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On Some Numerical Relations of d–gonal Linear Systems

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Abstract

Let $\mathcal{L}$ be a pencil of degree d on a curve C and let $e_1 \cdot \circ \cdot, e_{d-1}$ be scrolar invariants. We already prove that $e_k \leq e_{k+1} + e_{d-1} + 2$ for $k=1, \ldots d-2$ if $|\mathcal{L}^{\otimes e_{d-1}+2}|$ is birationally very ample. In this article, we extend the above result.

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0 Introduction

Let C be a complete non-singular curve defined over an algebraically closed field k with $\mathrm{char}(k) \neq 2$. Let $\mathcal{L}$ be a base point free invertible sheaf on C with $\dim\Gamma(C, \mathcal{L}) = 2$ and $\deg \mathcal{L} = d$. We assume that C is non-hyperelliptic of genus g. Let $\alpha_i = \dim\Gamma(C, \mathcal{L}^{\otimes i+1}) - \dim\Gamma(C, \mathcal{L}^{\otimes i})$. Then we have $1 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_i \leq \cdots \leq d$ because the kernel of $\Gamma(C, \mathcal{L}^{\otimes i}) \to \Gamma(C, \mathcal{L}^{\otimes i+1})/\Gamma(C, \mathcal{L}^{\otimes i})$ defined by the multiplication by some section of $\Gamma(C, \mathcal{L})$ is $\Gamma(C, \mathcal{L}^{\otimes i-1})$.

Definition A *Under the above assumptions, we put $f_i = \min\{j : \alpha_j \geq d - i + 1\} - 1$ for every $i = 1, \ldots, d-1$. and we put f_0=0. Moreover, for any $\sigma \in \mathbb{Z}/d\mathbb{Z}$, we put e_σ=f_i where σ=i mod d and $0 \leq i \leq d-1$. And we call $e_1, \cdots, e_{d-1}$ scrollar invariants.*

The following result is well-known (see [5] p.4591).

Theorem A $e_1 \geq \cdots \geq e_{d-1}$ *and* $e_1 + \cdots e_{d-1} = g - d + 1$.

And we proved the following result (see [5] p.4588).

Theorem B *Let $\mathcal{L}$ be a base point free line bundle with* $\dim\Gamma(C,\mathcal{L}) = 2$ *and* $\deg\mathcal{L} = d$ *with the scrollar invariants* $e_1, \ldots, e_{d-1}$. *If* $\mathcal{L}^{\otimes e_{d-1}+2}$ *is birationally very ample, then* $e_k \le e_{k+1} + e_{d-1} + 2$ *for* $k = 1, \ldots d-2$.

In this article, we extend Theorem B. The following is our result.

Theorem C *Let $\mathcal{L}$ be a base point free line bundle with* $\dim\Gamma(C,\mathcal{L}) = 2$ *and* $\deg\mathcal{L} = d$ *with the scrollar invariants* $e_1, \ldots, e_{d-1}$. *If* $\mathcal{L}^{\otimes e_{d-1}+2}$ *is birationally very ample, then* $e_j \le e_{j+s} + e_{d-s} + 2$ *for any* $d-1 \ge j \ge 2s$ *where* $1 \le s \le d-1$.

NOTATIONS
$\mathcal{O}_C$: The structure sheaf of a variety C
$\mathcal{O}(D)$: The line budle defined by a divisor D
$\Gamma(C,\mathcal{L})$: The vector space defined by $\{f \in K(C); (f) + D \ge 0\}$
$|D|$: The complete linear system defined by a divisor D

1 Proof of Theorem C

Let D be an effective divisor such that $\mathcal{L} \cong \mathcal{O}(D)$ and let $1, x$ be a basis of a vector space $\Gamma(C, \mathcal{O}(D))$. A basis of a vector space $\Gamma(C, \mathcal{O}(iD))$ for $i \in \mathbb{N}$ is

$$\Gamma(C, \mathcal{O}(iD)) = [1, x, \cdots, x^i]$$

for $i = 1, \cdots, e_{d-1} + 1$,

$$\Gamma(C, \mathcal{O}(iD)) = [1, x, \cdots, x^i, y_1, \cdots, y_1 x^{i-e_{d-1}-2}]$$

for $i = e_{d-1} + 2, \cdots, e_{d-2} + 1$,

$$\Gamma(C, \mathcal{O}(iD)) = [1, x, \cdots, x^i, y_1, \cdots, y_1 x^{i-e_{d-1}-2}, \cdots, y_j, \cdots, y_j x^{i-e_{d-j}-2}]$$

for $i = e_{d-j} + 2, \cdots, e_{d-(j+1)} + 1$ $(j = 1, 2, \cdots, d-2)$ and

$$\Gamma(C, \mathcal{O}(iD)) = [1, x, \cdots, x^i, y_1, \cdots, y_1 x^{i-e_{d-1}-2}, \cdots, y_{d-1}, \cdots, y_{d-1} x^{i-e_1-2}]$$

for $i \ge e_1 + 2$ by Definition of $e_1, \cdots, e_{d-1}$ where

$$y_j \in \Gamma(C, \mathcal{O}((e_{d-j}+2)D)) \text{ and } y_j \notin \Gamma(C, \mathcal{O}((e_{d-j}+1)D)).$$

If $|(e_{d-i}+2)D|$ is birationally very ample, then we can choose y_i such that

$$k(x, y_i) = k(C)$$

because $k(C)/k(x)$ has only finitely many subfields. As we assume that $\mathcal{L}^{\otimes e_{d-1}+2}$ is birationally very ample, so we have that $\mathcal{L}^{\otimes e_{d-1}+2}, \cdots, \mathcal{L}^{\otimes e_{d-s}+2}$ are birationally very ample. Hence we may assume that

$$k(x, y_1) = k(C), \cdots, k(x, y_s) = k(C).$$

Moreover we may assume that y_η satisfies $y_s = \mu_0(x) + \mu_1(x)y_\eta + \cdots + \mu_l(x)y_\eta^{d-1}$ for $\eta \neq s$. We assume that $e_{d-j} > e_{d-j+s} + e_{d-s} + 2$ for some $d-1 \geq j \geq 2s$. Then $\Gamma(C, \mathcal{O}((e_{d-(k-s)}+2)D)) \subset \Gamma(C, \mathcal{O}((e_{d-(j-s)}+2)D))$ for every k such that $s+1 \leq k \leq j$ (i.e.$1 \leq k-s \leq j-s$), hence

$$\Gamma(C, \mathcal{O}((e_{d-(k-s)}+2)D))\Gamma(C, \mathcal{O}((e_{d-s}+2)D)) \subset$$

$$\Gamma(C, \mathcal{O}((e_{d-(k-s)}+2+e_{d-s}+2)D)) \subset$$

$$\Gamma(C, \mathcal{O}((e_{d-(j-s)}+2+e_{d-s}+2)D)) \subset \Gamma(C, \mathcal{O}((e_{d-j}+1)D)).$$

Therefore there are $a_i(x), b_{ij}(x)$ such that

$$\begin{aligned} y_s y_1 &= a_1(x) + b_{1\,1}(x)y_1 + \cdots + b_{1\,j-1}(x)y_{j-1} \\ &\vdots \\ y_s y_{j-s} &= a_{j-s}(x) + b_{j-s\,1}(x)y_1 + \cdots + b_{j-s\,j-1}(x)y_{j-1}. \end{aligned}$$

Hence

$$-\begin{pmatrix} a_1 + b_{1j-1}y_{j-1} + \cdots + b_{1j-s-1}y_{j-s-1} \\ \vdots \\ a_{j-s} + b_{j-sj-1}y_{j-1} + \cdots + b_{j-sj-s+1}y_{j-s+1} \end{pmatrix} =$$

$$\begin{pmatrix} b_{11}-y_s & & * \\ & \ddots & \\ * & & b_{j-sj-s}-y_s \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{j-s} \end{pmatrix} \cdots (A)$$

where components of $*$ are polynomials of x. Let

$$B = \begin{pmatrix} b_{11}-y_s & & * \\ & \ddots & \\ * & & b_{j-sj-s}-y_s \end{pmatrix}$$

and

$$C = \begin{pmatrix} b_{1j-1} & \cdots & b_{1j-s+1} \\ \vdots & & \vdots \\ b_{j-sj-1} & & b_{j-sj-s+1} \end{pmatrix}$$

As C is a $(j-s, s-1)$ matrix with coefficients in $k(x)$, therefore there is a matrix $\tilde{C}$ such that some column vector of $\tilde{C}C$ is a zero vector because $j-s > s-1$ from the assumption. And $\det(B) \neq 0$. If $\det(B) = 0$, then y_s has a relation of degree $j-s$ over $k(x)$ because B is a square matrix of degree $j-s$, so $\det(B) = 0$ implies that y_s has a relation of degree at most $j-s$ over $k(x)$. As $k(C) = k(x, y_s)/k(x)$ and $j-s \leq d-2$, this is a contradiction. Hence $\det(B) \neq 0$. Hence we have

$$f_0(x)\det(B)y_\eta = T(y_s).$$

where $f_0(x)$ is a rational function of x, $\deg_{y_s}(T(y_s)) = j-s-1$ and $\deg_{y_s}\det(B) \leq j-s$. As y_η is a basis of $k(C)/k(x)$, so $y_s = \mu_0(x) + \mu_1(x)y_\eta + \cdots + \mu_l(x)y_\eta^{d-1}$. Therefore y_s has a relation of degree $(d-1)(j-s+1)$ over $k(x)$. Hence $d|(d-1)(j-s+1)$. This is a contradiction. So we prove Theorem C.

Q.E.D.

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