J. Math. Tokushima Univ.
Vol. 30 (1996), 81–87

Correction and Addendum to: On the Classical and the Quantum Mechanics in a Magnetic Field

By

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(Received September 13, 1996)

Abstract

The proof of the main results (a generalization of Helton's theorem) in the previous paper [6] is incorrect. This paper points out the error in [6], and derives a weakened conclusion concerning the difference spectrum of the Schrödinger operator in a magnetic field.

1991 *Mathematics Subject Classification.* Primary 58G25: Secondary 58F05

In the previous paper [6] we considered the relationship between the distribution of the energy levels of the quantum system in a magnetic field and the periodicity of the classical orbits of the system (a generalization of Helton's theorem). We, however, have found that the proof developed in it does not go well and "Theorem" should be weakened. This paper is devoted to the correction and an addendum to the paper [6].

After reviewing the geometrical setting of the mechanics in a magnetic field, we point out in §2 the error in the "proof", and derive a weakened conclusion (Proposition 2.2 and Theorem 2.4). Further in §3 we briefly offer some results concerning the relation between the quantum spectrum and the holonomies of the connection.

1. Geometrical setting of the mechanics in a magnetic field (cf. [6, §§2–3])

Let $\pi: P \to M$ be a principal $U(1)$-bundle over a compact C^∞ manifold M, where $U(1) = \{e^{it}; 0 \le t < 2\pi\}$. The cotangent bundle T^*P of P is endowed with the standard symplectic structure Ω, and the $U(1)$ action on P is naturally lifted to the symplectic action on (T^*P, Ω). We have the momentum map $J: T^*P \to \mathrm{u}(1)^*$ ($\mathrm{u}(1)^*$ being the dual space of the Lie algebra $\mathrm{u}(1)$ of $U(1)$) associated to this $U(1)$-action, and obtain the symplectic manifold $(P_\mu \equiv J^{-1}(\mu)/U(1), \Omega_\mu)$ for each $\mu \in \mathrm{u}(1)^*$ reduced from (T^*P, Ω) according to the Marsden-Weinstein reduction mechanism.

Given a connection $\tilde{\nabla}$ on the principal bundle P. Then we can construct a diffeomorphism $\Psi_\mu: P_\mu \to T^*M$ associated to the connection $\tilde{\nabla}$. Let Θ be the curvature form of $\tilde{\nabla}$ (which is a $\mathrm{u}(1)$-valued two form on M), and let $\Theta_\mu = \langle \mu, \Theta \rangle$ for $\mu \in \mathrm{u}(1)^*$. Then, Ψ_μ turns out to be a symplectic diffeomorphism between (P_μ, Ω_μ) and (T^*M, Ω_μ^M) with Ω_μ^M being given by $\Omega_0 + \pi_M^*\Theta_\mu$, where Ω_0 denotes the standard symplectic form on T^*M and π_M is the projection of T^*M onto M ([6, Proposition 2.4]). Take a Riemannian metric m on M, and m naturally induces the Hamiltonian function H on T^*M, locally expressed as $H(x, \eta) = \Sigma m^{jk}(x)\eta_j\eta_k$. Thus we have a Hamiltonian dynamical system (T^*M, Ω_μ^M, H), which is isomorphic with $(P_\mu, \Omega_\mu, H_\mu \equiv \Psi_\mu^* H)$, and we regard this Hamiltonian system to describe the motion of a classical particle with the "charge" $\mu (\in \mathrm{u}(1)^*)$ in the magnetic field Θ.

Next, we introduce the quantum system associated to (T^*M, Ω_μ^M, H). Let us consider the subset

$$\Lambda^* = \{\lambda \in \mathrm{u}(1)^*; \langle \lambda, \partial/\partial t \rangle \in \mathbf{Z}\},$$

of $\mathrm{u}(1)^*$, which is identified with $\mathbf{Z}$. For each $\lambda \in \Lambda^*$ define the irreducible unitary representation ρ_λ of $U(1)$ by

$$\rho_\lambda(e^{it}) = e^{-i\langle \lambda, \partial/\partial t \rangle t} \in S^1 \subset \mathbf{C} \setminus 0.$$

Then, we obtain the complex line bundle $\pi_\lambda: E_\lambda \to M$ associated to the principal $U(1)$ bundle P by the representation ρ_λ. Moreover, on the line bundle E_λ we have the linear connection $\tilde{\nabla}^{(\lambda)}$ induced from $\tilde{\nabla}$ on P. From the connection $\tilde{\nabla}^{(\lambda)}$ on E_λ and the Riemannian metric m on M we can naturally define a differential operator $L^{(\lambda)}$ called the *Bochner-Laplacian*, which is a second order, non-negative, formally self-adjoint elliptic operator acting on $C^\infty(E_\lambda)$ (the space of C^∞ sections of E_λ). Let $\alpha = \Sigma a_j dx^j \otimes \partial/\partial t$ be the connection form (defined on an open subset of M) of $\tilde{\nabla}$ with respect to a local frame of P. Then, $L^{(\lambda)}$ is locally expressed with respect to the local frame of E_λ (associated to the local frame of P) as

$$L^{(\lambda)} = -\sum_{j,k} m^{jk}(\nabla_j - i\lambda a_j)(\nabla_k - i\lambda a_k) \qquad (\lambda \in \mathbf{Z}),$$

where ∇ is the Levi-Civita connection associated to m. As the quantum object corresponding to the Hamiltonian system $(T^*M, \Omega_\lambda^M, H)$ $(\cong(P_\lambda, \Omega_\lambda, H_\lambda))$ we take the differential operator $L^{(\lambda)}$ on E_λ, which is called the *Schrödinger operator with magnetic potential* α. Note that the classical system $(T^*M, \Omega_\lambda^M, H)$ is quantized only for $\lambda \in \Lambda^* = \mathbf{Z}$.

2. Spectrum and classical orbits

Let us consider the spectrum $\{\kappa_1^{(\lambda)} \le \kappa_2^{(\lambda)} \le \cdots \le \kappa_j^{(\lambda)} \le \cdots\}$ of $L^{(\lambda)}$ and the classical trajectories of $(T^*M, \Omega_\lambda^M, H)$. It will be natural to consider instead of these data the spectrum $\{\nu_j^{(\lambda)} = \sqrt{\kappa_j^{(\lambda)}}\}_{j=1}^\infty$ of $\sqrt{L^{(\lambda)}}$ and the trajectories of the flow $\phi_s^{(\lambda)}$ on $T^*M \setminus 0$ associated to the Hamiltonian $\sqrt{H}$. Notice that the orbits of $\phi_s^{(\lambda)}$ is the same as those associated to H (ignoring the difference of parameters). Let $\Sigma^{(\lambda)}$ denote the set of cluster points of the set $\{\nu_j^{(\lambda)} - \nu_k^{(\lambda)}; j, k = 1, 2, \ldots\}$.

In the previous paper we asserted the following as Theorem [6, p. 20]:

"*Every trajectory of the flow* $\phi_s^{(\lambda)}$ *is periodic if* $\Sigma^{(\lambda)} \neq \mathbf{R}$."

However, the "proof" of this "Theorem" has not be carried out correctly, and the assertion must be weakened.

We review the "proof" developed there in [6, pp. 21–22]. Take an invariant metric on $U(1)$, and we can define the $U(1)$-invariant metric $\tilde{m}$ on P (the Kaluza-Klein metric) as follows: The metric on $U(1)$ induces a metric on the subspace V_p of T_pP $(p \in P)$ tangent to the fiber. Let $T_pP = V_p \oplus H_p$ be the splitting given by the connection on P. The metric on M induces a metric on the horizontal space H_p via π_*, and we require that V_p and H_p are orthogonal with respect to $\tilde{m}$. Let us consider the Hamiltonian system $(T^*P, \Omega, \sqrt{\tilde{H}})$ and the first order elliptic operator $\sqrt{\tilde{\Delta}}$ on P induced by the Riemannian metric $\tilde{m}$. The space of $U(1)$-equivariant functions, $C_\lambda^\infty(P)$, is identified with $C^\infty(E_\lambda)$, and on which $\sqrt{\tilde{\Delta}} = \sqrt{L^{(\lambda)} + |\lambda|^2}$ holds. Let $\tilde{\phi}_s$ be the Hamiltonian flow of $(T^*P, \Omega, \sqrt{\tilde{H}})$. The flow $\tilde{\phi}_s$ contained in the submanifold $J^{-1}(\lambda)$ induces the flow $\tilde{\phi}_s^{(\lambda)}$ on P_λ which is same as $\phi_s^{(\lambda)}$ (ignoring difference of parameter). For any $U(1)$-invariant pseudo-differential operator R on P and $f \in C_0^\infty(\mathbf{R})$ we consider

$$R_f = \int_{-\infty}^{\infty} \hat{f}(s) \exp(-is\sqrt{\tilde{\Delta}}) R \exp(is\sqrt{\tilde{\Delta}})\, ds.$$

If $\Sigma^{(\lambda)} \neq \mathbf{R}$ holds, then the operator R_f is a smoothing operator on $C^\infty(E_\lambda)$ (or a compact operator on $L^2(E_\lambda)$) for such f that $\operatorname{supp} f \subset \mathbf{R} \setminus \Sigma^{(\lambda)}$. In [6] we derived from this fact the following incorrect formula:

$$(*) \qquad \int_{-\infty}^{\infty} \hat{f}(s) r(\tilde{\phi}_s^{(\lambda)}(x, \xi))\, ds = 0 \qquad \text{for } \forall (x, \xi) \in P_\lambda,$$

r being the principal symbol of R (defined as a function on P_λ). Since the flow $\tilde{\phi}_s^{(\lambda)}(x, \xi)$ is not homogeneous in ξ, i.e., the orbits of $\tilde{\phi}_s^{(\lambda)}(x, c\xi)$ $(c \in \mathbf{R}^+)$ is not same as that of $\tilde{\phi}_s^{(\lambda)}(x, \xi)$, the formula $(*)$ must be replaced by

$$\lim_{T\to\infty}\left[\overline{\lim_{|\xi|\to\infty}}\ \sup_{x\in M}\left|\int_{-T}^{T}\hat{f}(s)r(\tilde{\phi}_s^{(\lambda)}(x, \xi))\,ds\right|\right] = 0. \tag{2.1}$$

Here we take $(x^1, \dots, x^n, t)$ $(0 \le t < 2\pi)$ as local coordinates of $\pi^{-1}(U) \cong U \times U(1)$ $(U \subset M)$, and $(x^1, \dots, x^n, t, \xi_1, \dots, \xi_n, \tau)$ denotes the canonical coordinates of $T^*\pi^{-1}(U)$. Then, we have

$$J^{-1}(\lambda) \cap T^*\pi^{-1}(U) = \{(x, t, \xi, \lambda);\ x \in U,\ 0 \le t < 2\pi,\ \xi \in \mathbf{R}^n\},$$

for each $\lambda \in \Lambda^*$, and we can take (x, ξ) as local coordinates of P_λ. The formula (2.1) is derived as follows. Put

$$R_f^T = \int_{-T}^{T}\hat{f}(s)\exp(-is\sqrt{\bar{\Delta}})\,R\,\exp(is\sqrt{\bar{\Delta}})\,ds.$$

Then, by Egorov's theorem R_f^T is a pseudo-differential operator of order zero with the leading symbol

$$r_f^T(x, \xi) = \int_{-T}^{T}\hat{f}(s)r(\tilde{\phi}_s^{(\lambda)}(x, \xi))\,ds.$$

The infimum of the operator norm $\|R_f^T - K\|$ when K varies over compact operators on $L^2(E_\lambda)$ is equal to $\overline{\lim}_{|\xi|\to\infty}\sup_x|r_f^T(x, \xi)|$ (cf. [3, Theorem 3.3]). Obviously $\|R_f^T - R_f\|$ tends to zero as $T \to \infty$, and we get (2.1). Unfortunately, we cannot derive from (2.1) the fact that every orbit of $\tilde{\phi}_s^{(\lambda)}$ is closed.

From the formula (2.1) together with some other consideration we will derive some properties of $\Sigma^{(\lambda)}$. Notice that the orbit $\tilde{\phi}_s^{(\lambda)}(x, \xi)$ on P_λ is induced from the orbit $\tilde{\phi}_s(x, t, \xi, \lambda)$ on T^*P (which is just the orbit of geodesic flow). Let U^*P denote the unit cotangent bundle relative to the metric on P, and let $D\colon T^*P \setminus 0 \to U^*P$ be the map defined by $(x, t, \xi, \tau) \mapsto (x, t, \xi/c, \tau/c)$ $(c = |(\xi, \tau)|)$. For the orbit $\tilde{\phi}_s(x, t, \xi, \tau)$ on T^*P we define the orbit $\tilde{\phi}_s^U(x, t, \xi, \tau)$ on U^*P as

$$\tilde{\phi}_s^U(x, t, \xi, \tau) = \tilde{\phi}_s(D(x, t, \xi, \tau)) = D[\tilde{\phi}_s(x, t, \xi, \tau)].$$

Lemma 2.1. *The orbit $\tilde{\phi}_s^U(x, t, c\xi, \lambda)$ converges to the orbit $\tilde{\phi}_s^U(x, t, \xi, 0)$ as $c \to \infty$.*

Proof. The initial point of the orbit $\tilde{\phi}_s^U(x, t, c\xi, \lambda)$ is

$$\left(x,\ t,\ \frac{c\xi}{|(c\xi, \lambda)|},\ \frac{\lambda}{|(c\xi, \lambda)|}\right) = \left(x,\ t,\ \frac{\xi}{|(\xi, \lambda/c)|},\ \frac{\lambda}{c|(\xi, \lambda/c)|}\right),$$

which tends to $(x, t, \xi/|\xi|, 0)$ as $c \to \infty$. Hence the lemma follows from the properties concerning initial conditions of ordinary differential equations. □

The orbit $\tilde{\phi}_s(x, t, \xi, 0)$ induces the orbit $\tilde{\phi}_s^{(0)}(x, \xi)$ on P_0 which is just the orbit $\phi_s^{(0)}(x, \xi)$ of geodesic flow on $(T^*M, \Omega_0, \sqrt{H})$. The function $r(x, \xi, \tau)$ on T^*P, the principal symbol of R, can be taken to satisfy $r(x, c\xi, c\tau) = r(x, \xi, \tau)$ $(c \in \mathbf{R}^+)$, hence we get from (2.1)

$$\int_{-\infty}^{\infty} \hat{f}(s) r(\tilde{\phi}_s^{(0)}(x, \xi))\, ds = 0 \qquad \text{for } \forall (x, \xi) \in P_0 = T^*M, \tag{2.2}$$

where r with $\tau = 0$ is regarded as a function on P_0. By means of the same argument in [2, pp. 487–489] we conclude from (2.2) that $\tilde{\phi}_s^{(0)}(x, \xi)$ is closed for every $(x, \xi) \in P_0$, and have the following.

Proposition 2.2. *If $\Sigma^{(\lambda)} \neq \mathbf{R}$ for some $\lambda \in \Lambda^*$, then every trajectory of the geodesic flow $\phi_s^{(0)}$ on $(T^*M, \Omega_0, \sqrt{H})$ is periodic. Moreover, the function ρ assigning $(x, \xi) \in T^*M \setminus 0$ to the fundamental period of the trajectory $\phi_s^{(0)}(x, \xi)$ is bounded, and*

$$\Sigma^{(\lambda)} \supset \bigcup_{(x,\xi) \in T^*M \setminus 0} \left\{ \frac{2\pi n}{\rho(x, \xi)} ;\, n \in \mathbf{Z} \right\}$$

holds good.

By virtue of Wadsley's theorem [8] (see [1, p. 183 or Appendix A], also) we see that the orbits $\phi_s^{(0)}(x, \xi)$ have a least common period, i.e., the Riemannian manifold (M, m) is a P_ℓ-manifold (ℓ being the least common period). On the other hand, Guillemin showed the following result concerning ℓ and $\Sigma^{(0)}$ (for the Laplace-Beltrami operator).

Proposition 2.3 ([2, Theorem 6]). *Suppose every trajectory of the geodesic flow $\phi_s^{(0)}$ is periodic with least common period ℓ. Then, $\Sigma^{(0)} = \{2\pi n/\ell ;\, n \in \mathbf{Z}\}$ holds good.*

Combining the above results, we obtain the following theorem.

Theorem 2.4. *Suppose $\Sigma^{(\lambda_0)} \neq \mathbf{R}$ for some $\lambda_0 \in \Lambda^* \setminus 0$. Then, every trajectory of the geodesic flow $\phi_s^{(0)}$ on $(T^*M, \Omega_0, \sqrt{H})$ is periodic with a least common period denoted by ℓ, and moreover*

$$\Sigma^{(0)} = \left\{ \frac{2\pi n}{\ell} ;\, n \in \mathbf{Z} \right\} \subset \Sigma^{(\lambda)}$$

holds for every $\lambda \in \Lambda^* \setminus 0$.

Corollary 2.5. $\Sigma^{(\lambda)} = \mathbf{R}$ *for every* $\lambda \in \Lambda^*$ *if and only if the geodesic flow* $\phi_s^{(0)}$ *is not periodic* (*i.e., at least one geodesic is not closed*).

3. Further consideration—Spectrum and holonomy

In [4] we considered the spectrum of the Bochner-Laplacian on the vector bundle over a $C_{2\pi}$-manifold. The arguments developed there is directly applicable to the line bundle E_λ over M when (M, m) is a P_ℓ-manifold. In this section we briefly offer some results about the structure of $\Sigma^{(\lambda)} (\lambda \in \Lambda^*)$ in terms of the holonomies of the connection $\tilde{\nabla}^{(\lambda)}$. The details of this section and further investigations will appear in the forthcoming paper [7].

When (M, m) is a P_ℓ-manifold, we see that the spectrum of $\sqrt{L^{(\lambda)}}$ is distributed in a neighborhood of the lattice points

$$\mu_k = \frac{2\pi}{\ell}\left(k + \frac{\beta}{4}\right), \qquad k = 1, 2, 3, \dots,$$

where β is the Maslov index of the periodic trajectories. We clarified in [4] that the distribution of the eigenvalues of $\sqrt{L^{(\lambda)}}$ in the interval $[\mu_k, \mu_{k+1})$ as $k \to \infty$ is described by the holonomies of the connection $\tilde{\nabla}^{(\lambda)}$. For each $(x, \xi) \in U^*M$ we denote by $Q_{\tilde{\nabla}}^{(\lambda)}(x, \xi) \in S^1$ the holonomy of $\tilde{\nabla}^{(\lambda)}$ along the closed geodesic $\gamma_t(x, \xi)$ $(0 \leq t \leq \ell)$ on M with the initial condition (x, ξ). We say that the spectrum $\{\nu_j^{(\lambda)}\}_{j=1}^{\infty}$ *makes cluster* if for any $\varepsilon > 0$ there is only finitely many eigenvalues lying the outside of

$$\bigcup_{k=1}^{\infty} [\mu_k + c - \varepsilon, \mu_k + c + \varepsilon], \tag{3.1}$$

where c is some constant with $0 \leq c < 2\pi/\ell$. Then, we have the following (see [4, Theorem 5.1 and Proposition 5.2]).

Proposition 3.1 *The spectrum of* $\sqrt{L^{(\lambda)}}$ *makes cluster if and only if* $Q_{\tilde{\nabla}}^{(\lambda)}\colon U^*M \to S^1$ *is a constant function. Moreover, in such case* $Q_{\tilde{\nabla}}^{(\lambda)} \equiv 1$ *or* $\equiv -1$ *according to* $c = 0$ *or* $= \pi/\ell$, *respectively, in* (3.1).

We can see that $\Sigma^{(\lambda)} = \{2\pi n/\ell\,;\, n \in \mathbf{Z}\}$ holds if and only if the spectrum of $\sqrt{L^{(\lambda)}}$ makes cluster. Moreover, we note the following property of the holonomy:

$$Q_{\tilde{\nabla}}^{(m\lambda)}(x, \xi) = (Q_{\tilde{\nabla}}^{(\lambda)}(x, \xi))^m \qquad (m \in \mathbf{Z}).$$

As a consequence, we have the following.

Theorem 3.2. *Suppose* $\Sigma^{(\lambda_0)} = \{2\pi n/\ell\,;\, n \in \mathbf{Z}\}$ *for some* $\lambda_0 \in \Lambda^* \setminus 0$. *Then,*
(1) $\Sigma^{(\lambda)} = \{2\pi n/\ell\,;\, n \in \mathbf{Z}\}$ *for every* $\lambda \in \Lambda^*$,

(2) (M, m) *is a* P_ℓ*-manifold, and*
(3) $Q_{\tilde{\nabla}}^{(\lambda)}$ *is a constant function* $(\equiv 1$ *or* $\equiv -1)$ *for every* $\lambda \in \Lambda^* \setminus 0$.

By virtue of further considerations of holonomies of the connection we have the following theorem concerning the structure of $\Sigma^{(\lambda)} (\lambda \in \Lambda^*)$.

Theorem 3.3. (0) *The structure of* $\Sigma^{(\lambda)} (\lambda \in \Lambda^*)$ *is one of the following three types*:

I: $\Sigma^{(\lambda)} = \mathbf{R}$ *for every* $\lambda \in \Lambda^*$,

II: $\Sigma^{(0)} = \{2\pi n/\ell;\, n \in \mathbf{Z}\}$ *and there is* $\lambda_0 \in \Lambda^* \setminus 0$ *such that* $\Sigma^{(\lambda)} = \mathbf{R}$ *for any* λ *with* $|\lambda| \geq |\lambda_0|$, *and*

III: $\Sigma^{(\lambda)} = \{2\pi n/\ell;\, n \in \mathbf{Z}\}$ *for every* $\lambda \in \Lambda^*$.

(1) *Type* I *occurs if and only if the geodesic flow* $\phi_s^{(0)}$ *is not periodic* (*i.e.*, (M, m) *is not a* P_ℓ*-manifold*).

(2) *Type* II *occurs if and only if* (M, m) *is a* P_ℓ*-manifold and the function* $Q_{\tilde{\nabla}}^{(\lambda_0)}$ *takes a value not* ± 1 *for some* $\lambda_0 \in \Lambda^* \setminus 0$.

(3) *Type* III *occurs if and only if* (M, m) *is a* P_ℓ*-manifold and* $Q_{\tilde{\nabla}}^{(\lambda)}$ *is a constant function* $(\equiv 1$ *or* $\equiv -1)$ *for every* $\lambda \in \Lambda^* \setminus 0$.

Remarks. 1. In the case of Type II the measure of $\Sigma^{(\lambda)}$ for $0 < |\lambda| < |\lambda_0|$ is not equal to zero.

2. Type III actually occurs when the principal bundle is the Hopf bundle $S^{2n+1} \to (\mathbf{C}P^n, m_0)$ with m_0 being the standard metric and the $\tilde{\nabla}$ is the harmonic connection, in that case every trajectory of the flow $\phi_s^{(\lambda)}$ is periodic for every $\lambda \in \Lambda^*$ (cf. [5]). The following question is left for future considerations:

"Does every trajectory of the flow $\phi_s^{(\lambda)}$ is periodic for every $\lambda \in \Lambda^*$ when (or only when) $\Sigma^{(\lambda)}$ is of Type III?"

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