J. Math. Tokushima Univ.
Vol. 31 (1997), 69–80

Existence and Uniqueness of Quasiperiodic Solutions to Van der Pol type Equations

By

Zulfikar ALI, Yoshitane SHINOHARA, Hitoshi IMAI, Atsuhito KOHDA
Kuniya OKAMOTO, Hideo SAKAGUCHI and Haruo MIYAMOTO

Department of Mathematics,
Faculty of Engineering,
Tokushima University, Tokushima 770, *Japan*

(Received September 12, 1997)

Abstract

This paper is concerned with the existence and uniqueness of quasiperiodic solutions to Van der Pol type equations driven by two or more distinct frequency input signals from the viewpoint of numerical analysis.

A numerical result given in the previous paper [13] is corrected.

1991 Mathematics Subject Classification. Primary 34C27

0. Introduction

The most fundamental problem in nonlinear oscillations is to find the periodic or quasiperiodic solutions to the nonlinear ordinary differential equations such as

$$\ddot{x}+\alpha\dot{x}+\beta x+\gamma x^3=\sum_{k=1}^{m}(a_k\cos\nu_k t+b_k\sin\nu_k t)\qquad (\cdot=\frac{d}{dt}),$$

$$\ddot{x}-2\lambda(1-x^2)\,\dot{x}+x=0,$$

and

$$\ddot{x}-2\lambda(1-x^2)\,\dot{x}+x=\sum_{k=1}^{m}(a_k\cos\nu_k t+b_k\sin\nu_k t),$$

where α, β, γ, λ, a_k, b_k, ν_k ($k=1, 2, \ldots, m$) are all positive constants. But, it is very difficult, in general, to find the exact solutions in analytical form. Thus, we are obliged to study the solutions by numerical methods. As for the periodic solutions to nonlinear periodic systems and also to nonlinear autonomous systems, we refer to the papers [1], [8], [16], [17], [22], [25] and [9], [10], [11], [23], [24].

From a practical viewpoint, a harmonic balance analysis of nonlinear quasiperiodic microwave circuits has been given by Maas [5] in view of qualitative applications, but he is concerned with neither the existence analysis nor the error analysis.

Chua and Ushida [2] have presented two efficient algorithms for obtaining steady-state solutions to nonlinear quasiperiodic circuits and systems driven by two or more distinct frequency input signals. They have calculated some approximate solutions to Duffing type equations with two frequency input signals and they have given error estimation, but they are not concerned with the existence analysis of the exact solutions.

In the present paper, we will show that we can indeed verify the existence and uniqueness of an exact solution and know the error bound of the approximate solution to nonlinear quasiperiodic differential equations driven by two or more distinct frequency input signals. By making use of the generalized exponential dichotomy, we will be able to strengthen the error estimation of the approximate solutions.

As for the Duffing type equations we refer to the papers [4], [14]. In the paper a numerical example concerned with the Van der Pol type equation is given in revised form.

1. Existence and uniqueness theorem

A function $f(t) \in C(R\,;\,R^d)$, where R denotes the real line hereafter, is said to be quasiperiodic with periods $\omega_1, \omega_2, \ldots, \omega_m$ if $f(t)$ is represented as

$$f(t) = f_0(t, t, \ldots, t) \tag{1.1}$$

for some continuous periodic function $f_0(u_1, u_2, \ldots, u_m)$ with period ω_i, in each u_i. Without any loss of generality we may assume that $\omega_1, \omega_2, \ldots, \omega_m$ are all positive and further that reciprocals of these periods are rationally linearly independent (see [17]). A function $f(t)$ is said to be almost periodic if from every sequence $\{\alpha_n\}$ one can extract a subsequence $\{\alpha_n'\}$ such that $\{f(t+\alpha_n')\}$ is uniformly convergent on R. We assume that all functions considered in the present paper are continuous on R. It is known in [3] that the limit value

$$a(f, \sigma) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T f(t)\, e^{-i\sigma t}\, dt$$

exists for any almost periodic function $f(t)$ and any real σ and that there is a countable set Σ of real numbers such that $a(f, \sigma) = 0$ if $\sigma \notin \Sigma$. The module of f, $M(f)$, is defined to be the smallest additive group of real numbers that contains the set Σ for which $a(f, \sigma) \neq 0$ if $\sigma \in \Sigma$.

According to the results of papers [4],[13] we have the following propositions.

PROPOSITION 1. *The limit function $f(t)$ of a uniformly convergent sequence $\{f_n(t)\}$ of quasiperiodic functions is also quasiperiodic.*

Consider a linear differential operator

$$Lz=dz/dt-A(t)z, \tag{1.2}$$

where $A(t)$ is an almost periodic or quasiperiodic matrix.

Let $\Phi(t)$ be the fundamental matrix of the linear homogeneous equation

$$Lz=0 \tag{1.3}$$

satisfying the initial condition $\Phi(0)=E$ (unit matrix).

The linear homogeneous equation (1.3) is called to satisfy the generalized exponential dichotomy if there exist a projection P, positive constants σ_1, σ_2 and non-negative functions $C_1(t,s)$, $C_2(t,s)$ such that

(i) $\quad \| \Phi(t) P\Phi^{-1}(s) \| \leq C_1(t,s)e^{-\sigma_1(t-s)}$ *for* $t\geq s$,

(ii) $\quad \| \Phi(t)(E-P)\Phi^{-1}(s) \| \leq C_2(t,s)\, e^{-\sigma_2(s-t)}$ *for* $t<s$,

(iii) *the integral*

$$\int_{-\infty}^{t} C_1(t,s)\, e^{-\sigma_1(t-s)}\, ds+\int_{t}^{+\infty} C_2(t,s)e^{-\sigma_2(s-t)}\, ds \tag{1.4}$$

is bounded on R by a positive number M.

Here we introduce the ℓ_∞ norm $\| \cdot \|$ in Euclidean space and denote that $\|f\| =\sup_{t\in R} \|f(t)\|$ for any bounded function $f=f(t)$.

We have the following two propositions.

PROPOSITION 2 ([13]). *Let $A(t)$ be an almost periodic matrix. Suppose that the equation (1.3) satisfies the generalized exponential dichotomy and that $f(t)$ is an almost periodic function. Then there is a unique almost periodic solution $z(t)$ of the inhomogeneous equation*

$$Lz=f(t) \tag{1.5}$$

and the modules satisfy the relation

$$\mathbf{Mod}\ (z) \subset \mathbf{Mod}\ (A,f), \tag{1.6}$$

where **Mod** (A, f) *is the smallest additive group of real numbers that contains the countable set Σ for which $a(f,\sigma)=0$ and $a(a_{ij},\sigma)=0$ if $\sigma\notin\Sigma$ and $A=(a_{ij})$.*

PROPOSITION 3 ([13]). *Let $A(t)$ be a quasiperiodic square matrix* with *periods $\omega_1,\omega_2, \ldots,\omega_m$. Suppose that the equation (1.3) satisfies the generalized exponential dichotomy. Then for any quasiperiodic function $f(t)$*

with periods $\omega_1, \omega_2, \ldots, \omega_m$ the inhomogeneous equation (1.5) has a unique quasiperiodic solution $z(t)$ with the same periods $\omega_1, \omega_2, \ldots, \omega_m$ given by

$$z(t) = \int_{-\infty}^{+\infty} G(t, s) f(s)\, ds, \tag{1.7}$$

where

$$G(t, s) = \begin{cases} \Phi(t) P \Phi^{-1}(s) & \text{for } t \geq s, \\ -\Phi(t)(E-P)\Phi^{-1}(s) & \text{for } t < s. \end{cases}$$

Moreover the solution $z(t)$ satisfies the relation (1.6) and

$$\| z \| \leq M \| f \| . \tag{1.8}$$

Our numerical analysis of the quasiperiodic Van der Pol type equation is based on the following existence and uniqueness theorem.

THEOREM 1 ([13]). *Given a nonlinear differential equation*

$$\frac{dz}{dt} = X(t, z), \tag{1.9}$$

where z and $X(t, z)$ are vectors and $X(t, z)$ is quasiperiodic in t with periods $\omega_1, \omega_2, \ldots, \omega_m$ and is continuously differentiable with respect to z belonging to a region D of z-space.

Suppose that there is a continuously differentiable quasiperiodic function $z_0(t)$ with periods $\omega_1, \omega_2, \ldots, \omega_m$ such that

$$z_0(t) \in D,$$

$$\left\| \frac{dz_0(t)}{dt} - X[t, z_0(t)] \right\| \leq r$$

for all $t \in R$. Further suppose that there are a positive number δ, a non-negative number $\kappa < 1$ and quasiperiodic matrix $A(t)$ with periods $\omega_1, \omega_2, \ldots, \omega_m$ such that

$$\begin{cases} \text{(i)} & \text{the linear equation (1.3) satisfies the generalized exponential dichotomy,} \\ \text{(ii)} & D_\delta = \{ z \,;\, \| z - z_0(t) \| < \delta \quad \text{for some } t \in R \} \subset D, \\ \text{(iii)} & \| \Psi(t, z) - A(t) \| \leq \dfrac{\kappa}{M} \quad \text{whenever } \| z - z_0(t) \| \leq \delta, \\ \text{(iv)} & \dfrac{Mr}{1-\kappa} \leq \delta. \end{cases} \tag{1.10}$$

Here $\Psi(t,z)$ is the Jacobian matrix of $X(t,z)$ with respect to z and the quantity M is given in (1.8).

Then the given equation (1.9) possesses a solution $z = \hat{z}(t)$ quasiperiodic in t with periods $\omega_1, \omega_2, \ldots, \omega_m$ such that

$$\| z_0(t) - \hat{z}(t) \| \leq \frac{Mr}{1-\kappa} \tag{1.11}$$

for all $t \in R$. *Furthermore, to equation (1.9) there is no other quasiperiodic solution belonging to* D_δ *besides* $z = \hat{z}(t)$.

2. Quasiperiodic solution to the Van der Pol type equation

We shall first consider the following linear differential equation

$$\frac{d^2 x}{dt^2} + 2\mu \frac{dx}{dt} + \nu^2 x = f(t), \tag{2.1}$$

where μ, ν are constants such that $\nu > 0$, $\mu \neq 0$, and $f(t)$ is quasiperiodic with periods $\omega_1, \omega_2, \ldots, \omega_m$. Putting $y = dx/dt$,

$$z = \begin{pmatrix} x \\ y \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 \\ -\nu^2 & -2\mu \end{pmatrix},\ F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix},$$

equation (2.1) can be written in the vector form as follows :

$$\frac{dz}{dt} = Az + F(t). \tag{2.2}$$

Let L be the differential operator defined by

$$Lz = \frac{dz}{dt} - Az, \tag{2.3}$$

then the fundamental matrix $\Phi(t)$ of the linear system $Lz = 0$ such that $\Phi(0) = E$ is given by $\Phi(t) = \exp tA$, which will be called the matrizant of L. In what follows, we denote by $\| \cdot \|$ the following ℓ_∞ norm of vectors and matrices :

$\| v \| = \max_i |v_i|$ *for vector* v *with components* v_i,

$\| \Phi \| = \max_i \sum_j |\phi_{ij}|$ *for matrix* Φ *with components* ϕ_{ij}.

The matrizant satisfies the inequality

$$\| \Phi(t) \| \leq K_0 e^{-\sigma_0 t}. \tag{2.4}$$

Here K_0 and σ_0 are quantities specified in the following three cases :

(i) when $|\mu| > \nu$,

$$K_0 = \begin{cases} \max\left(\frac{1}{\beta-\alpha}(1+|\beta|),\ \frac{1}{\beta-\alpha}(\nu^2+2|\mu|)\right) & \text{if } \mu<0, \\ \max\left(\frac{1}{\beta-\alpha}(1+|\alpha|),\ \frac{1}{\beta-\alpha}(\nu^2+2|\mu|)\right) & \text{if } \mu>0, \end{cases}$$

$$\sigma_0 = \begin{cases} -\alpha & \text{if } \mu<0, \\ -\beta & \text{if } \mu>0, \end{cases}$$

(ii) when $|\mu| = \nu$, (2.5)

$$K_0 = K_0(t) = \max(1+|\mu t|+|t|,\ 1+|\mu t|+|\mu^2 t|),$$

$$\sigma_0 = \mu,$$

(iii) and when $|\mu|<\nu$,

$$K_0=\frac{\nu+1}{\sqrt{\nu^2-\mu^2}}\max(1,\nu),$$

$$\sigma_0=\mu,$$

where $\alpha=-\mu-\sqrt{\mu^2-\nu^2}$, $\beta=-\mu+\sqrt{\mu^2-\nu^2}$.

Remark that K_0 is not a constant number when $|\mu|=\nu$.

From (2.4) and Proposition 2, the equation (1.3) satisfies the generalized exponential dichotomy, because we can choose the matrix P such that

$$P=\begin{cases} 0 & \text{when } \mu<0, \\ E & \text{when } \mu>0. \end{cases}$$

Consequently, we get the following theorem.

THEOREM 2 ([13]). *If $\mu\neq 0$, the equation (1.3) satisfies the generalized exponential dichotomy and the unique quasiperiodic solution $z=z_0(t)={}^t(x_0(t), y_0(t))$ with periods $\omega_1, \omega_2, \ldots, \omega_m$ to the equation (2.2) is given by*

$$z_0(t)=\int_{-\infty}^{+\infty}G(t,s)F(s)ds,$$

where the Green function $G(t,s)$ is specified in the following two cases :

(i) *when $\mu>0$,*

$$G(t,s)=\begin{cases} \Phi(t-s) & \text{for } t\geq s, \\ 0 & \text{for } t<s, \end{cases}$$

(ii) *when $\mu<0$,*

$$G(t,s)=\begin{cases} 0 & \text{for } t\geq s, \\ -\Phi(t-s) & \text{for } t<s. \end{cases}$$

Moreover, we have

$$\| G(t,s) \| \leq K_0 e^{-\sigma|t-s|},$$

where

$$\sigma=|\sigma_0|.$$

Now, consider the Van der Pol type equation with quasiperiodic forcing term such as

$$\frac{d^2x}{dt^2}-2\lambda(1-x^2)\frac{dx}{dt}+x=\sum_{k=1}^{m}(a_k\cos\nu_k t+b_k\sin\nu_k t), \tag{2.6}$$

where λ and ν_k ($k=1, 2, \ldots, m$) are all positive parameters.

In the following, we may assume that the reciprocals of periods $\omega_k=2\pi/\nu_k$ ($k=1, 2, \ldots, m$) are rationally independent.

The equation (2.6) can be written into the vector form

$$\frac{dz}{dt}=Az+\phi(t)+\lambda\eta(z), \tag{2.7}$$

where

$$z=\binom{x}{y},\quad A=\begin{pmatrix}0 & 1\\ -1 & -2\lambda\end{pmatrix},\quad \phi(t)=\begin{pmatrix}0\\ \sum_{k=1}^{m}(a_k\cos\nu_k t+b_k\sin\nu_k t)\end{pmatrix},$$

$$\eta(z)=\begin{pmatrix}0\\ -2x^2y\end{pmatrix}.$$

Let L be the differential operator defined by

$$Lw=\frac{dw}{dt}-Aw, \tag{2.8}$$

then Theorem 2 tells us that the equation (1.3) defined by (2.8) satisfies the generalized exponential dichotomy for $\mu\neq 0$ and that the linear operator G defined by $G\phi=w$, which means

$$\int_{-\infty}^{+\infty}G(t,s)\phi(s)ds=w(t),$$

satisfies the inequality

$$\|G\|\leq M, \tag{2.9}$$

where

$$M=\begin{cases}\dfrac{1+2\lambda}{2\left(\lambda-\sqrt{\lambda^2-1}\right)\sqrt{\lambda^2-1}} & \text{if } \lambda>1,\\[2mm] 3 & \text{if } \lambda=1,\\[2mm] \dfrac{\sqrt{2+2\lambda}}{\lambda\sqrt{1-\lambda^2}} & \text{if } 0<\lambda<1.\end{cases} \tag{2.10}$$

Remark that the value of M for $\lambda>1$ is corrected for the paper [13].
The quasiperiodic solution of the linear equation

$$Lz=\phi(t) \tag{2.11}$$

is given by $z=z_0(t)=\binom{x_0(t)}{y_0(t)}$, where

$$x_0(t)=\sum_{k=1}^{m}(\alpha_k\cos\nu_k t+\beta_k\sin\nu_k t), \tag{2.12}$$

$$y_0(t)=\frac{d}{dt}x_0(t),$$

and

$$\alpha_k=\frac{a_k(1-\nu_k^2)+2b_k\lambda\nu_k}{(1-\nu_k^2)^2+4\lambda^2\nu_k^2},\quad \beta_k=\frac{-2a_k\lambda\nu_k+b_k(1-\nu_k^2)}{(1-\nu_k^2)^2+4\lambda^2\nu_k^2}.$$

Since

$$\sqrt{\alpha_k^2+\beta_k^2} = \frac{\sqrt{a_k^2+b_k^2}}{\sqrt{(1-\nu_k^2)^2+4\lambda^2\nu_k^2}}$$

we have the following estimation

$$|x_0(t)|,\ |y_0(t)| \leq K \tag{2.13}$$

for all $t \in R$, where

$$K=\max\left(\sum_{k=1}^{m}\frac{\sqrt{a_k^2+b_k^2}}{\sqrt{(1-\nu_k^2)^2+4\lambda^2\nu_k^2}},\ \sum_{k=1}^{m}\frac{|\nu_k|\sqrt{a_k^2+b_k^2}}{\sqrt{(1-\nu_k^2)^2+4\lambda^2\nu_k^2}}\right).$$

Using the estimate (2.13), we can estimate the residual function for $z_0(t)$ as follows :

$$\left\|\frac{dz_0(t)}{dt}-Az_0(t)-\phi(t)-\lambda\eta(z_0(t))\right\| = \|-\lambda\eta(z_0(t))\|$$
$$=2\lambda|x_0^2(t)\,y_0(t)|\leq 2\lambda K^3.$$

Accordingly, we can choose

$$r=2\lambda K^3. \tag{2.14}$$

Let $D_k=\{z\ ;\ \|z\|\leq 2K\}$, $D'=\bigcup_{t\in R}\{z\ ;\ \|z-z_0(t)\|\leq K\}$. It is clear that $z_0(t)\in D_k$ *for any* $t\in R$ *and* $D'\subset D_k$.

Let us denote the Jacobian matrix of the right-hand side of (2.7) with respect to z by $\Psi(z)$. Then we have the inequality

$$\|\Psi(z)-A\|=2\lambda(2|x|+|y|)|x|\leq 24\lambda K^2 \tag{2.15}$$

for all $z\in D'$.

In order to apply Theorem 1 to the present case, we have to check with the inequalities in (1.10). The question is *"Is it possible to take a non-negative number* $\kappa<1$ *satisfying both inequalities*

$$24\lambda K^2\leq\frac{\kappa}{M} \text{ and } 2\lambda K^3M\leq(1-\kappa)K?".$$

From the inequalities $24\lambda K^2M\leq\kappa<1$ and $2\lambda K^2M\leq 1-\kappa$, we have the inequalities $24\lambda K^2M\leq\kappa\leq 1-2\lambda K^2M$ and $26\lambda K^2M\leq 1$. Hence we have

$$0<\lambda\leq\frac{1}{26K^2M} \tag{2.16}$$

or

$$K \leq \sqrt{\frac{1}{26\lambda M}} \quad . \tag{2.17}$$

Consequently, we have the following existence and uniqueness theorem of a quasiperiodic solution to the quasiperiodic Van der Pol type equation.

THEOREM 3. *If the parameter λ and the constant number K satisfy (2.16) or (2.17), the given equation (2.6) possesses a quasiperiodic solution $z=\hat{z}(t)$ with periods $\omega_1, \omega_2, \ldots, \omega_m$ such that*

$$\| \hat{z}(t) - z_0(t) \| \leq K \tag{2.18}$$

for all $t \in R$.

If the inequalities (2.16) or (2.17) do not hold, or the error estimation (2.18) is too crude, we should compute a more accurate approximation than $z_0(t)$. For this purpose we have considered an approximate quasiperiodic solution written in the form

$$x_n(t) = \alpha(0,0) + \sum_{r=1}^{n+1} \sum_{|p|=r} \{\alpha_p \cos(p,\nu)t + \beta_p \sin(p,\nu)t\},$$

$$y_n(t) = \frac{d}{dt} x_n(t),$$

where $(p,\nu) = \sum_{k=1}^{m} p_k \nu_k$, $|p| = \sum_{k=1}^{m} |p_k|$, and we have determined the unknown coefficients $\alpha(0,0)$, α_p, β_p by means of the Galerkin method.

For the computed Galerkin approximation of n-th order as

$$\bar{x}_n(t) = \bar{\alpha}(0,0) + \sum_{r=1}^{n+1} \sum_{|p|=r} \{\bar{\alpha}_p \cos(p,\nu)t + \bar{\beta}_p \sin(p,\nu)t\},$$

we consider the residual function

$$r(t) = \frac{d^2 \bar{x}_n(t)}{dt^2} - 2\lambda \frac{d\bar{x}_n(t)}{dt} + \bar{x}_n(t) + 2\lambda \bar{x}_n^2(t) \frac{d\bar{x}_n(t)}{dt} - \sum_{k=1}^{m} (a_k \cos \nu_k t + b_k \sin \nu_k t)$$

which can be expanded into the finite double Fourier series as

$$r(t) = f(0,0) + \sum_{r=1}^{3(n+1)} \sum_{|p|=r} \{f_p \cos(p,\nu)t + g_p \sin(p,\nu)t\},$$

where 3 $(n+1)$ is considered as sufficiently large when n is large.

Put

$$r=|f(0,0)|+\sum_{r=1}^{3(n+1)}\sum_{|p|=r}\{|f_p|+|g_p|\}, \tag{2.19}$$

then we have $|r(t)|\leq r$ for all $t\in R$. Define

$$\Omega=|\bar{a}(0,0)|+\sum_{r=1}^{n+1}\sum_{|p|=r}\{|\bar{a}_p|+|\bar{\beta}_p|\} \tag{2.20}$$

and

$$\Omega'=\sum_{r=1}^{n+1}\sum_{|p|=r}\{|\bar{a}_p|+|\bar{\beta}_p|\}|(p,\nu)|, \tag{2.21}$$

then we have the inequalities $\Omega\geq\sup_{t\in R}|\bar{x}_n(t)|$ and $\Omega'\geq\sup_{t\in R}|\bar{y}_n(t)|$.

For z which lies in the δ-neighbourhood of $\bar{z}_n(t)={}^t(\bar{x}_n(t),\bar{y}_n(t))$, we have

$$\|\Psi(z)-A\|\leq 2\lambda\{\Omega(2\Omega'+\Omega)+2(\Omega'+2\Omega)\delta+3\delta^2\}.$$

If there exist a non-negative number $\kappa<1$ and a positive number δ satisfying both inequalities

$$2\lambda\{\Omega(2\Omega'+\Omega)+2(\Omega'+2\Omega)\delta+3\delta^2\}\leq\frac{\kappa}{M} \text{ and } \frac{r}{1-\kappa}\mathrm{M}\leq\delta, \text{ then}$$

from Theorem 1 the exact quasiperiodic solution $\hat{z}(t)={}^t(\hat{x}(t),\hat{y}(t))$ with periods ω_k ($k=1, 2, \dots, m$) exists and an error estimation of $\bar{z}_n(t)$ is given by

$$\|\bar{z}_n(t)-\hat{z}(t)\|\leq\frac{Mr}{1-\kappa},$$

that is,

$$|\bar{x}_n(t)-\hat{x}(t)|,\ \left|\frac{d}{dt}\bar{x}_n(t)-\frac{d}{dt}\hat{x}(t)\right|\leq\frac{Mr}{1-\kappa}$$

for all $t\in R$.

3. Numerical example

We shall consider the Van der Pol type equation (2.6) with $\nu_1=\sqrt{2}$, $\nu_2=\sqrt{5}$, $\nu_k=0$ $(k\geq 3)$[13].

As for the case $\lambda=10, a_1=a_2=1/32, a_k=0$ $(k\geq3)$ and $b_k=0$ $(k\geq1)$, we have

$$\begin{aligned}x_0(t)=2\{&-0.00001950686\cos\nu_1 t-0.0005517378\sin\nu_1 t\\&-0.00003100200\cos\nu_2 t-0.0003466129\sin\nu_2 t\}.\end{aligned}$$

In order to find a more accurate approximation, we have used the Galerkin method. After 2 iterations starting with $x_0(t)$, we have a Galerkin approximation of 7-th order as

$$\begin{aligned}\bar{x}_7(t)=2\{&-0.0000195\cos\nu_1 t-0.0005517\sin\nu_1 t\\&-0.0000310\cos\nu_2 t-0.0003466\sin\nu_2 t\},\end{aligned}$$

where all the terms whose coefficients are smaller than 10^{-7} in magnitude are omitted.

By (2.19), (2.20) and (2.21), we take $r=0.73\times 10^{-10}$, $\Omega=0.001897723$ and $\Omega'=0.003304484$.
If we take $\delta=K=0.0067$, we have

$$2\lambda\{\Omega(2\Omega'+\Omega)+2(\Omega'+2\Omega)\delta+3\delta^2\}\leq 0.004919048\leq\frac{\kappa}{M},$$

and

$$\kappa\geq 0.004919048M=0.1035579,$$

where $M=21.06$. Remark that the value of K and M are corrected. Hence we can choose κ as 0.11, then we have

$$\frac{Mr}{1-\kappa}=\frac{21.06\times 0.73\times 10^{-10}}{0.89}=17.27\ldots\times 10^{-10}<0.18\times 10^{-8}.$$

From the above calculation we have an error estimation

$$\| z_7(t)-\hat{z}(t)\|<0.18\times 10^{-8}.$$

References

[1] R. Bouc, Sur la methode de Galerkin-Urabe pour les systèmes différentieles périodiques. Intern. J. Non-Linear Mech., 7 (1972), 175–188.

[2] L.O.Chua and A. Ushida, Algorithms for computing almost-periodic steady-state response of nonlinear systems to multiple input frequencies. Memorandum No. UCB/ERL M80/55, Electronics Research Laboratory, UC, Berkeley, 1980.

[3] A.M.Fink, Almost periodic differential equations. Lecture Notes in Math.,377, Springer-Verlag, 1974.

[4] A. Kohda and Y.Shinohara, Numerical analysis of the quasiperiodic solutions to Duffing type equations. Japan J. Indust. Appl. Math.,10 (1993), 367-378.

[5] S. Maas, Nonlinear microwave circuits. Artech House Inc., 1988.

[6] T. Mitsui, Investigation of numerical solutions of some nonlinear quasiperiodic differential equations. Publ. RIMS, Kyoto Univ., 13 (1977), 793-820.

[7] F. Nakajima, Existence of quasi-periodic solutions of quasiperiodic systems. Funkcial. Ekvac.,15 (1972), 61-73.

[8] Y. Shinohara, A geometric method of numerical solution of nonlinear equations and its application to nonlinear oscillations. Publ. RIMS, Kyoto Univ., 8 (1972), 13-42.

[9] Y. Shinohara, Numerical analysis of periodic solutions and their periods to autonomous differential systems. J. Math. Tokushima Univ., 11 (1977), 11-32.

[10] Y. Shinohara, Galerkin method for autonomous differential equations. J. Math.Tokushima Univ., 15 (1981), 53–85.

[11] Y. Shinohara and N.Yamamoto, Galerkin approximation of periodic solution and its period to Van der Pol equation. J. Math .Tokushima Univ., 12 (1978), 19-42.

[12] Y. Shinohara, A. Kohda and T. Mitsui, On quasiperiodic solutions to Van der Pol equation. J. Math. Tokushima Univ., 18 (1984), 1-9.

[13] Y. Shinohara, M. Kurihara and A.Kohda, Numerical analysis of quasiperiodic

solutions to nonlinear differential equations. Japan J. Appl. Math., 3(1986), 315-330.

[14] Y. Shinohara, A. Kohda and H. Imai, Existence and uniqueness of quasiperiodic solutions to quasiperiodic nonlinear differential equations. Numerical analysis of ordinary differential equations and its applications. T. Mitsui and Y. Shinohara, eds., World Scientific, 1995, 147-163.

[15] M. Urabe, Galerkin's procedure for nonlinear periodic systems. Arch. Rational Mech. Anal., 20 (1965), 120-152.

[16] M. Urabe, Numerical investigation of subharmonic solutions to Duffing's equation. Trudy Pjator Mezdunarodnoi Konferencii po Nelineinyn Kolebanijam, Inst. Mat. Akad. Nauk USSR., Kiev, Tom 4, 1970, 21-67.

[17] M. Urabe, Green functions of pseudoperiodic differential operators. Japan-US Seminar on Ordinary Differential and Functional Equations, Lecture Notes in Math., 243, Springer-Verlag, 1971.

[18] M. Urabe, Existence theorem of quasiperiodic solutions to nonlinear differential systems. Funkcial. Ekvac., 15 (1972), 75-100.

[19] M. Urabe, On the existence of quasiperiodic solutions to nonlinear quasiperiodic differential equations. Proc. 6th ICNO, Polish Akad. Sci., Warsaw, 1972, Poznan, 1-38.

[20] M. Urabe, On a modified Galerkin's procedure for nonlinear quasiperiodic differential systems. Equations Différentielles et Fonctionnelles Non Linéares, Actes de la Conference Internationale 《Equa-Diff 73》, Hermann, Paris, 1973, 223-258.

[21] M. Urabe, On the existence of quasiperiodic solutions to nonlinear quasiperiodic differential equations. Nonlinear Vibration Problems, Zagadnienia Drgan Nieliniowych, Warsaw, 1974, 85-93.

[22] M. Urabe and A. Reiter, Numerical computation of nonlinear forced oscillations by Galerkin's procedure. J. Math.Anal Appl., 14 (1966), 107-140.

[23] N. Yamamoto, An error analysis of Galerkin approximations of periodic solution and its period to autonomous differential system. J. Math. Tokushima Univ.,13 (1979), 53–77.

[24] N. Yamamoto, Galerkin method for autonomous differential equations with unknown parameters. J. Math. Tokushima Univ., 16 (1982), 55-93.

[25] N. Yamamoto, A remark to Galerkin method for nonlinear periodic systems with unknown parameters. J. Math. Tokushima Univ., 16 (1982), 95-126.