ belongs to L^2 by virtue of the assumption on μ. $Y(x)$ proves to be a stationary and continuous random field and, what is more, it becomes a slowly increasing, stationary and continuous random field. Therefore, we have, for $\phi \in \underset{\sim}{A}$,

$$\begin{aligned}\int Y(x)\phi(x)\mathrm{d}x &= \int \phi(x)\int e^{-2\pi i\lambda\cdot x}\mathrm{d}M(\lambda)\mathrm{d}x\\ &= \int \hat{\phi}(\lambda)\mathrm{d}M(\lambda) = X(\phi),\end{aligned}$$

which implies that $X(\phi)$ is induced by a slowly increasing, stationary and continuous random field Y. Q.E.D.

In the proof of the theorem above, we have the following.

Corollary. *A slowly increasing, stationary and continuous random field and a stationary and continuous random field are identical.*

By a similar way to Theorem 4.4, we obtain the following.

Theorem 5.4. *In the case of a real and stationary Fourier hyperfield, the spectral random hypomeasure M is hermitiansymmetric, i.e., $M(E) = \overline{M(-E)}$.*

§6. Derivatives of stationary Fourier hyperfields

Any random Fourier hyperfield has derivatives of any order, which are also random Fourier hyperfields.

Theorem 6.1. *Let X be a stationary Fourier hyperfield with the spectral measure μ and the spectral random hypomeasure M. Then $X^{(k)} (= D^k X = D_1^{k_1} \cdots D_d^{k_d} X)$ is also a stationary Fourier hyperfield whose spectral measure $\mu_k = \mu_{(k_1,\dots,k_d)}$ and spectral random hypomeasure $M_k = M_{(k_1,\dots,k_d)}$ are given by*

$$d\mu_k(\lambda) = (2\pi\lambda)^{2k} d\mu(\lambda),\ dM_k(\lambda) = (2\pi i\lambda)^k dM(\lambda),$$

where we put

$$\begin{aligned}(2\pi\lambda)^{2k} &= (2\pi\lambda_1)^{2k_1} \cdots (2\pi\lambda_d)^{2k_d},\\ (2\pi i\lambda)^k &= (2\pi i\lambda_1)^{k_1} \cdots (2\pi i\lambda_d)^{k_d}.\end{aligned}$$

Proof. We have, by definition,

$$X^{(k)}(\phi) = (-1)^{|k|} X(\phi^{(k)}) = (-1)^{|k|} \int \hat{\phi}^{(k)}(\lambda) \mathrm{d} M(\lambda)$$
$$= \int (2\pi i \lambda)^k \hat{\phi}(\lambda) \mathrm{d} M(\lambda),$$

since we have, for $\phi \in \underset{\sim}{A}$,

$$\hat{\phi}^{(k)}(\lambda) = (-1)^{|k|} (2\pi i \lambda)^k \hat{\phi}(\lambda).$$

In the above, we put

$$|k| = k_1 + \cdots + k_d.$$

Thus $X^{(k)}$ proves to be a stationary Fourier hyperfield satisfying the conditions above. Q.E.D.

By Theorem 5.3, we have the following.

Theorem 6.2. *Let X be as in Theorem* 6.1. *In order that $X^{(k)}$ is a stationary and continuous random field, it is necessary and sufficient that the spectral measure μ of X satisfies the following inequality*:

$$\int \lambda^{2k} \mathrm{d} \mu(\lambda) < \infty.$$

§7. The structure theorem

In this section we consider the operation of local operators with constant coefficients on random Fourier hyperfields and prove the structure theorem of stationary Fourier hyperfields. Here a local operator with constant coefficients is defined to be an infinite order differential operator with constant coefficients of the form

$$J(D) = \sum_{k=0}^{\infty} b_k D^k$$

with

$$\lim_{|k| \to \infty} \sqrt[|k|]{|b_k| \, k!} = 0.$$

Then we have the following.

Theorem 7.1. *Let X be a stationary Fourier hyperfield with the spectral measure μ and the spectral random hypomeasure M. Then, for any local operator with constant coefficients $J(D)$, $J(D)X$ is also a stationary Fourier hyperfield whose spectral measure μ_J and spectral random hypomeasure M_J are given by*

$$d\mu_J(\lambda) = |J(2\pi i\lambda)|^2 d\mu(\lambda),$$
$$dM_J(\lambda) = J(2\pi i\lambda)dM(\lambda).$$

Proof. We have, by definition,

$$\begin{aligned}(J(D)X)(\varphi) &= X(J(-D)\varphi) \\ &= \int (J(-D)\varphi)\hat{}\,\mathrm{d}M(\lambda) = \int J(2\pi i\lambda)\hat{\varphi}(\lambda)\mathrm{d}M(\lambda),\end{aligned}$$

since we have, for $\varphi \in \underset{\sim}{A}$,

$$(J(-D)\varphi)\hat{}(\lambda) = J(2\pi i\lambda)\varphi(\lambda).$$

Thus $J(D)X$ proves to be a stationary Fourier hyperfield satisfying the above conditions. Q.E.D.

By Theorem 5.3, we have the following.

Theorem 7.2. *We use the notation in Theorem* 7.1. *In order that $J(D)X$ is a stationary and continuous random field, it is necessary and sufficient that the spectral measure μ of X satisfies the following inequality*:

$$\int |J(2\pi i\lambda)|^2 \mathrm{d}\mu(\lambda) < \infty.$$

From Theorem 7.1, we can see that, for any stationary and continuous random field $V(x)$ and for any local operator with constant coefficients $J(D)$, $X = J(D)V$ becomes a stationary Fourier hyperfield. Furthermore we can see that all stationary Fourier hyperfield is of the form above. Thus we have the following structure theorem of stationary Fourier hyperfields.

Theorem 7.3. *An arbitrary stationary Fourier hyperfield is of the form $J(D)V$, where $V(x)$ is a certain stationary and continuous random field and $J(D)$ is a certain local operator with constant coefficients.*

Proof. Let X be a stationary Fourier hyperfield. Then, by the vector-valued version of Theorem 8.4.9 of Kaneko [16], we can find some continuous random field $V(x)$ and some local operator with constant coefficients such that $X = J(D)V$. Since X is stationary, we can see that V is stationary. In fact, we have, for any φ and ψ in $\underset{\sim}{A}$,

$$(\tau_h X(\varphi), \tau_h X(\psi)) = (X(\varphi), X(\psi))$$

by virtue of the assumption. Then, we have

$$(\tau_h V(J(-D)\varphi), \tau_h V(J(-D)\psi)) = (V(J(-D)\varphi), V(J(-D)\psi)).$$

But, similarly to the proof of Theorem 8.4.9 of Kaneko [16], we can select $J(D)$ so that $J(-D)$ becomes a continuous mapping of $\underset{\sim}{A}$ onto itself. Thus, by Definition 2.2, we can see that V is stationary. Q.E.D.

At all last we mention the vector-valued version of Theorem 8.4.9 of Kaneko [16].

Theorem 7.4. *Every H-valued Fourier hyperfunction on* $\boldsymbol{D}^d$ *can be represented as* $J(D)f(x)$ *for some slowly increasing, H-valued, continuous function* $f(x)$ *and for some local operator* $J(D)$ *with constant coefficients.*

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