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Theory of Fourier Microfunctions of Several Types (II)

Dedicated to Professor Takeshi Hirai on his 60*th birthday*

By

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Abstract

In this paper, we define the concept of modified and mixed Fourier microfunctions and investigate their structures. Thereby we obtain the decomposition of singularity of modified and mixed Fourier hyperfunctions. Then we can deduce the qualitative and quantitative property of modified and mixed Fourier hyperfunctions by examining only their singularity spectrums. We also investigate their vector-valued versions.

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Introduction

This paper is the second part of the series of papers on the theory of Fourier microfunctions of several types.

In this paper, we define the concept of modified and mixed Fourier microfunctions and their vector-valued versions, and study their fundamental properties. We can investigate these in a similar way to S.K.K. [22] and Ito [4].

Sheaf homomorphisms $\tilde{\tilde{\mathscr{A}}} \to \tilde{\tilde{\mathscr{B}}}$ and $\mathscr{A}^\# \to \mathscr{B}^\#$ are defined and they become injections. Thereby, the concept of modified and mixed Fourier hyperfunctions can be considered as a generalization of the concept of slowly increasing and real-analytic functions. One purpose of this paper is to analyse the structure of the quotient sheaves $\tilde{\tilde{\mathscr{B}}}/\tilde{\tilde{\mathscr{A}}}$ and $\mathscr{B}^\#/\mathscr{A}^\#$. We can analyse this structure by a similar way to the theory of Sato and Fourier microfunctions. For Sato and Fourier microfunctions, we refer the reader to Kaneko [8], [9], [10], Kashiwara-Kawai-Kimura [11], Morimoto [13], [14], Sato [18], [19], [20], [21], Sato-Kawai-Kashiwara [22] and Ito [4]. The first target is to show that we can define the sheaves $\tilde{\tilde{\mathscr{C}}}$ and $\mathscr{C}^\#$ of modified and mixed Fourier microfunctions over S^*M, respectively, which is the cosphere bundle over M, the radial compactification of $\boldsymbol{R}^{|n|}$, and that we can have the fundamental exact sequences

$$0 \longrightarrow \tilde{\tilde{\mathscr{A}}} \longrightarrow \tilde{\tilde{\mathscr{B}}} \longrightarrow \pi_* \tilde{\tilde{\mathscr{C}}} \longrightarrow 0,$$

and

$$0 \longrightarrow \mathscr{A}^\# \longrightarrow \mathscr{B}^\# \longrightarrow \pi_* \mathscr{C}^\# \longrightarrow 0,$$

where $\pi: S^*M \to M$ is the projection and $\pi_* \tilde{\tilde{\mathscr{C}}}$ and $\pi_* \mathscr{C}^\#$ denote the direct images of $\tilde{\tilde{\mathscr{C}}}$ and $\mathscr{C}^\#$ with respect to π, respectively.

Further we investigate more precise structures of modified and mixed Fourier microfunctions. Until now, the flabbiness of the sheaves $\tilde{\tilde{\mathscr{C}}}$ and $\mathscr{C}^\#$ is not yet known.

Next, we consider a similar construction of the theory of vector-valued modified and mixed Fourier microfunctions.

At last we note that modified and mixed Fourier microfunctions on an open set in $S^*\boldsymbol{R}^{|n|}$ are nothing else but Sato microfunctions and vector-valued Sato microfunctions, respectively, where $S^*\boldsymbol{R}^{|n|}$ is the cosphere bundle over $\boldsymbol{R}^{|n|}$.

In section 4, we construct the theory of modified Fourier microfunctions.

In section 5, we construct the theory of vector-valued modified Fourier microfunctions.

In section 6, we construct the theory of mixed Fourier microfunctions.

In section 7, we construct the theory of vector-valued mixed Fourier microfunctions.

4. Theory of modified Fourier microfunctions

4.1. Modified Fourier hyperfunctions. In this section we apply the general theory of section 1 of Ito [4] to certain special situation and construct the theory of modified Fourier microfunctions.

In this subsection we recall the notion of modified Fourier hyperfunctions following Saburi [16] and Ito [2].

Let $\boldsymbol{R}^n$ be an n-dimensional Euclidean space and $\boldsymbol{R}_n$ be its dual space. Let $\boldsymbol{D}^n = \boldsymbol{R}^n \sqcup S_\infty^{n-1}$ be the radial compactification of $\boldsymbol{R}^n$ in the sense of Kawai [11], Definition 1.1.1, p. 468. We denote this $\boldsymbol{D}^n$ by $\tilde{\tilde{\boldsymbol{R}}}^n$ and put $M = \tilde{\tilde{\boldsymbol{R}}}^n$ and $X = \tilde{\tilde{\boldsymbol{C}}}^n = \boldsymbol{D}^{2n}$ which is the radial compactification of $\boldsymbol{C}^n$ identified with $\boldsymbol{R}^{2n}$.

Let $\tilde{\tilde{\mathscr{O}}}$ be the sheaf of slowly increasing and holomorphic functions on X following Saburi [16] and Ito [2], and put $\tilde{\tilde{\mathscr{A}}} = \tilde{\tilde{\mathscr{O}}}|_M$. Then $\tilde{\tilde{\mathscr{A}}}$ is the sheaf of slowly increasing and real-analytic functions on M. Then we have $\tilde{\tilde{\mathscr{A}}} = \iota^{-1}\tilde{\tilde{\mathscr{O}}}$, where $\iota: M \to X$ is the canonical injection.

As in Saburi [16] and Ito [2], we define the sheaf of modified Fourier hyperfunctions on M.

Definition 4.1. The sheaf $\tilde{\tilde{\mathscr{B}}}$ is, by definition,

$$\tilde{\tilde{\mathscr{B}}} = \mathscr{H}^n_M(\tilde{\tilde{\mathcal{O}}}) = \mathrm{Dist}^n(M, \tilde{\tilde{\mathcal{O}}}),$$

where the symbol in the right hand side of the above equality is due to Sato [17], p. 405. A section of $\tilde{\tilde{\mathscr{B}}}$ is called a Fourier hyperfunction.

As stated in Saburi [16] and Ito [2], we have $\mathscr{H}^k_M(\tilde{\tilde{\mathcal{O}}}) = 0$ for $k \neq n$ and $\tilde{\tilde{\mathscr{B}}}$ constitutes a flabby sheaf on M.

Now we apply Lemma 1.1 of Ito [4] to this case where $\mathscr{F}$, X and Y correspond to $\tilde{\tilde{\mathcal{O}}}$, X and M respectively. Then we obtain the sheaf homomorphism

$$\tilde{\tilde{\mathscr{A}}} \longrightarrow \tilde{\tilde{\mathscr{B}}},$$

which will be proved to be injective later. This injection allows us to consider modified Fourier hyperfunctions as a generalization of slowly increasing and real-analytic functions. The purpose of this section is to analyse the structure of the quotient sheaf $\tilde{\tilde{\mathscr{B}}}/\tilde{\tilde{\mathscr{A}}}$ by a similar way to S.K.K [22].

4.2. Definition of modified Fourier microfunctions. Here we use the notation in subsection 1.1 of Ito [4]. Suppose that $M = \tilde{\tilde{\boldsymbol{R}}}^n$ and $X = \tilde{\tilde{\boldsymbol{C}}}^n$. Then we have the following isomorphisms

$$T(X \cap \boldsymbol{C}^n)|_{\boldsymbol{R}^n} \cong T\boldsymbol{R}^n \oplus iT\boldsymbol{R}^n, \ T\boldsymbol{R}^n \cong \boldsymbol{R}^n \times \boldsymbol{R}^n,$$

$$T^*(X \cap \boldsymbol{C}^n)|_{\boldsymbol{R}^n} \cong T^*\boldsymbol{R}^n \oplus iT^*\boldsymbol{R}^n, \ T^*\boldsymbol{R}^n \cong \boldsymbol{R}^n \times \boldsymbol{R}_n$$

by the complex structure of $X \cap \boldsymbol{C}^n = \boldsymbol{C}^n$. Here i denotes $\sqrt{-1}$ and $\boldsymbol{R}_n$ is the dual space of $\boldsymbol{R}^n$. Hence we have the isomorphisms

$$T_{\boldsymbol{R}^n}(X \cap \boldsymbol{C}^n) \cong T\boldsymbol{R}^n, \ S_{\boldsymbol{R}^n}(X \cap \boldsymbol{C}^n) \cong S\boldsymbol{R}^n \cong \boldsymbol{R}^n \times S^{n-1},$$

$$T^*_{\boldsymbol{R}^n}(X \cap \boldsymbol{C}^n) \cong T^*\boldsymbol{R}^n, \ S^*_{\boldsymbol{R}^n}(X \cap \boldsymbol{C}^n) \cong S^*\boldsymbol{R}^n \cong \boldsymbol{R}^n \times S_{n-1}$$

by the identification $i\xi \leftrightarrow \xi$. Since X is the radial compactification of $X \cap \boldsymbol{C}^n$ along the real subspace, we can take the radial compactification of $T(X \cap \boldsymbol{C}^n)|_{\boldsymbol{R}^n}$ and $T^*(X \cap \boldsymbol{C}^n)|_{\boldsymbol{R}^n}$ along the base space. Hence we obtain the isomorphisms

$$TX|_M \cong TM \oplus iTM, \ TM \cong M \times \boldsymbol{R}^n,$$

$$T^*X|_M \cong T^*M \oplus iT^*M, \ T^*M \cong M \times \boldsymbol{R}_n.$$

Hence we have the isomorphisms

$$T_MX \cong TM, \ S_MX \cong SM \cong M \times S^{n-1},$$

$$T^*_MX \cong T^*M, \ S^*_MX \cong S^*M \cong M \times S_{n-1}.$$

Taking account of this fact, we denote S_MX and S^*_MX by iSM and iS^*M, respectively. The point of iSM (resp. iS^*M) is frequently denoted by $x + i\xi 0$

(resp. $(x, i\langle \eta, dx\rangle\infty) = (x, i\eta\infty)$), where $\xi \in S^{n-1}$ (resp. $\eta \in S_{n-1}$).

We use the general discussions of subsection 1.1 of Ito [4] to this special case. We denote

$$DM = \{(x + i\xi 0, (x, i\eta\infty)) \in iSM \underset{M}{\times} iS^*M;\ \langle i\xi, i\eta\rangle = -\langle \xi, \eta\rangle \geqq 0\},$$

$$IM = \{(x + i\xi 0, (x, i\eta\infty)) \in iSM \underset{M}{\times} iS^*M;\ \langle i\xi, i\eta\rangle = -\langle \xi, \eta\rangle > 0\}.$$

Let ${}^M\tilde{X}$ be the real-monoidal transform of X with center M. Put ${}^M\tilde{X}^+ = (X - M) \sqcup DM$ and ${}^M\tilde{X}^* = (X - M) \sqcup iS^*M$. Similarly to the diagram (1.3) of Ito [4], we have the following diagram:

$$\begin{tikzcd}
 & {}^M\tilde{X}^+ \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & DM \arrow[l, hook'] \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & \\
{}^M\tilde{X} \arrow[dr, "\tau"'] & iSM \arrow[l, hook'] \arrow[dr, "\tau"] & {}^M\tilde{X}^* \arrow[dl, "\pi"'] & iS^*M \arrow[l, hook'] \arrow[dl, "\pi"] \\
 & X & M \arrow[l, hook'] &
\end{tikzcd}$$

Theorem 4.2. *We have* $\mathscr{H}^k_{iSM}(\tau^{-1}\tilde{\tilde{\mathscr{O}}}) = 0$ *for* $k \neq 1$, *where* $\tau\colon {}^M\tilde{X} \to X$ *is the canonical projection.*

Proof. Let $x = x_0 + i\xi 0$ be a point of iSM. Then we have

$$\mathscr{H}^k_{iSM}(\tau^{-1}\tilde{\tilde{\mathscr{O}}})_x \cong \lim\operatorname{ind}_{\tilde{U}\ni x} H^{k-1}(\tilde{U} - iSM, \tilde{\tilde{\mathscr{O}}}), \qquad \text{for } k > 1,$$

and we have the exact sequence

$$0 \longrightarrow \mathscr{H}^0_{iSM}(\tau^{-1}\tilde{\tilde{\mathscr{O}}})_x \longrightarrow \tilde{\tilde{\mathscr{O}}}_{x_0} \xrightarrow{\alpha} \lim\operatorname{ind}_{\tilde{U}\ni x} H^0(\tilde{U} - iSM, \tilde{\tilde{\mathscr{O}}}),$$

where $\tilde{U}$ runs over the neighborhoods of x. Since $\tilde{U} - iSM \neq \emptyset$, α is injective by the property of unique continuation of holomorphic functions. Therefore $\mathscr{H}^0_{iSM}(\tau^{-1}\tilde{\tilde{\mathscr{O}}})_x = 0$. On the other hand, there is a fundamental system of neighborhoods $\{\tilde{U}\}$ of x such that $\tilde{U} - iSM$ is an $\tilde{\tilde{\mathscr{O}}}$-pseudoconvex open set. It follows from the Oka-Cartan-Kawai Theorem B that we have

$$\mathscr{H}^k_{iSM}(\tau^{-1}\tilde{\tilde{\mathscr{O}}})_x = 0 \qquad \text{for } k > 1. \qquad \text{(Q.E.D.)}$$

The following theorem is the most essential one in the theory of Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 4.3. *We have* $\mathscr{H}^k_{iS^*M}(\pi^{-1}\tilde{\tilde{\mathscr{O}}}) = 0$ *for* $k \neq n$, *where* $\pi\colon {}^M\tilde{X}^* \to X$ *is the canonical projection.*

Proof. Let $x = (x_0, i\eta\infty) \in iS^*M$. Then, by Proposition 1.4 of Ito [4], we have

$$\mathscr{H}^k_{iS^*M}(\pi^{-1}\tilde{\tilde{\mathcal{O}}})_x \cong \lim \operatorname{ind}_Z H^k_Z(X, \tilde{\tilde{\mathcal{O}}}),$$

where Z runs over the family of

$$Z = \{z = x + iy \in U;\ \langle y, \eta_j\rangle \leqq 0\ (1 \leqq j \leqq n)\}$$

where U is a neighborhood of $\pi(x) = x_0$ in X and $\eta_1,\dots,\eta_n \in S_{n-1}$ are chosen so that the convex hull of $\eta_0(=\eta)$, $\eta_1,\dots,\eta_n$ contains a certain neighborhood of the origin. Moreover, we have

$$\lim \operatorname{ind}_Z H^k_Z(X, \tilde{\tilde{\mathcal{O}}}) = \lim \operatorname{ind}_G \mathscr{H}^k_G(\tilde{\tilde{\mathcal{O}}})_{x_0},$$

where G runs all over the family of

$$G = \{z = x + iy \in X;\ \langle y, \eta_j\rangle \leqq 0\ (1 \leqq j \leqq n)\},$$

for $\eta_1,\dots,\eta_n$ varying in a neighborhood of $-\eta$. By the "Edge of the Wedge" Theorem (see the following theorem 4.4), we have

$$\mathscr{H}^k_G(\tilde{\tilde{\mathcal{O}}})_{x_0} = 0 \qquad \text{for } k \neq n.$$

Therefore, we have

$$\mathscr{H}^k_{iS^*M}(\pi^{-1}\tilde{\tilde{\mathcal{O}}}) = 0 \qquad \text{for } k \neq n. \qquad \text{(Q.E.D.)}$$

Theorem 4.4 (the "Edge of the Wedge" Theorem). *Put* $G = \{z = x + iy \in X;\ y_j \geqq 0\ (1 \leqq j \leqq n)\}^{c1}$. *Then we have, for each* $x \in M$,

$$\mathscr{H}^k_G(\tilde{\tilde{\mathcal{O}}})_x = 0 \qquad \textit{for } k \neq n.$$

In the proof of Theorem 4.3, we have only use the linear transformation of G in Theorem 4.4.

The proof of Theorem 4.4 goes in a similar way to Kashiwara-Kawai-Kimura [10], Theorem 2.2.2, p. 60.

Definition 4.5. We define the sheaf $\tilde{\tilde{\mathscr{C}}}$ on iS^*M by

$$\tilde{\tilde{\mathscr{C}}} = \mathscr{H}^n_{iS^*M}(\pi^{-1}\tilde{\tilde{\mathcal{O}}})^a,$$

where we denote by a the antipodal map $iS^*M \to iS^*M$, and by $\mathscr{F}^a$ the inverse image under a of a sheaf $\mathscr{F}$ on iS^*M. A section of $\tilde{\tilde{\mathscr{C}}}$ is called a modified Fourier microfunction.

Now we define the sheaves $\tilde{\tilde{\mathscr{Q}}}$, $\tilde{\tilde{\mathcal{O}}}^\beta$ and $\tilde{\tilde{\mathscr{A}}}^\beta$ by

$$\tilde{\tilde{\mathscr{Q}}} = \mathscr{H}^1_{iSM}(\tau^{-1}\tilde{\tilde{\mathcal{O}}}),$$
$$\tilde{\tilde{\mathcal{O}}}^\beta = j_*(\tilde{\tilde{\mathcal{O}}}|_{X-M}),$$
$$\tilde{\tilde{\mathscr{A}}}^\beta = \tilde{\tilde{\mathcal{O}}}^\beta|_{iSM},$$

where $j: X - M \hookrightarrow {}^M\tilde{X}$, $\pi: {}^M\tilde{X}^* \to X$ and $\tau: {}^M\tilde{X} \to X$ are canonical maps.

By Proposition 1.3 of Ito [4] and Theorems 4.2 and 4.3, we have the following.

Proposition 4.6. *We have*

$$R^k\tau_*\pi^{-1}\tilde{\tilde{\mathscr{Q}}} = \begin{cases} \tilde{\tilde{\mathscr{C}}}^a & (k = n-1) \\ 0 & (k \neq n-1) \end{cases}.$$

Theorem 4.7. *We have*

$$R^k\pi_*\tilde{\tilde{\mathscr{C}}} = R^{k+n-1}\tau_*\tilde{\tilde{\mathscr{Q}}} = 0 \qquad \textit{for } k \neq 0,$$

and we have the exact sequence

$$0 \longrightarrow \tilde{\tilde{\mathscr{A}}} \longrightarrow \tilde{\tilde{\mathscr{B}}} \longrightarrow \pi_*\tilde{\tilde{\mathscr{C}}} \longrightarrow 0.$$

Proof. $R^k\pi_*\tilde{\tilde{\mathscr{C}}} = R^{k+n-1}\tau_*\tilde{\tilde{\mathscr{Q}}}$ is the trivial corollary of the preceding proposition. The triangle obtained in Proposition 1.6 of Ito [4] implies immediately $R^k\pi_*\tilde{\tilde{\mathscr{C}}} = 0$ for $k \neq 0$ and yields the exact sequence in the theorem. (Q.E.D.)

This is the required decomposition of singularity of Fourier hyperfunctions.

Corollary 4.8. *We have the exact sequence*

$$0 \longrightarrow \tilde{\tilde{\mathscr{A}}}(M) \xrightarrow{\lambda} \tilde{\tilde{\mathscr{B}}}(M) \xrightarrow{\mathrm{sp}} \tilde{\tilde{\mathscr{C}}}(iS^*M) \longrightarrow 0.$$

Definition 4.9. Let $u \in \tilde{\tilde{\mathscr{B}}}(M)$. We call $\mathrm{sp}(u) \in \tilde{\tilde{\mathscr{C}}}(iS^*M)$ a spectrum of u. We denote by $\mathrm{S.S.}u$ the support $\mathrm{supp\,sp}(u)$ of $\mathrm{sp}(u)$ and call it a singularity spectrum of u. $\pi(\mathrm{S.S.}u)$ is evidently the subset where u is not slowly increasing nor real-analytic and is called the singular support of u.

Corollary 4.10. *Let $u \in \tilde{\tilde{\mathscr{B}}}(M)$. Then u is a slowly increasing and real-analytic function on M if and only if* $\mathrm{S.S.}u \neq \emptyset$.

Since $\mathscr{A} = \tilde{\tilde{\mathscr{A}}}|_{\boldsymbol{R}^n}$, $\mathscr{B} = \tilde{\tilde{\mathscr{B}}}|_{\boldsymbol{R}^n}$ and $\mathscr{C} = \tilde{\tilde{\mathscr{C}}}|_{iS^*\boldsymbol{R}^n}$ hold in the notation of S.K.K [22], we have the following Corollary by restricting the exact sequence in Theorem 4.7.

Corollary 4.11. *Let $\pi: iS^*\boldsymbol{R}^n \to \boldsymbol{R}^n$ be the canonical projection. Then we have the exact sequence*

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \pi_*\mathscr{C} \longrightarrow 0.$$

4.3. Fundamental diagram on $\tilde{\tilde{\mathscr{C}}}$. We apply the arguments in the subsection 1.2 of Ito [4] to a special case. At first we apply Proposition 1.10 of Ito [4] to the situation $\mathscr{F} = \tilde{\tilde{\mathscr{Q}}}^a$, $X = M$, $S = iSM$. Then $\mathscr{G} = \tilde{\tilde{\mathscr{C}}}$, $\mathscr{E} = \pi_*\tilde{\tilde{\mathscr{C}}}$. We obtain the following.

Proposition 4.12. *We have*

$$R^k\pi_*\tau^{-1}\tilde{\tilde{\mathscr{C}}} = 0 \qquad \text{for } k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow \tilde{\tilde{\mathscr{Q}}} \longrightarrow \tau^{-1}\pi_*\tilde{\tilde{\mathscr{C}}} \longrightarrow \pi_*\tau^{-1}\tilde{\tilde{\mathscr{C}}} \longrightarrow 0.$$

Now we apply the same proposition to the case where $\mathscr{F} = \tilde{\tilde{\mathscr{A}}}^\beta$. Thus we obtain a homomorphism

$$(4.1) \qquad \tilde{\tilde{\mathscr{A}}}^\beta \longrightarrow \tau^{-1}R^{n-1}\tau_*\tilde{\tilde{\mathscr{A}}}^\beta,$$
$$\tilde{\tilde{\mathscr{A}}}^\beta = \boldsymbol{R}j_*(\tilde{\tilde{\mathscr{O}}}|_{X-M})|_{iSM},$$

where $j\colon X - M \hookrightarrow {}^M\tilde{X}$ is the canonical injection, which implies that

$$R^{n-1}\tau_*\tilde{\tilde{\mathscr{A}}}^\beta = R^{n-1}(\tau\circ j)_*(\tilde{\tilde{\mathscr{O}}}|_{X-M}).$$

Hence we can define the canonical map

$$R^{n-1}\tau_*\tilde{\tilde{\mathscr{A}}}^\beta \longrightarrow \tilde{\tilde{\mathscr{B}}}.$$

It yields, together with (4.1), a homomorphism $\tilde{\tilde{\mathscr{A}}}^\beta \to \tau^{-1}\tilde{\tilde{\mathscr{B}}}$. Summing up, we have obtained the following.

Theorem 4.13. *We have the following diagram of exact sequences of sheaves on iSM:*

$$(4.2) \qquad \begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau^{-1}\tilde{\tilde{\mathscr{A}}} & \longrightarrow & \tilde{\tilde{\mathscr{A}}}^\beta & \longrightarrow & \tilde{\tilde{\mathscr{Q}}} & \longrightarrow & 0 \\
 & & \| & & \lambda\downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau^{-1}\tilde{\tilde{\mathscr{A}}} & \longrightarrow & \tau^{-1}\tilde{\tilde{\mathscr{B}}} & \longrightarrow & \tau^{-1}\pi_*\tilde{\tilde{\mathscr{C}}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \pi_*\tau^{-1}\tilde{\tilde{\mathscr{C}}} & = & \pi_*\tau^{-1}\tilde{\tilde{\mathscr{C}}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Proof. It has already been proved that the rows are exact. The right column is exact by Proposition 4.12. Hence it follows that the middle column is exact. (Q.E.D.)

Let us transform the diagram (4.2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'\pi'^{-1}$, where τ', π' are projections $IM \to iS^*M$ and $IM \to iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F} \cong \boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F} \cong \pi^{-1}\mathscr{F}[1-n].$$

By Proposition 1.7 of Ito [4],

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1}\tilde{\tilde{\mathscr{C}}} \cong \boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1}\tilde{\tilde{\mathscr{C}}} \cong \tilde{\tilde{\mathscr{C}}}[1-n].$$

By operating $\boldsymbol{R}\tau'_!\pi'^{-1}$ on exact columns in (4.2), we obtain

$$R^k\tau'_!\pi'^{-1}\tilde{\tilde{\mathscr{Q}}} = 0 \qquad \text{for } k \neq n-1,$$
$$R^k\tau'_!\pi'^{-1}\tilde{\tilde{\mathscr{A}}}^\beta = 0 \qquad \text{for } k \neq n-1.$$

We define the sheaves $\tilde{\tilde{\mathscr{A}}}^\vee$ and $\tilde{\tilde{\mathscr{Q}}}^\vee$ on iS^*M by

$$\tilde{\tilde{\mathscr{A}}}^\vee = R^{n-1}\tau'_!\pi'^{-1}\tilde{\tilde{\mathscr{A}}}^\beta,$$
$$\tilde{\tilde{\mathscr{Q}}}^\vee = R^{n-1}\tau'_!\pi'^{-1}\tilde{\tilde{\mathscr{Q}}}.$$

Then, in this way, we obtain the following theorem.

Theorem 4.14. *We have the diagram of exact sequences of sheaves on iS^*M:*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^{-1}\tilde{\tilde{\mathscr{A}}} & \longrightarrow & \tilde{\tilde{\mathscr{A}}}^\vee & \longrightarrow & \tilde{\tilde{\mathscr{Q}}}^\vee & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^{-1}\tilde{\tilde{\mathscr{A}}} & \longrightarrow & \pi^{-1}\tilde{\tilde{\mathscr{B}}} & \longrightarrow & \pi^{-1}\pi_*\tilde{\tilde{\mathscr{C}}} & \longrightarrow & 0 \\
 & & & & {\scriptstyle \mathrm{sp}}\downarrow & & \downarrow & & \\
 & & & & \tilde{\tilde{\mathscr{C}}} & = & \tilde{\tilde{\mathscr{C}}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & ,
\end{array} \tag{4.3}$$

and the diagram (4.2) *and the diagram* (4.3) *are mutually transformed by the functors $\boldsymbol{R}\tau'_!\pi'^{-1}[n-1]$ and $\boldsymbol{R}\pi_*\tau^{-1}$.*

We give a direct application of Theorem 4.13, which gives a relation between singularity spectrum and the domain of the defining function of a modified Fourier hyperfunction.

By using the similar notation to Proposition 2.16 of Ito [4], we can state the following proposition.

Proposition 4.15. *Let U be an open subset of iSM with convex fiber, and V a convex hull of U. Then we have*

(1) *If* $\varphi \in \Gamma(U, \tilde{\tilde{\mathscr{A}}}^{\beta})$, *then* $\mathrm{S.S.}(\lambda(\varphi)) \subset U^{\circ}$. *Conversely, if* $f(x) \in \Gamma(\tau U, \tilde{\tilde{\mathscr{B}}})$ *satisfies* $\mathrm{S.S.}(f) \subset U^{\circ}$, *then there exists a unique* $\varphi \in \Gamma(U, \tilde{\tilde{\mathscr{A}}}^{\beta})$ *such that* $f = \lambda(\varphi)$. *Namely, we have the exact sequence*

$$0 \longrightarrow \tilde{\tilde{\mathscr{A}}}^{\beta}(U) \xrightarrow{\lambda} \tilde{\tilde{\mathscr{B}}}(\tau U) \xrightarrow{\mathrm{sp}} \tilde{\tilde{\mathscr{C}}}(iS^*M - U^{\circ}).$$

(2) $\Gamma(V, \tilde{\tilde{\mathscr{A}}}^{\beta}) \to \Gamma(U, \tilde{\tilde{\mathscr{A}}}^{\beta})$ *is an isomorphism.*

Proof. Consider the exact sequence

$$0 \longrightarrow \tilde{\tilde{\mathscr{A}}}^{\beta} \longrightarrow \tau^{-1}\tilde{\tilde{\mathscr{B}}} \longrightarrow \pi_* \tau^{-1}\tilde{\tilde{\mathscr{C}}} \longrightarrow 0.$$

From this, we have the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \Gamma(V, \tilde{\tilde{\mathscr{A}}}^{\beta}) & \longrightarrow & \Gamma(V, \tau^{-1}\tilde{\tilde{\mathscr{B}}}) & \longrightarrow & \Gamma(V, \pi_*\tau^{-1}\tilde{\tilde{\mathscr{C}}}) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \Gamma(U, \tilde{\tilde{\mathscr{A}}}^{\beta}) & \longrightarrow & \Gamma(U, \tau^{-1}\tilde{\tilde{\mathscr{B}}}) & \longrightarrow & \Gamma(U, \pi_*\tau^{-1}\tilde{\tilde{\mathscr{C}}}) \end{array} \tag{4.4}$$

with exact rows. Since $V \to \tau V = \tau U$ and $U \to \tau U$ are open mappings with convex fibers, we have

$$\Gamma(V, \tau^{-1}\tilde{\tilde{\mathscr{B}}}) = \Gamma(U, \tau^{-1}\tilde{\tilde{\mathscr{B}}}) = \Gamma(\tau U, \tilde{\tilde{\mathscr{B}}}).$$

Since $\pi^{-1}V \to \tau\pi^{-1}V = iS^*M - V^{\circ} = iS^*M - U^{\circ}$ is an open mapping with connected fiber, we have

$$\begin{aligned} \Gamma(V, \pi_*\tau^{-1}\tilde{\tilde{\mathscr{C}}}) &= \Gamma(\pi^{-1}V, \tau^{-1}\tilde{\tilde{\mathscr{C}}}) \cong \Gamma(\tau\pi^{-1}V, \tilde{\tilde{\mathscr{C}}}) \\ &= \Gamma(iS^*M - U^{\circ}, \tilde{\tilde{\mathscr{C}}}). \end{aligned}$$

On the other hand,

$$\Gamma(iS^*M - U^{\circ}, \tilde{\tilde{\mathscr{C}}}) \longrightarrow \Gamma(\pi^{-1}U, \tau^{-1}\tilde{\tilde{\mathscr{C}}}^{a}) = \Gamma(U, \pi_*\tau^{-1}\tilde{\tilde{\mathscr{C}}}^{a})$$

is injective. Summing up, the middle arrow in the diagram (4.4) is an isomorphism and the right one is injective. Hence it follows that the left one is isomorphic. Moreover,

$$0 \longrightarrow \Gamma(U, \tilde{\tilde{\mathscr{A}}}^{\beta}) \longrightarrow \Gamma(\tau U, \tilde{\tilde{\mathscr{B}}}) \longrightarrow \Gamma(iS^*M - U^{\circ}, \tilde{\tilde{\mathscr{C}}})$$

is exact, which completes the proof. (Q.E.D.)

Definition 4.16. We say $u \in \tilde{\tilde{\mathscr{B}}}(\Omega)$ to be micro-analytic at $(x, i\eta\infty)$ in iS^*M if $(x, i\eta\infty) \notin \mathrm{S.S.}\,u$. This is equivalent to being represented as

$$u = \Sigma_j \lambda(\varphi_j), \quad \varphi_j \in \tilde{\tilde{\mathscr{A}}}^B(U_j), \quad (x, i\eta\infty) \notin U_j^\circ.$$

5. Vector-valued and mocified Fourier microfunctions

5.1. Vector-valued and modified Fourier hyperfunctions. In this section we recall the notion of vector-valued and modified Fourier hyperfunctions following Ito [2].

We use the similar notation to subsection 4.1. Let E be a Fréchet space over the complex number field.

Let ${}^E\tilde{\tilde{\mathscr{O}}}$ be the sheaf of E-valued, slowly increasing and holomorphic functions on X following Ito [2], Definition 2.1.1, p. 75, and put ${}^E\tilde{\tilde{\mathscr{A}}} = {}^E\tilde{\tilde{\mathscr{O}}}|_M$. Then ${}^E\tilde{\tilde{\mathscr{A}}}$ is the sheaf of E-valued, slowly increasing and real-analytic functions on M. Then we have ${}^E\tilde{\tilde{\mathscr{A}}} = \imath^{-1}\,{}^E\tilde{\tilde{\mathscr{O}}}$, where $\imath\colon M \hookrightarrow X$ is the canonical injection.

As in Ito [2], we define the sheaf of E-valued and modified Fourier hyperfunctions on M.

Definition 5.1. The sheaf ${}^E\tilde{\tilde{\mathscr{B}}}$ is, by definition,

$${}^E\tilde{\tilde{\mathscr{B}}} = \mathscr{H}^n_M({}^E\tilde{\tilde{\mathscr{O}}}) = \mathrm{Dist}^n(M, {}^E\tilde{\tilde{\mathscr{O}}}).$$

A section of ${}^E\tilde{\tilde{\mathscr{B}}}$ is called an E-valued and modified Fourier hyperfunction.

As stated in Ito [2], we have $\mathscr{H}^k_M({}^E\tilde{\tilde{\mathscr{O}}}) = 0$ for $k \neq n$ and ${}^E\tilde{\tilde{\mathscr{B}}}$ constitutes a flabby sheaf on M.

Now we apply Lemma 1.1 of Ito [4] to this case where $\mathscr{F}$, X and Y correspond to ${}^E\tilde{\tilde{\mathscr{O}}}$, X and M respectively. Then we obtain the sheaf homomorphism

$${}^E\tilde{\tilde{\mathscr{A}}} \longrightarrow {}^E\tilde{\tilde{\mathscr{B}}},$$

which will be proved to be injective later. This injection allows us to consider E-valued and modified Fourier hyperfunctions as a generalzation of E-valued, slowly increasing and real-analytic functions. The purpose of this chapter is to analyse the structure of the quotient sheaf ${}^E\tilde{\tilde{\mathscr{B}}}/{}^E\tilde{\tilde{\mathscr{A}}}$ by a similar way to S.K.K [22].

5.2. Definition of vector-valued and modified Fourier microfunctions. We use the similar notation to subsection 4.2. Let E be a Fréchet space over the complex number field. We denote by ${}^E\tilde{\tilde{\mathscr{O}}}$ the sheaf of E-valued, slowly increasing and holomorphic functions defined on X. We have the following.

Theorem 5.2. *We have $\mathscr{H}^k_{iSM}(\tau^{-1}\,{}^E\tilde{\tilde{\mathscr{O}}}) = 0$ for $k \neq 1$, where $\tau\colon {}^M\tilde{X} \to X$ is the canonical projection.*

Proof. It goes in a similar way to Theorem 4.2. (Q.E.D.)

The following theorem is the most essential one in the theory of E-valued and modified Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 5.3. *We have* $\mathscr{H}^k_{iS^*M}(\pi^{-1}{}^E\tilde{\tilde{\mathcal{O}}})=0$ *for* $k\neq n$, *where* $\pi\colon {}^M\tilde{X}^*\to X$ *is the canonical projection.*

Proof. It goes in a similar way to Theorem 4.3. (Q.E.D.)

In the above proof, the following theorem is essential.

Theorem 5.4 (the "Edge of the Wedge" Theorem). *Put* $G=\{z=x+iy\in X;\ y_j\geqq 0\ (1\leqq j\leqq n)\}^{c1}$. *Then we have, for each* $x\in M$,

$$\mathscr{H}^k_G({}^E\tilde{\tilde{\mathcal{O}}})_x=0 \qquad \textit{for}\ \ k\neq n.$$

Definition 5.5. We define the sheaf ${}^E\tilde{\tilde{\mathscr{C}}}$ in iS^*M by

$${}^E\tilde{\tilde{\mathscr{C}}}=\mathscr{H}^n_{iS^*M}(\pi^{-1}{}^E\tilde{\tilde{\mathcal{O}}})^a.$$

A section of ${}^E\tilde{\tilde{\mathscr{C}}}$ is called an E-valued and modified Fourier microfunctions.

Now we define the sheaves ${}^E\tilde{\tilde{\mathscr{Q}}}$, ${}^E\tilde{\tilde{\mathcal{O}}}{}^\beta$, ${}^E\tilde{\tilde{\mathscr{A}}}{}^\beta$ by

$$\begin{aligned}
{}^E\tilde{\tilde{\mathscr{Q}}}&=\mathscr{H}^1_{iSM}(\tau^{-1}{}^E\tilde{\tilde{\mathcal{O}}}),\\
{}^E\tilde{\tilde{\mathcal{O}}}{}^\beta&=j_*({}^E\tilde{\tilde{\mathcal{O}}}|_{X-M}),\\
{}^E\tilde{\tilde{\mathscr{A}}}{}^\beta&={}^E\tilde{\tilde{\mathcal{O}}}{}^\beta|_{iSM},
\end{aligned}$$

where $j\colon X-M\hookrightarrow {}^M\tilde{X}$, $\pi\colon {}^M\tilde{X}^*\to X$ and $\tau\colon {}^M\tilde{X}\to X$ are canonical maps.

By proposition 1.3 of Ito [4] and Theorems 5.2 and 5.3, we have the following.

Proposition 5.6. *We have*

$$R^k\tau_*\pi^{-1}{}^E\tilde{\tilde{\mathscr{Q}}}=\begin{cases}{}^E\tilde{\tilde{\mathscr{C}}}{}^a, & (\textit{for}\ \ k=n-1),\\ 0, & (\textit{for}\ \ k\neq n-1).\end{cases}$$

Theorem 5.7. *We have*

$$R^k\pi_*{}^E\tilde{\tilde{\mathscr{C}}}=R^{k+n-1}\tau_*{}^E\tilde{\tilde{\mathscr{Q}}}=0\ \textit{for}\ \ k\neq 0,$$

and we have the exact sequence

$$0\longrightarrow {}^E\tilde{\tilde{\mathscr{A}}}\longrightarrow {}^E\tilde{\tilde{\mathscr{B}}}\longrightarrow \pi_*{}^E\tilde{\tilde{\mathscr{C}}}\longrightarrow 0.$$

Proof. It goes in a similar way to Theorem 4.7. (Q.E.D.)

This is the required decomposition of singularity of E-valued and modified Fourier hyperfunctions.

Corollary 5.8. *We have the exact sequence*

$$0 \longrightarrow \tilde{\tilde{\mathscr{A}}}(M; E) \xrightarrow{\lambda} \tilde{\tilde{\mathscr{B}}}(M; E) \xrightarrow{\mathrm{sp}} \tilde{\tilde{\mathscr{C}}}(iS^*M; E) \longrightarrow 0.$$

Definition 5.9. Let $u \in \tilde{\tilde{\mathscr{B}}}(M; E)$. We call $\mathrm{sp}(u) \in \tilde{\tilde{\mathscr{C}}}(iS^*M; E)$ a spectrum of u. We denote by $\mathrm{S.S.}\,u$ the support $\mathrm{supp\,sp}(u)$ of $\mathrm{sp}(u)$ and call it a singularity spectrum of u. $\pi(\mathrm{S.S.}\,u)$ is evidently the subset where u is not slowly increasing nor real-analytic and is called the singular support of u.

Corollary 5.10. *Let $u \in \tilde{\tilde{\mathscr{B}}}(M; E)$. Then u is an E-valued, slowly increasing and real-analytic function on M if and only if* $\mathrm{S.S.}\,u = \emptyset$.

We have ${}^E\mathscr{A} = {}^E\tilde{\tilde{\mathscr{A}}}|_{R^n}$, ${}^E\mathscr{B} = {}^E\tilde{\tilde{\mathscr{B}}}|_{R^n}$ and ${}^E\mathscr{C} = {}^E\tilde{\tilde{\mathscr{C}}}|_{iS^*R^n}$ in the notation of Corollary 3.11 of Ito [4]. Then we have the following Corollary by restricting the exact sequence in Theorem 5.7.

Corollary 5.11. *Let $\pi: iS^*R^n \to R^n$ be the canonical projection. Then we have the exact sequence*

$$0 \longrightarrow {}^E\mathscr{A} \longrightarrow {}^E\mathscr{B} \longrightarrow \pi_* {}^E\mathscr{C} \longrightarrow 0.$$

5.3. Fundamental diagram on ${}^E\tilde{\tilde{\mathscr{C}}}$. We apply the arguments in the subsection 1.2 of Ito [4] to a special case. At first we apply Proposition 1.10 of Ito [4] to the situation $\mathscr{F} = {}^E\tilde{\tilde{\mathscr{Q}}}{}^a$, $X = M$, $S = iSM$. Then $\mathscr{G} = {}^E\tilde{\tilde{\mathscr{C}}}$, $\mathscr{E} = \pi_* {}^E\tilde{\tilde{\mathscr{C}}}$. We obtain the following.

Proposition 5.12. *We have*

$$R^k\pi_*\tau^{-1\,E}\tilde{\tilde{\mathscr{C}}} = 0 \qquad \textit{for}\ \ k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow {}^E\tilde{\tilde{\mathscr{Q}}} \longrightarrow \tau^{-1}\pi_* {}^E\tilde{\tilde{\mathscr{C}}} \longrightarrow \pi_*\tau^{-1\,E}\tilde{\tilde{\mathscr{C}}} \longrightarrow 0.$$

Now we apply the same proposition to the case where $\mathscr{F} = {}^E\tilde{\tilde{\mathscr{A}}}{}^\beta$. Thus we obtain a homomorphism

$$\text{(5.1)} \qquad {}^E\tilde{\tilde{\mathscr{A}}}{}^\beta \longrightarrow \tau^{-1}R^{n-1}\tau_* {}^E\tilde{\tilde{\mathscr{A}}}{}^\beta,$$

$$ {}^E\tilde{\tilde{\mathscr{A}}}{}^\beta = \boldsymbol{R}j_*({}^E\tilde{\tilde{\mathscr{O}}}|_{X-M})|_{iSM},$$

where $j: X - M \hookrightarrow {}^M\tilde{X}$ is the canonical injection, which implies that

$$R^{n-1}\tau_* {}^E\tilde{\tilde{\mathscr{A}}}{}^\beta = R^{n-1}(\tau \circ j)_*({}^E\tilde{\tilde{\mathscr{O}}}|_{X-M}).$$

Hence we can define the canonical map

$$R^{n-1}\tau_* {}^E\tilde{\tilde{\mathscr{A}}}{}^\beta \longrightarrow {}^E\tilde{\tilde{\mathscr{B}}}.$$

It yields, together with (5.1), a homomorphism ${}^E\tilde{\tilde{\mathscr{A}}}{}^\beta \to \tau^{-1\,E}\tilde{\tilde{\mathscr{B}}}$. Summing up, we have obtained the following.

Theorem 5.13. *We have the following diagram of exact sequences of sheaves on iSM:*

$$(5.2)\qquad \begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tau^{-1}{}^{E}\tilde{\tilde{\mathscr{A}}} & \longrightarrow & {}^{E}\tilde{\tilde{\mathscr{A}}}{}^{\beta} & \longrightarrow & {}^{E}\tilde{\tilde{\mathscr{Q}}} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\lambda} & & \downarrow & & \\ 0 & \longrightarrow & \tau^{-1}{}^{E}\tilde{\tilde{\mathscr{A}}} & \longrightarrow & \tau^{-1}{}^{E}\tilde{\tilde{\mathscr{B}}} & \longrightarrow & \tau^{-1}\pi_{*}{}^{E}\tilde{\tilde{\mathscr{C}}} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \pi_{*}\tau^{-1}{}^{E}\tilde{\tilde{\mathscr{C}}} & = & \pi_{*}\tau^{-1}{}^{E}\tilde{\tilde{\mathscr{C}}} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Proof. It has already been proved that the rows are exact. The right column is exact by Proposition 5.12. Hence it follows that the middle column is exact. (Q.E.D.)

Let us transform the diagram (5.2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'_!\pi'^{-1}$, where τ', π' are projections $IM \to iS^*M$ and $IM \to iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F} \cong \boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F} \cong \pi^{-1}\mathscr{F}[1-n].$$

By Proposition 1.7 of Ito [4],

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1}{}^{E}\tilde{\tilde{\mathscr{C}}} \cong \boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1}{}^{E}\tilde{\tilde{\mathscr{C}}} \cong {}^{E}\tilde{\tilde{\mathscr{C}}}[1-n].$$

By operating $R\tau'_!\pi'^{-1}$ on exact columns in (5.2), we obtain

$$R^k\tau'_!\pi'^{-1}{}^{E}\tilde{\tilde{\mathscr{Q}}} = 0 \qquad \text{for } k \neq n-1,$$
$$R^k\tau'_!\pi'^{-1}{}^{E}\tilde{\tilde{\mathscr{A}}}{}^{\beta} = 0 \qquad \text{for } k \neq n-1.$$

We define the sheaves ${}^{E}\tilde{\tilde{\mathscr{A}}}{}^{\vee}$ and ${}^{E}\tilde{\tilde{\mathscr{Q}}}{}^{\vee}$ on iS^*M by

$${}^{E}\tilde{\tilde{\mathscr{A}}}{}^{\vee} = R^{n-1}\tau'_!\pi'^{-1}{}^{E}\tilde{\tilde{\mathscr{A}}}{}^{\beta},$$
$${}^{E}\tilde{\tilde{\mathscr{Q}}}{}^{\vee} = R^{n-1}\tau'_!\pi'^{-1}{}^{E}\tilde{\tilde{\mathscr{Q}}}.$$

Then, in this way, we obtain the following theorem.

Theorem 5.14. *We have the diagram of exact sequences of sheaves on iS^*M:*

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi^{-1}\,{}^{E}\tilde{\tilde{\mathscr{A}}} \arrow[r] \arrow[d, equal] & {}^{E}\tilde{\tilde{\mathscr{A}}}^{\vee} \arrow[r] \arrow[d] & {}^{E}\tilde{\tilde{\mathscr{Q}}}^{\vee} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^{-1}\,{}^{E}\tilde{\tilde{\mathscr{A}}} \arrow[r] & \pi^{-1}\,{}^{E}\tilde{\tilde{\mathscr{B}}} \arrow[r] \arrow[d, "\mathrm{sp}"'] & \pi^{-1}\pi_*\,{}^{E}\tilde{\tilde{\mathscr{C}}} \arrow[r] \arrow[d] & 0 \\
 & & {}^{E}\tilde{\tilde{\mathscr{C}}} \arrow[r, equal] \arrow[d] & {}^{E}\tilde{\tilde{\mathscr{C}}} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{5.3}
$$

and the diagram (5.2) *and the diagram* (5.3) *are mutually transformed by the functors $\boldsymbol{R}\tau_!\pi'^{-1}[n-1]$ and $\boldsymbol{R}\pi_*\tau^{-1}$.*

We give a direct application of Theorem 5.13, which gives a relation between singularity spectrum and the domain of the defining function of an E-valued and modified Fourier hyperfunction.

By using the similar notation to Proposition 2.16 of Ito [4], we can state the following proposition.

Proposition 5.15. *Let U be an open subset of iSM with convex fiber, and V a convex hull of U. Then we have*

(1) *If $\varphi \in \Gamma(U, {}^{E}\tilde{\tilde{\mathscr{A}}}^{\beta})$, then* $\mathrm{S.S.}(\lambda(\varphi)) \subset U^{\circ}$. *Conversely, if $f(x) \in \Gamma(\tau U, {}^{E}\tilde{\tilde{\mathscr{B}}})$ such that $f = \lambda(\varphi)$. Namely, we have the exact sequence*

$$0 \longrightarrow \tilde{\tilde{\mathscr{A}}}^{\beta}(U; E) \xrightarrow{\lambda} \tilde{\tilde{\mathscr{B}}}(\tau U; E) \xrightarrow{\mathrm{sp}} \tilde{\tilde{\mathscr{C}}}(iS^*M - U^{\circ}; E).$$

(2) $\Gamma(V, {}^{E}\tilde{\tilde{\mathscr{A}}}^{\beta}) \to \Gamma(U, {}^{E}\tilde{\tilde{\mathscr{A}}}^{\beta})$ *is an isomorphism.*

Proof. It goes in a similar way to Proposition 2.16 of Ito [4]. (Q.E.D.)

Definition 5.16. We say $u \in \tilde{\tilde{\mathscr{B}}}(\Omega; E)$ to be micro-analytic at $(x, i\eta\infty)$ in iS^*M if $(x, i\eta\infty) \notin \mathrm{S.S.}\,u$. This is equivalent to being represented as

$$u = \Sigma_j \lambda(\varphi_j), \quad \varphi_j \in \tilde{\tilde{\mathscr{A}}}^{\beta}(U_j; E), \quad (x, i\eta\infty) \notin U_j^{\circ}.$$

6. Mixed Fourier microfunctions

6.1. Mixed Fourier hyperfunctions. In this subsection we recall the notion of mixed Fourier hyperfunctions following Ito [2].

Let $n = (n_1, n_2)$ be a pair of nonnegative integers with $|n| = n_1 + n_2 \neq 0$. We denote the product spaces $\tilde{C}^{n_1} \times \tilde{\tilde{C}}^{n_2}$ and $\tilde{R}^{n_1} \times \tilde{\tilde{R}}^{n_2}$ by $C^{\#,n}$ and $R^{\#,n}$ respectively. Also put $C^{|n|} = C^{n_1} \times C^{n_2}$, $X = C^{\#,n}$ and $M = R^{\#,n}$. Then M is the closure of $R^{|n|} = R^{n_1} \times R^{n_2}$ in X. We denote $z = (z', z'') \in C^{|n|}$ so that $z' = (z_1, \dots, z_{n_1})$, $z'' = (z_{n_1+1}, \dots, z_{|n|})$.

Let $\tilde{\tilde{\mathcal{O}}}{}^{\#}$ be the sheaf of slowly increasing and holomorphic functions on X following Ito [2], Definition 5.1.1, p. 25. Put $\mathscr{A}^{\#} = \mathcal{O}^{\#}|_M$. Then $\mathscr{A}^{\#}$ is the sheaf of slowly increasing and real-analytic functions on M. Then we have $\mathscr{A}^{\#} = \iota^{-1}\mathcal{O}^{\#}$, where $\iota\colon M \hookrightarrow X$ is the canonical injection.

As in Ito[2], we define the sheaf of mixed Fourier hyperfunctions on M.

Definition 6.1. The sheaf $\mathscr{B}^{\#}$ is, by definition,

$$\mathscr{B}^{\#} = \mathscr{H}^n_M(\mathcal{O}^{\#}) = \mathrm{Dist}^{|n|}(M, \mathcal{O}^{\#}),$$

where the symbol in the right hand side of the above equality is due to Sato [17], p. 405. A section of $\mathscr{B}^{\#}$ is called a mixed Fourier hyperfunction.

As stated in Ito [2], $\mathscr{H}^k_M(\mathcal{O}^{\#}) = 0$ for $k \neq |n|$ and $\mathscr{B}^{\#}$ constitutes a flabby sheaf on M.

Now we apply Lemma 1.1 of Ito [4] to this case where $\mathscr{F}$, X and Y correspond to $\mathcal{O}^{\#}$, X and M respectively. Then we obtain the sheaf homomorphism

$$\mathscr{A}^{\#} \longrightarrow \mathscr{B}^{\#},$$

which will be proved to be injective later. This injection allows us to consider mixed Fourier hyperfunctions as a generalization of slowly increasing and real-analytic functions. The purpose of this chapter is to analyse the structure of the quotient sheaf $\mathscr{B}^{\#}/\mathscr{A}^{\#}$ by a similar way to S.K.K [22].

6.2. Definition of mixed Fourier microfunctions. Suppose $M = R^{\#,n}$ and $X = C^{\#,n}$, where $n = (n_1, n_2)$ is a pair of nonnegative integers with $|n| = n_1 + n_2 \neq 0$. We denote by $\mathcal{O}^{\#}$ the sheaf of slowly increasing and holomorphic functions defined on X. The (co-) sphere bundle iSM (resp. iS^*M) are defined similarly to subsection 2.2 of Ito [4]. We also use a similar notation to subsection 2.2 of Ito [4]. Then we have the following diagram:

$$\begin{tikzcd}
 & {}^M\tilde{X}^+ \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & DM \arrow[l, hook'] \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & \\
{}^M\tilde{X} \arrow[dr, "\tau"'] & iSM \arrow[l, hook'] \arrow[dr, "\tau"] & {}^M\tilde{X}^* \arrow[dl, "\pi"'] & iS^*M \arrow[l, hook'] \arrow[dl, "\pi"] \\
 & X & M \arrow[l, hook'] &
\end{tikzcd}$$

Theorem 6.2. *We have* $\mathscr{H}^k_{iSM}(\tau^{-1}\mathcal{O}^{\#}) = 0$ *for* $k \neq 1$, *where* $\tau\colon {}^M\tilde{X} \to X$ *is the canonical projection.*

The following theorem is the most essential one in the theory of mixed Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 6.3. *We have* $\mathscr{H}^k_{iS^*M}(\pi^{-1}\mathcal{O}^{\#}) = 0$ *for* $k \neq |n|$, *where* $\pi\colon {}^M\tilde{X}^* \to X$ *is the canonical projection.*

In the proof of the above theorem, the following theorem is essential.

Theorem 6.4 (the "Edge of the Wedge" Theorem). *Put* $G = \{z = x + iy \in \boldsymbol{C}^{|n|};\ y_j \geqq 0\ (1 \leqq j \leqq |n|)\}^{c1}$. *Then we have, for each* $x \in M$,

$$\mathscr{H}^k_G(\mathcal{O}^{\#})_x = 0 \qquad \text{for } k \neq |n|.$$

Definition 6.5. We define the sheaf $\mathscr{C}^{\#}$ on iS^*M by

$$\mathscr{C}^{\#} = \mathscr{H}^{|n|}_{iS^*M}(\pi^{-1}\mathcal{O}^{\#})^a,$$

where we denote by a the antipodal map $iS^*M \to iS^*M$, and by $\mathscr{F}^a$ the inverse image under a of a sheaf $\mathscr{F}$ on iS^*M. A section of $\mathscr{C}^{\#}$ is called a mixed Fourier microfunction.

Now we define the sheaves $\mathscr{Q}^{\#}$, $\mathcal{O}^{\#,\beta}$ and $\mathscr{A}^{\#,\beta}$ by

$$\mathscr{Q}^{\#} = \mathscr{H}^1_{iSM}(\tau^{-1}\mathcal{O}^{\#}),$$
$$\mathcal{O}^{\#,\beta} = j_*(\mathcal{O}^{\#}|_{X-M}),$$
$$\mathscr{A}^{\#,\beta} = \mathcal{O}^{\#,\beta}|_{iSM},$$

where $j\colon X - M \hookrightarrow {}^M\tilde{X}$, $\pi\colon {}^M\tilde{X}^* \to X$ and $\tau\colon {}^M\tilde{X} \to X$ are canonical maps.

By Proposition 1.3 of Ito [4] and Theorems 6.2 and 6.3, we have the following.

Proposition 6.6. *We have*

$$R^k\tau_*\pi^{-1}\mathscr{Q}^{\#} = \begin{cases} (\mathscr{C}^{\#})^a & (k = |n| - 1) \\ 0 & (k \neq |n| - 1) \end{cases}.$$

Theorem 6.7. *We have*

$$R^k\pi_*\mathscr{C}^{\#} = R^{k+|n|-1}\tau_*\mathscr{Q}^{\#} = 0 \qquad \text{for } k \neq 0,$$

and we have the exact sequence

$$0 \longrightarrow \mathscr{A}^{\#} \longrightarrow \mathscr{B}^{\#} \longrightarrow \pi_*\mathscr{C}^{\#} \longrightarrow 0.$$

This is the required decomposition of singularity of mixed Fourier hyperfunctions.

Corollary 6.8. *We have the exact sequence*

$$0 \longrightarrow \mathscr{A}^{\#}(M) \xrightarrow{\lambda} \mathscr{B}^{\#}(M) \xrightarrow{\mathrm{sp}} \mathscr{C}^{\#}(iS^*M) \longrightarrow 0.$$

Definition 6.9. Let $u \in \mathscr{B}^{\#}(M)$. We call $\mathrm{sp}(u) \in \mathscr{C}^{\#}(iS^*M)$ a spectrum of u. We denote by S.S.u the support $\mathrm{supp}\,\mathrm{sp}(u)$ of $\mathrm{sp}(u)$ and call it a singularity spectrum of u. $\pi(\text{S.S.}u)$ is evidently the subset where u is not slowly increasing nor real-analytic and is called the singular support of u.

Corollary 6.10. *Let $u \in \mathscr{B}^{\#}(M)$. Then u is a slowly increasing and real-analytic function on M if and only if* S.S.$u = \emptyset$.

Since $\mathscr{A} = \mathscr{A}^{\#}|_{\boldsymbol{R}^n}$, $\mathscr{B} = \mathscr{B}^{\#}|_{\boldsymbol{R}^n}$ and $\mathscr{C} = \mathscr{C}^{\#}|_{iS^*\boldsymbol{R}^n}$ hold in the notation of S.K.K [22], we have the following Corollary by restricting the exact sequence in Theorem 6.7.

Corollary 6.11. *Let $\pi: iS^*\boldsymbol{R}^n \to \boldsymbol{R}^n$ be the canonical projection. Then we have the exact sequence*

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \pi_*\mathscr{C} \longrightarrow 0.$$

6.3. Fundamental diagram on $\mathscr{C}^{\#}$. We apply the arguments in the subsection 1.2 of Ito [4] to a special case. At first we apply Proposition 1.10 of Ito [4] to the situation $\mathscr{F} = (\mathscr{Q}^{\#})^a$, $X = M$, $S = iSM$. Then $\mathscr{G} = \mathscr{C}^{\#}$, $\mathscr{E} = \pi_*\mathscr{C}^{\#}$. We obtain the following.

Proposition 6.12. *We have*

$$R^k\pi_*\tau^{-1}\mathscr{C}^{\#} = 0 \qquad \text{for } k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow \mathscr{Q}^{\#} \longrightarrow \tau^{-1}\pi_*\mathscr{C}^{\#} \longrightarrow \pi_*\tau^{-1}\mathscr{C}^{\#} \longrightarrow 0.$$

Now we apply the same proposition to the case where $\mathscr{F} = \mathscr{A}^{\#,\beta}$. Thus we obtain a homomorphism

$$\mathscr{A}^{\#,\beta} \longrightarrow \tau^{-1}R^{n-1}\tau_*\mathscr{A}^{\#,\beta}, \tag{6.1}$$

$$\mathscr{A}^{\#,\beta} = \boldsymbol{R}j_*(\mathscr{O}^{\#}|_{X-M})|_{iSM},$$

where $j: X - M \hookrightarrow {}^M\tilde{X}$ is the canonical injection, which implies that

$$R^{|n|-1}\tau_*\mathscr{A}^{\#,\beta} = R^{|n|-1}(\tau \circ j)_*(\mathscr{O}^{\#}|_{X-M}).$$

Hence we can define the canonical map

$$R^{|n|-1}\tau_*\mathscr{A}^{\#,\beta} \longrightarrow \mathscr{B}^{\#}.$$

It yields, together with (6.1), a homomorphism $\mathscr{A}^{\#,\beta} \to \tau^{-1}\mathscr{B}^{\#}$. Summing up, we have obtained the following.

Theorem 6.13. *We have the following diagram of exact sequences of sheaves on iSM:*

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tau^{-1}\mathscr{A}^{\#} \arrow[r] \arrow[d, equal] & \mathscr{A}^{\#,\beta} \arrow[r] \arrow[d, "\lambda"'] & \mathscr{Q}^{\#} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tau^{-1}\mathscr{A}^{\#} \arrow[r] & \tau^{-1}\mathscr{B}^{\#} \arrow[r] \arrow[d] & \tau^{-1}\pi_*\mathscr{C}^{\#} \arrow[r] \arrow[d] & 0 \\
 & & \pi_*\tau^{-1}\mathscr{C}^{\#} \arrow[r, equal] \arrow[d] & \pi_*\tau^{-1}\mathscr{C}^{\#} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{6.2}
$$

Let us transform the diagram (6.2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'_!\pi'^{-1}$, where τ', π' are projections $IM \to iS^*M$ and $IM \to iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F} \cong \boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F} \cong \pi^{-1}\mathscr{F}[1-|n|].$$

By Proposition 1.7 of Ito [4],

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1}\mathscr{C}^{\#} \cong \boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1}\mathscr{C}^{\#} \cong \mathscr{C}^{\#}[1-|n|].$$

By operating $R\tau'_!\pi'^{-1}$ on exact columns in (6.2), we obtain

$$R^k\tau'_!\pi'^{-1}\mathscr{Q}^{\#} = 0 \qquad \text{for } k \neq |n|-1,$$

$$R^k\tau'_!\pi'^{-1}\mathscr{A}^{\#,\beta} = 0 \qquad \text{for } k \neq |n|-1.$$

We define the sheaves $\mathscr{A}^{\#,\vee}$ and $\mathscr{Q}^{\#,\vee}$ on iS^*M by

$$\mathscr{A}^{\#,\vee} = R^{|n|-1}\tau'_!\pi'^{-1}\mathscr{A}^{\#,\beta},$$

$$\mathscr{Q}^{\#,\vee} = R^{|n|-1}\tau'_!\pi'^{-1}\mathscr{Q}^{\#}.$$

Then, in this way, we obtain the following theorem.

Theorem 6.14. *We have the diagram of exact sequences of sheaves on iS^*M*

$$(6.3)\qquad \begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi^{-1}\mathscr{A}^{\#} & \longrightarrow & \mathscr{A}^{\#,\vee} & \longrightarrow & \mathscr{Q}^{\#,\vee} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi^{-1}\mathscr{A}^{\#} & \longrightarrow & \pi^{-1}\mathscr{B}^{\#} & \longrightarrow & \pi^{-1}\pi_{*}\mathscr{C}^{\#} & \longrightarrow & 0 \\ & & & & {\scriptstyle sp}\downarrow & & \downarrow & & \\ & & & & \mathscr{C}^{\#} & = & \mathscr{C}^{\#} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & . \end{array}$$

and the diagram (6.2) *and the diagram* (6.3) *are mutually transformed by the functors* $R\tau_!'\pi'^{-1}[|n|-1]$ *and* $R\pi_*\tau^{-1}$.

We give a direct application of Theorem 6.13, which gives a relation between singularity spectrum and the domain of the defining function of a mixed Fourier hyperfunction.

By using the similar notation to Proposition 2.16 of Ito [4], we can state the following proposition.

Proposition 6.15. *Let U be an open subset of iSM with convex fiber, and V a convex hull of U. Then we have*

(1) *If* $\varphi\in\Gamma(U,\mathscr{A}^{\#,\beta})$, *then* S.S. $(\lambda(\varphi))\subset U^\circ$. *Conversely, if* $f(x)\in\Gamma(\tau U,\mathscr{B}^{\#})$ *satisfies* S.S. $(f)\subset U^\circ$, *then there exists a unique* $\varphi\in\Gamma(U,\mathscr{A}^{\#,\beta})$ *such that* $f=\lambda(\varphi)$. *Namely, we have the exact sequence*

$$0\longrightarrow\mathscr{A}^{\#,\beta}(U)\xrightarrow{\ \lambda\ }\mathscr{B}^{\#}(\tau U)\xrightarrow{\ sp\ }\mathscr{C}^{\#}(iS^*M-U^\circ).$$

(2) $\Gamma(V,\mathscr{A}^{\#,\beta})\to\Gamma(U,\mathscr{A}^{\#,\beta})$ *is an isomorphism.*

Definition 6.16. We say $u\in\mathscr{B}^{\#}(\Omega)$ to be micro-analytic at $(x, i\eta\infty)$ in iS^*M if $(x, i\eta\infty)\notin$ S.S. u. This is equivalent to being represented as

$$u=\sum_j\lambda(\varphi_j),\quad \varphi_j\in\mathscr{A}^{\#,\beta}(U_j),\quad (x, i\eta\infty)\notin U_j^\circ.$$

7. Vector-valued and mixed Fourier microfunctions

7.1. Vector-valued and mixed Fourier hyperfunctions. In this subsection we recall the notion of vector-valued and mixed Fourier hyperfunctions following Ito [2], which is somewhat different from the one defined by Nagamachi [15].

We use a similar notation to subsection 6.1. Let E be a Fréchet space over the complex number field.

Let ${}^E\mathcal{O}^\#$ be the sheaf of E-valued, slowly increasing and holomorphic functions on X following Ito [2], Definition 6.1.1, p. 37, and put ${}^E\mathcal{A}^\# = {}^E\mathcal{O}^\#|_M$. Then ${}^E\mathcal{A}^\#$ is the sheaf of E-valued, slowly increasing and real-analytic functions on M. Then we have ${}^E\mathcal{A}^\# = \iota^{-1}\,{}^E\mathcal{O}^\#$, where $\iota\colon M \hookrightarrow X$ is the canonical injection.

As in Ito [2], we define the sheaf of E-valued and mixed Fourier hyperfunctions on M:

Definition 7.1. The sheaf ${}^E\mathcal{B}^\#$ is, by definition,

$$ {}^E\mathcal{B}^\# = \mathcal{H}_M^{|n|}({}^E\mathcal{O}^\#) = \mathrm{Dist}^{|n|}(M, {}^E\mathcal{O}^\#). $$

A section of ${}^E\mathcal{B}^\#$ is called an E-valued and mixed Fourier hyperfunction.

As stated in Ito [2], $\mathcal{H}_M^k({}^E\mathcal{O}^\#) = 0$ for $k \neq |n|$ and ${}^E\mathcal{B}^\#$ constitutes a flabby sheaf on M.

Now we apply Lemma 1.1 of Ito [4] to this case where $\mathcal{F}$, X and Y correspond to ${}^E\mathcal{O}^\#$, X and M respectively. Then we obtain the sheaf homomorphism

$$ {}^E\mathcal{A}^\# \longrightarrow {}^E\mathcal{B}^\#, $$

which will be proved to be injective later. This injection allows us to consider E-valued and mixed Fourier hyperfunctions as a generalization of E-valued, slowly increasing and real-analytic functions. The purpose of this chapter is to analyse the structure of the quotient sheaf ${}^E\mathcal{B}^\#/{}^E\mathcal{A}^\#$ by a similar way to S.K.K [22].

7.2. Definition of vector-valued and mixed Fourier microfunctions. We use a similar notation to subsection 6.2. Let E be a Fréchet space over the complex number field. We denote by ${}^E\mathcal{O}^\#$ the sheaf of E-valued, slowly increasing and holomorphic functions defined on X. We have the following.

Theorem 7.2. *We have $\mathcal{H}_{iSM}^k(\tau^{-1}\,{}^E\mathcal{O}^\#) = 0$ for $k \neq 1$, where $\tau\colon {}^M\tilde{X} \to X$ is the canonical projection.*

The following theorem is the most essential one in the theory of E-valued and mixed Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 7.3. *We have $\mathcal{H}_{iS^*M}^k(\pi^{-1}\,{}^E\mathcal{O}^\#) = 0$ for $k \neq |n|$, where $\pi\colon {}^M\tilde{X}^* \to X$ is the canonical projection,*

In the proof of the above theorem, the following theorem is essential.

Theorem 7.4 (the "Edge of the Wedge" Theorem). *Put $G = \{z = x + iy \in \mathbf{C}^{|n|};\ y_j \geq 0\ (1 \leq j \leq |n|)\}^{\mathrm{cl}}$. Then we have, for each $x \in M$,*

$$\mathscr{H}_G^k({}^E\mathcal{O}^\#)_x = 0 \qquad \textit{for} \ \ k \neq |n|.$$

Definition 7.5. We define the sheaf ${}^E\mathscr{C}^\#$ in iS^*M by

$$^E\mathscr{C}^\# = \mathscr{H}^{|n|}_{iS^*M}(\pi^{-1}\,{}^E\mathcal{O}^\#)^a.$$

A section of ${}^E\mathscr{C}^\#$ is called an E-valued and mixed Fourier microfunctions.

Now we define the sheaves ${}^E\mathscr{Q}^\#$, ${}^E\mathcal{O}^{\#,\beta}$, ${}^E\mathscr{A}^{\#,\beta}$ by

$$\begin{aligned} {}^E\mathscr{Q}^\# &= \mathscr{H}^1_{iSM}(\tau^{-1}\,{}^E\mathcal{O}^\#), \\ {}^E\mathcal{O}^{\#,\beta} &= j_*({}^E\mathcal{O}^\#|_{X-M}), \\ {}^E\mathscr{A}^{\#,\beta} &= {}^E\mathcal{O}^{\#,\beta}|_{iSM}, \end{aligned}$$

where $j\colon X - M \hookrightarrow {}^M\widetilde{X}^* \to X$ and $\tau\colon {}^M\widetilde{X} \to X$ are canonical maps.

By proposition 1.3 of Ito [4] and Theorems 7.2 and 7.3, we have the following.

Proposition 7.6. *We have*

$$R^k\tau_*\pi^{-1}\,{}^E\mathscr{Q}^\# = \begin{cases} ({}^E\mathscr{C}^\#)^a, & (\textit{for} \ \ k = |n|-1), \\ 0, & (\textit{for} \ \ k \neq |n|-1). \end{cases}$$

Theorem 7.7. *We have*

$$R^k\pi_*\,{}^E\mathscr{C}^\# = R^{k+|n|-1}\tau_*\,{}^E\mathscr{Q}^\# = 0 \qquad \textit{for} \ \ k \neq 0,$$

and we have the exact sequence

$$0 \longrightarrow {}^E\mathscr{A}^\# \longrightarrow {}^E\mathscr{B}^\# \longrightarrow \pi_*\,{}^E\mathscr{C}^\# \longrightarrow 0.$$

This is the required decomposition of singularity of E-valued and mixed Fourier hyperfunctions.

Corollary 7.8. *We have the exact sequence*

$$0 \longrightarrow \mathscr{A}^\#(M; E) \xrightarrow{\ \lambda\ } \mathscr{B}^\#(M; E) \xrightarrow{\ \mathrm{sp}\ } \mathscr{C}^\#(iS^*M; E) \longrightarrow 0.$$

Definition 7.9. Let $u \in \mathscr{B}^\#(M; E)$. We call $\mathrm{sp}(u) \in \mathscr{C}^\#$ $(iS^*M; E)$ a spectrum of u. We denote by $\mathrm{S.S.}\,u$ the support $\mathrm{supp\,sp}(u)$ of $\mathrm{sp}(u)$ and call it a singularity spectrum of u. $\pi(\mathrm{S.S.}\,u)$ is evidently the subset where u is not slowly increasing nor real-analytic and it is called the singular support of u.

Corollary 7.10. *Let $u \in \mathscr{B}^\#(M; E)$. Then u is an E-valued, slowly increasing and real-analytic function on M if and only if* $\mathrm{S.S.}\,u = \emptyset$.

Put ${}^E\mathscr{A} = {}^E\mathscr{A}^\#|_{\boldsymbol{R}^n}$, ${}^E\mathscr{B} = {}^E\mathscr{B}^\#|_{\boldsymbol{R}^n}$ and ${}^E\mathscr{C} = {}^E\mathscr{C}^\#|_{iS^*\boldsymbol{R}^n}$. Then we have the following Corollary by restricting the exact sequence in Theorem 7.7.

Corollary 7.11. *Let* $\pi\colon iS^*\boldsymbol{R}^n \to \boldsymbol{R}^n$. *Then we have the exact sequence*

$$0 \longrightarrow {}^E\mathscr{A} \longrightarrow {}^E\mathscr{B} \longrightarrow \pi_* {}^E\mathscr{C} \longrightarrow 0.$$

7.3. Fundamental diagram on ${}^E\mathscr{C}^\#$**.** We apply the arguments in the subsection 1.2 of Ito [4] to a special case. At first we apply Proposition 1.10 of Ito [4] to the situation $\mathscr{F} = ({}^E\mathscr{Q}^\#)^a$, $X = M$, $S = iSM$. Then $\mathscr{G} = {}^E\mathscr{C}^\#$, $\mathscr{E} = \pi_* {}^E\mathscr{C}^\#$. We obtain the following.

Proposition 7.12. *We have*

$$R^k \pi_* \tau^{-1} {}^E\mathscr{C}^\# = 0 \qquad \textit{for } k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow {}^E\mathscr{Q}^\# \longrightarrow \tau^{-1}\pi_* {}^E\mathscr{C}^\# \longrightarrow \pi_* \tau^{-1} {}^E\mathscr{C}^\# \longrightarrow 0.$$

Now we apply the same proposition to the case where $\mathscr{F} = {}^E\mathscr{A}^{\#,\beta}$. Thus we obtain a homomorphism

$$(7.1) \qquad {}^E\mathscr{A}^{\#,\beta} \longrightarrow \tau^{-1} R^{|n|-1} \tau_* {}^E\mathscr{A}^{\#,\beta},$$

$${}^E\mathscr{A}^{\#,\beta} = \boldsymbol{R}j_*({}^E\mathscr{O}^\#|_{X-M})|_{iSM},$$

where $j\colon X - M \hookrightarrow {}^M\tilde{X}$ is the canonical injection, which implies that

$$R^{|n|-1} \tau_* {}^E\mathscr{A}^{\#,\beta} = R^{|n|-1}(\tau \circ j)_*({}^E\mathscr{O}^\#|_{X-M}).$$

Hence we can define the canonical map

$$R^{|n|-1} \tau_* {}^E\mathscr{A}^{\#,\beta} \longrightarrow {}^E\mathscr{B}^\#.$$

It yields, together with (7.1), a homomorphism ${}^E\mathscr{A}^{\#,\beta} \to \tau^{-1} {}^E\mathscr{B}^\#$. Summing up, we have obtained the following.

Theorem 7.13. *We have the following diagram of exact sequences of sheaves on* iSM:

$$
(7.2)\qquad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tau^{-1}\,{}^{E}\mathscr{A}^{\#} \arrow[r] \arrow[d, equal] & {}^{E}\mathscr{A}^{\#,\beta} \arrow[r] \arrow[d, "\lambda"'] & {}^{E}\mathscr{Q}^{\#} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tau^{-1}\,{}^{E}\mathscr{A}^{\#} \arrow[r] & \tau^{-1}\,{}^{E}\mathscr{B}^{\#} \arrow[r] \arrow[d] & \tau^{-1}\pi_{*}\,{}^{E}\mathscr{C}^{\#} \arrow[r] \arrow[d] & 0 \\
 & & \pi_{*}\tau^{-1}\,{}^{E}\mathscr{C}^{\#} \arrow[r, equal] \arrow[d] & \pi_{*}\tau^{-1}\,{}^{E}\mathscr{C}^{\#} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

Let us transform the diagram (7.2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'_!\pi'^{-1}$, where τ', π' are projections $IM \to iS^*M$ and $IM \to iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F} \cong \boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F} \cong \pi^{-1}\mathscr{F}[1-|n|].$$

By Proposition 1.7 of Ito [4],

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1}\,{}^{E}\mathscr{C}^{\#} \cong \boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1}\,{}^{E}\mathscr{C}^{\#} \cong {}^{E}\mathscr{C}^{\#}[1-|n|].$$

By operating $R\tau'_!\pi'^{-1}$ on exact columns in (7.2), we obtain

$$R^k\tau'_!\pi'^{-1}\,{}^{E}\mathscr{Q}^{\#} = 0 \qquad \text{for } k \neq |n|-1,$$
$$R^k\tau'_!\pi'^{-1}\,{}^{E}\mathscr{A}^{\#,\beta} = 0 \qquad \text{for } k \neq |n|-1.$$

We define the sheaves ${}^{E}\mathscr{A}^{\#,\vee}$ and ${}^{E}\mathscr{Q}^{\#,\vee}$ on iS^*M by

$${}^{E}\mathscr{A}^{\#,\vee} = R^{|n|-1}\tau'_!\pi'^{-1}\,{}^{E}\mathscr{A}^{\#,\beta},$$
$${}^{E}\mathscr{Q}^{\#,\vee} = R^{|n|-1}\tau'_!\pi'^{-1}\,{}^{E}\mathscr{Q}^{\#,\beta}.$$

Then, in this way, we obtain the following theorem.

Theorem 7.14. *We have the diagram of exact sequences of sheaves on iS^*M*

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi^{-1}{}^{E}\mathscr{A}^{\#} \arrow[r] \arrow[d, equal] & {}^{E}\mathscr{A}^{\#,\vee} \arrow[r] \arrow[d] & {}^{E}\mathscr{Q}^{\#,\vee} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^{-1}{}^{E}\mathscr{A}^{\#} \arrow[r] & \pi^{-1}{}^{E}\mathscr{B}^{\#} \arrow[r] \arrow[d, "\mathrm{sp}"'] & \pi^{-1}\pi_{*}{}^{E}\mathscr{C}^{\#} \arrow[r] \arrow[d] & 0 \\
 & & {}^{E}\mathscr{C}^{\#} \arrow[r, equal] \arrow[d] & {}^{E}\mathscr{C}^{\#} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{7.3}
$$

and the diagram (7.2) *and the diagram* (7.3) *are mutually transformed by the functors* $\boldsymbol{R}\tau'_{!}\pi'^{-1}[|n|-1]$ *and* $\boldsymbol{R}\pi_{*}\tau^{-1}$.

We give a direct application of Theorem 7.13, which gives a relation between singularity spectrum and the domain of the defining function of an E-valued and modified Fourier hyperfunction.

By using the similar notation to Proposition 2.16 of Ito [4], we can state the following proposition.

Proposition 7.15. *Let U be an open subset of iSM with convex fiber, and V a convex hull of U. Then we have*

(1) *If* $\varphi \in \Gamma(U, {}^{E}\mathscr{A}^{\#,\beta})$, *then* $\mathrm{S.S.}(\lambda(\varphi)) \subset U^{\circ}$. *Conversely, if* $f(x) \in \Gamma(\tau U, {}^{E}\mathscr{B}^{\#})$ *satisfies* $\mathrm{S.S.}(f) \subset U^{\circ}$, *then there exists a unique* $\varphi \in \Gamma(U, {}^{E}\mathscr{A}^{\#,\beta})$ *such that* $f = \lambda(\varphi)$. *Namely, we have the exact sequence*

$$0 \longrightarrow \mathscr{A}^{\#,\beta}(U; E) \xrightarrow{\lambda} \mathscr{B}^{\#}(\tau U; E) \xrightarrow{\mathrm{sp}} \mathscr{C}^{\#}(iS^{*}M - U^{\circ}; E).$$

(2) $\Gamma(V, {}^{E}\mathscr{A}^{\#,\beta}) \to \Gamma(U, {}^{E}\mathscr{A}^{\#,\beta})$ *is an isomorphism.*

Definition 7.16. We say $u \in \mathscr{B}^{\#}(\Omega; E)$ to be micro-analytic at $(x, i\eta\infty)$ in $iS^{*}M$ if $(x, i\eta\infty) \notin \mathrm{S.S.}\,u$. This is equivalent to being represented as

$$u = \sum_{j} \lambda(\varphi_j), \quad \varphi_j \in \mathscr{A}^{\#,\beta}(U_j; E), \quad (x, i\eta\infty) \notin U_j^{\circ}.$$

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