J. Math. Tokushima Univ.
Vol. 29 (1995), 1–7

On the Construction of a Special Divisor of Some Special Curve

Dedicate to Professor Yoshihiro Ichijyô on his 65th birthday

By

Akira Ohbuchi

Department of Mathematical Sciences,
Faculty of Integrated Arts and Sciences,
The University of Tokushima,
1-1, Minamijosanjima-cho, Tokushima 770, JAPAN
email address: ohbuchi@ias.tokushima-u.ac.jp
(Recieved September 14, 1995)

Abstract

Let $\pi : X \to C$ be a triple covering of curves where C is Brill-Noether general. We prove the existence of a base point free pencil of degree $d = g - [\frac{3h+1}{2}] - 2$ on a curve X which is not composed with π.

1980 Mathematical Subject Classification (1985 Revision). 14H45, 14H10, 14C20

0 Introduction

Let C be a non-singular curve of genus h and let $\pi : X \to C$ be a triple covering i.e. $deg(\pi) = 3$ and X is a non-singular curve. Let g be the genus of X. In this paper, we investigate the problem of the existence of base point free pencils of relatively low degree on X. By a simple application of the Castelnuovo-Severi bound, we have the following:

Lemma A (Castelnuobo-Severi bound) *For any integer $n \geq \frac{g-3h}{2}$, a base point free pencil g_n^1 is a composed with π i.e. the morphism $\pi : X \to \mathbb{P}^1$ induced by π is always factors through π.*

On the other hand, not many things have been known about this problem. In [3], we prove the following:

Theorem A *If $h \geq 1$ and $g \geq (2[\frac{3h+1}{2}]+1)([\frac{3h+1}{2}+1])$ and C is general (in the sense of Bril-Noether), then there exists a base point free pencil of degree $d \geq g - [\frac{3h+1}{2}] - 1$ which is not composed with π.*

Our aim is to give an extension of Theorem A. The main result of this paper is to prove the following result.

Theorem B *Under the same assumption of Theorem A, there exists a base point free pencil of degree* $d = g - [\frac{3h+1}{2}] - 2$ *which is not composed with* π.

NOTATIONS
$\mathcal{O}_A$: The structure sheaf of a variety A
f^*: The pull back defined by a morphism f
f_*: The direct image defined by a morphism f
$\deg(f)$: The degree of a finite morphism f
$|\mathcal{L}|$: The complete linear system defined by an invertible sheaf $\mathcal{L}$
ϕ_V: The rational map defined by a linear system V
$\mathcal{O}_A(D)$: The invertible sheaf associated with a divisor D
$\Gamma(A, \mathcal{F})$: The global sections of a sheaf $\mathcal{F}$
K_A: A canonical divisor on a non-singular variety A
ω_A: The canonical invertible sheaf on a non-singular variety A
$\mathbb{P}(\delta)$: The projective bundle $\mathrm{Proj}(\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n \delta)$ defined by a locally free sheaf δ

1 The proof of Theorem B

First we prove the following:

Proposition 1 *Under the same assumption of Theorem A, we assume that* $h = 2e$. *Then there is a component* Z *of* $W^1_{g-3e-2}(X)_{red}$ *such that*

$$Z \not\subset \pi^* W^1_l(C) + W_{g-3e-2-3l}(X)$$

for any $l \geq 1$.

To prove the above result, we need the following lemma.

Lemma 1 *Let* $\pi_*(\mathcal{O}_X) \cong \mathcal{O}_C \oplus \mathcal{E}$ *and let* $\delta = -(C_0^2)$ *where* C_0 *is a minimal section of* $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. *Then there are two line bundles* $\mathcal{L}$, $\mathcal{M}$ *on* C *such that*

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L} \to 0$$

is exact and

$$deg(\mathcal{M}) = -(\frac{g-3h}{2} + \frac{2-\delta}{2})$$
$$deg(\mathcal{L}) = -(\frac{g-3h}{2} + \frac{2+\delta}{2})$$
$$-h \leq \delta \leq \frac{g-3h+2}{3}.$$

Proof. See [3].

Q.E.D.

Proof of Proposition 1 . Let D be a divisor of degree $5e+2$ on C. Then

$$dim\Gamma(X, \mathcal{O}(\pi^*D)) = dim\Gamma(C, \mathcal{O}(D)) = 3e + 3$$

$$dim\Gamma(X, \mathcal{O}(\pi^*(D + P + Q))) = dim\Gamma(C, \mathcal{O}(D + P + Q)) = 3e + 5$$

for any $P, Q \in C$ by Lemma 1. Hence we have

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D)) = g - 12e - 4$$

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*(D) - \pi^*(P + Q))) = g - 12e - 8$$

for any $P, Q \in C$. Now we must prove that there are $P_1, \cdots, P_{g-12e-6} \in X$ such that

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - P_1 - \cdots - P_{g-12e-6})) \geq 2$$

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - P_1 - \cdots - P_{g-12e-6} - \pi^*(P + Q))) = 0$$

for any $P, Q \in C$. We put

$$X_{g-12e-6} \supset U_{P+Q} = \{F | dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - \pi^*(P + Q) - F) = 0\}$$

and put

$$Z_{P+Q} = X_{g-12e-6} \setminus U_{P+Q}.$$

Now we consider $\Phi_{P+Q} : X_{g-12e-6} \longrightarrow W_{g-12e-6}(X)$ and $\phi : X_3 \times X_{g-12e-9} \longrightarrow X_{g-12e-6}$ by

$$\Phi_{P+Q}(F) = \mathcal{O}(K_X - \pi^*D - \pi^*(P + Q) = F, \phi(A, B) = A + B.$$

Let

$$S = S_{P+Q} = \Phi^{-1}(W^1_{g-12e-6}(X))$$
$$W = W_{P+Q} = X_{g-12e-6} \setminus S_{P+Q}$$

and let $p_2 : X_3 \times X_{g-12e-9} \longrightarrow X_{g-12e-9}$ be the second projection. Then

$$p_2(\phi^{-1}(Z_{P+Q})) \supset W.$$

Because $K_X - \pi^*D - \pi^*(P + Q) = F$ is linearly equivalent to an effective divisor E and $deg(E) = g - 3e - 5 \geq 3$. Therefore we can take an effective divisor $A_1 + A_2 + A_3 \subset E$. Hence

$$(A_1 + A_2 + A_3, F) \in \phi^{-1}(Z_{P+Q}).$$

So we have

$$p_2(\phi^{-1}(Z_{P+Q})) = X_{g-12e-9}.$$

Therefore

$$dim\phi^{-1}(Z_{P+Q}) = dimX_{g-12e-9} = g - 12e - 9.$$

We now put

$$Z^1_{P+Q} = \{F \in Z_{P+Q} | dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - \pi^*(P+Q) - F)) \geq 2\}.$$

By the same argument of [1] p.163 (1.7)Lemma, we can easily prove that no component of Z_{P+Q} is entirely contained in Z^1_{P+Q}. So $F \in Z_{P+Q} \setminus Z^1_{P+Q}$ implies $K_X - \pi^*(D) - \pi^*(P+Q) - F$ is linearly equivalent to some effective divisor E_0 such that $dim\Gamma(X, \mathcal{O}(E_0)) = 1$. Therefore we can take a divisor $x_1 + x_2 + x_3 \subset F$ such that

$$dim\Gamma(X, \mathcal{O}(x_1 + x_2 + x_3 + E_0)) = 1.$$

We put $F_0 = F - x_1 - x_2 - x_3$. Then

$$\phi(x_1 + x_2 + x_3, F_0) = F$$

and

$$F_0 \in W.$$

Hence

$$\phi^{-1}(Z_{P+Q} \setminus Z^1_{P+Q}) \cap p_2^{-1}(W) \to Z_{P+Q}$$

is dominating. So we have a component $T \subset \phi^{-1}(Z_{P+Q})$ such that

$$\phi(T) = Z_{P+Q}, \ \ p_2(T) = X_{g-12e-9}.$$

Hence $dim Z_{P+Q} = g - 12e - 9$. Now we consider the locus

$$\bigcup_{P+Q \in X_2} Z_{P+Q} \subset X_{g-12e-6}.$$

As $dim Z_{P+Q} = g - 12e - 9$, therefore

$$dim \overline{\bigcup_{P+Q \in X_2} Z_{P+Q}} \leq g - 12e - 7.$$

Hence

$$\overline{\bigcup_{P+Q \in X_2} Z_{P+Q}} \subsetneqq X_{g-12e-6}.$$

So we can take an $F = P_1 + \cdots P_{g-12e-6} \in X \setminus \overline{\bigcup_{P+Q \in X_2} Z_{P+Q}}$. By the definition of Z_{P+Q}, F satisfies that

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - F) \geq 2$$

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - F - \pi^*(P+Q)) = 0$$

for any $P, Q \in C$. This means $\mathcal{O}(K_X - \pi^*D - F) \in W^1_{g-3e-2}(X)$ and $\mathcal{O}(K_X - \pi^*D - F) \notin \pi^*(W^1_l(C)) + W_{g-3e-2}(X)$ for any $l \geq 1$ and $P, Q \in C$. Therefore we have Proposition 1.

Q.E.D.

Next we prove the following:

Proposition 2 *Under the same assumption of Theorem A, we assume that $h = 2e + 1$. Then there is a component Z of $W^1_{g-3e-4}(X)_{red}$ such that*

$$Z \not\subset \pi^* W^1_l(C) + W_{g-3e-4-3l}(X)$$

for any $l \geq 1$.

Proof of Proposition 2. Let D be a divisor of degree $5e + 5$ on C. Then

$$dim\Gamma(X, \mathcal{O}(\pi^*D)) = dim\Gamma(C, \mathcal{O}(D)) = 3e + 5$$

$$dim\Gamma(X, \mathcal{O}(\pi^*(D + P + Q))) = dim\Gamma(C, \mathcal{O}(D + P + Q)) = 3e + 7$$

for any $P, Q \in C$ by Lemma 1. Hence we have

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D)) = g - 12e - 11$$

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*(D) - \pi^*(P + Q))) = g - 12e - 15$$

for any $P, Q \in C$. Now we must prove that there are $P_1, \cdots, P_{g-12e-13} \in X$ such that

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - P_1 - \cdots - P_{g-12e-13})) \geq 2$$

$$dim\Gamma(X, \mathcal{O}(K_X - \pi^*D - P_1 - \cdots - P_{g-12e-13} - \pi^*(P + Q))) = 0$$

for any $P, Q \in C$. But the proof is completely the same method of the proof of Proposition 1.

Q.E.D.

We now prove the following result.

Proposition 3 *Under the same assumption of Theorem A, we assume that $h = 2e$. Then every component Z of $W^1_{g-3e-2}(X)_{red}$ whose general element has base point is of the form $\pi^* W^1_l(C) + W_{g-3e-2-3l}(X)$ for some $l \geq 1$.*

To prove the above result, we need the following results.

Theorem 1 *If Y is a general curve of genus g, then*

$$dim W^r_d(Y) \geq g - (r + 1)(g - d + r).$$

Proof. See [1] p.206 (1.1)Theorem.

Q.E.D.

Theorem 2 *Let Y be a curve of genus g, $\mathcal{L} \in W_n^r(Y) \setminus W_n^{r+1}(Y)$ and let $\mu : \Gamma(Y, \mathcal{L}) \otimes \Gamma(Y, \omega_Y \otimes \mathcal{L}^{\otimes -1}) \to \Gamma(Y, \omega_Y)$ be the cup product map. Then*

$$T_{\mathcal{L}}(W_n(Y)) \cong (im(\mu))^{\perp}$$

where $T_{\mathcal{L}}(W_n(Y)) \subset H^1(Y, \mathcal{O})$ is a tangent space of $W_n(Y)$ and $(im(\mu))^{\perp}$ is a dual space of $im\mu$ by the Serre duality pairing.

Proof. See [1] p.189 (4.2)Proposition.

Q.E.D.

Theorem 3 *Let Y be a non-hyperelliptic curve of genus $g \geq 3$, let d be an integer such that $2 \leq d \leq g-1$ and let r be an integer such that $0 < 2r \leq d$. Then*

$$dim W_d^r(Y) \leq d - 2r - 1.$$

Proof. See [1] p.191 (5.1)Theorem.

Q.E.D.

Proof of Proposition 3. Let $\Sigma \subset W_{g-3e-2}^1(X)$ be a component whose general element has a base point. Thus

$$\Sigma = \Sigma_\beta^1 + W_{g-3e-2-\beta}(X)$$

for some $\beta \geq 1$ where Σ_β^1 is a subvariety of $W_\beta^1(X)$ whose general element is a base point free. As

$$dim\ W_\beta^1(X) \geq g - 6e - 6$$

by Theorem 1, we have that

$$dim\ \Sigma_\beta^1(X) \geq \beta - 3e - 4.$$

By the assumption of Theorem A and Lemma A, $\beta - 3e - 4 \geq 0$. Let $\mathcal{L} \in \Sigma_\beta^1$ be a general element and let $mu : \Gamma(X, \mathcal{L}) \otimes \Gamma(X, \omega \otimes \mathcal{L}^{\otimes -1}) \to \Gamma(X, \omega_X)$ be the cup product map. Then

$$dim\ (im\mu)^{\perp} = dim\ T_{\mathcal{L}}(W_\beta^1(X)) \geq \beta - 3e - 4$$

by Theorem 2. By the base point free pencil trick,

$$dim\ \Gamma(X, \mathcal{L}^{\otimes 2}) - 3 \geq dim\ (im\mu)^{\perp}.$$

Therefore we have

$$dim\ \Sigma_\beta^1 \geq \beta - 3e - 4$$
$$dim\ \Gamma(X, \mathcal{L}^{\otimes 2}) \geq \beta - 3e - 1 \geq 3.$$

We consider a finite map

$$\phi : \Sigma_\beta^1 \to W_{2\beta}^{\beta-3e-2}(X)$$

by $\phi(\mathcal{L}) = \mathcal{L}^{\otimes 2}$. So we have

$$dim W_{2\beta}^{\beta-3e-2}(X) \geq \beta - 3e - 4.$$

Hence

$$dim\ W^1_{\beta+3e+3}(X) = dim\ W^1_{2\beta-(\beta-3e-3)}(X) \geq 2(\beta - 3e) - 7.$$

As $\beta \leq g - 3e - 3$, $\beta + 3e + 3 \leq g$. We assume that $\beta + 3e + 3 = g$. We first consider

$$\rho : W_{\beta+3e+3}(X) \rightarrow W_{g-2}(X)$$

by

$$\rho(\mathcal{L}) = \omega_X \otimes \mathcal{L}^{\otimes -1}.$$

This is a finite map. Therefore we have

$$dim\ W^1_{\beta+3e+3}(X) = dim\ W^1_{g-2}(X) = g - 2.$$

Hence

$$g - 2 \geq 2(\beta - 3e) - 7.$$

By the assumption of Theorem A, this is a contradiction. Now we assume that $\beta+3e+3 \leq g-1$. Then

$$dim\ W^1_{\beta+3e+3}(X) \leq \beta + 3e + 3 - 2 - 1$$

by Theorem 3. Hence

$$\beta \leq 9e + 7$$

By the assumption of Theorem A and Lemma A, this is a contradiction.

Q.E.D.

The following result is also proved by the same method of the proof of proposition 3

Proposition 4 *Under the same assumption of Theorem A, we assume that $h = 2e + 1$. Then every component Z of $W^1_{g-3e-4}(X)_{red}$ whose general element has base point is of the form $\pi^* W^1_l(C) + W_{g-3e-4-3l}(X)$ for some $l \geq 1$.*

Proof of Theorem B. By Proposition 2, Proposition 1, Proposition 3 and Proposition 4, we have Theorem B.

Q.E.D.

References

[1] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, Geometry of Algebraic Curves I, Springer Verlag, 1985

[2] R.Hartshorne, Algebraic Geometry, Springer Verlag, 1974

[3] T. Kato, C. Keem, A. Ohbuchi, On triple coverings of irrational curves, preprint