J. Math. Tokushima Univ.
Vol. 32 (1998), 43-51

New Axiom of Quantum Mechanics —Hilbert's 6th Problem—

By

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(Received September 14, 1998)

Abstract

In this article, we give one solution of Hilbert's 6th problem. We give the new axiom of quantum mechanics. Thereby we construct the mathematically reasonable framework which is consistent with the physical interpretation. This is an answer to the Einstein and Bohr's controversy.

1991 Mathematics Subject Classification. Primary 81P05, 81P10

Introduction

In this article we give one solution of Hilbert's 6th problem. This problem is the mathematical treatment of axioms of the theories of physics. As for this problem it is the question what is a solution. Here, if we consider the problem as the axiomatic treatment of the various theories of physics, some ones have been already solved and some others are not yet solved.

Now we consider the new axiom of quantum mechanics as the problem. As for this problem, somebody considers the von Neumann's theory as a solution. But in the old quantum mechanics, we have the dissociation of the mathematical calculation and the physical interpretation. So that in this paper we construct the new theory of quantum mechanics, and thereby we construct the mathematically reasonable framework which is consistent with the physical interpretation.

This is an answer to the controversy of Einstein and Bohr.

1. Orthogonal measures and orthogonal Radon measures

In this section, we mention the orthogonal measures and the orthogonal Radon measures.

Definition 1.1 We call a triplet $(\Lambda, \mathcal{B}, \mu)$ over a set Λ a **measure space** if the following two conditions are satisfied:

1. $\mathcal{B}$ is a σ-ring of subsets of Λ.

2. μ is a σ-finite completely additive nonnegative measure over $\mathcal{B}$.

A measure space $(\Lambda, \mathcal{B}, \mu)$ is also denoted by the symbol $\Lambda = \Lambda(\mathcal{B}, \mu)$.

Definition 1.2. Let a triplet $(\Lambda, \mathcal{B}, \mu)$ over a set Λ be a measure space and ξ an $\mathcal{H}$-valued set function over the subring

$$\mathcal{B}_\mu = \{B; B \in \mathcal{B} \text{ and } \mu(B) < \infty\}.$$

Then we call ξ an **$\mathcal{H}$-valued completely additive orthogonal measure** over $(\Lambda, \mathcal{B}, \mu)$ if ξ satisfies the relation

$$(\xi(A), \xi(B)) = \mu(A \cap B), \ (A, B \in \mathcal{B}_\mu).$$

Here the measure μ_ξ defined by the relation

$$\mu_\xi(A) = |\xi(A)|^2, \ (A \in \mathcal{B}_\mu)$$

is called a **nonnegative measure** of ξ. Then we have the relation $\mu_\xi = \mu|_{\mathcal{B}_\mu}$ in Definition 1.2.

Theorem 1.3. *Let a triplet $(\Lambda, \mathcal{B}, \mu)$ is a measure space over a set Λ and ξ an $\mathcal{H}$-valued completely additive orthogonal measure over $(\Lambda, \mathcal{B}, \mu)$. Then we have the following:*

(1) *For a sequence $A_k \in \mathcal{B}_\mu$, $(k = 1, 2, \cdots)$ such that $A_j \cap A_k = \emptyset (j \neq k)$ and $\cup_{k=1}^\infty A_k \in \mathcal{B}_\mu$, we have the relation*

$$\xi(\cup_{k=1}^\infty A_k) = \Sigma_{k=1}^\infty \xi(A_k), \ (in\ \mathcal{H})$$

(2) *For A, $B \in \mathcal{B}_\mu$ such that $A \cap B = \emptyset$, we have the relation $\xi(A) \perp \xi(B)$.*

Let ξ be an $\mathcal{H}$-valued completely additive orthogonal measure and $L_{2,\mu} = L_2(\Lambda, \mathcal{B}, \mu)$ the space of all complex valued square integrable functions on Λ. Then we can define the integral of $\phi \in L_{2,\mu}$;

$$\xi(\phi) = \int_\Lambda \phi(\lambda)\xi(d\lambda)$$

which has the following two conditions:

(1) $\xi(\phi) \in \mathcal{H}$, for $\phi \in L_{2,\mu}$,

(2) $(\xi(\phi), \xi(\psi)) = (\phi, \psi)_\mu$ for $\phi, \psi \in L_{2,\mu}$, where $(,)_\mu$ denotes the inner product in $L_{2,\mu}$.

Now we mention the definition of orthogonal Radon measures and their properties.

Let Λ be an open set in $\boldsymbol{R}^n$. Let $\mathcal{K}(\Lambda) = C_c(\Lambda)$ be the space of all continuous functions with compact support on Λ. We assume that $\mathcal{K}(\Lambda)$ is endowed with the canonical LF-topology.

We call a continuous linear form on $\mathcal{K}(\Lambda)$ a **Radon measure**. We call a continuous linear map from $\mathcal{K}(\Lambda)$ to a Hilbert space $\mathcal{H}$ an **$\mathcal{H}$-valued Radon measure.**

Let μ be a **positive Radon measure**, namely, $\mu(\phi) \geq 0$ for every $\phi \in \mathcal{K}(\Lambda)$ such as $\phi \geq 0$.

Definition 1.4. We use the above notation. Let μ be a positive Radon measure on Λ and ξ an $\mathcal{H}$-valued Radon measure on Λ. Then we call ξ an **orthogonal Radon measure** on (Λ, μ) if it satisfies the relation

$$(\xi(\phi), \xi(\psi)) = \mu(\phi^* \psi), \ (\phi, \psi \in \mathcal{K}(\Lambda)).$$

Here we define ϕ^* by the relation $\phi^*(\lambda) = \phi(\lambda)^*$ and the symbol $\phi(\lambda)^*$ denotes the complex conjugate of $\phi(\lambda)$.

Corollary 1.5. *We use the notation in Definition 1.4. Then we have the relation*

$$(\xi(\phi), \xi(\psi)) = 0, \ if \ \mathrm{supp}(\phi) \cap \mathrm{supp}(\psi) = \emptyset.$$

We can extend the domain of a positive Radon measure μ so that it includes defining functions of some kind of subsets of Λ. Then we define the family $\mathcal{B}^*$ of subsets of Λ by the relation

$$\mathcal{B}^* = \{A \subset \Lambda; \mu(\chi_A) < \infty\}.$$

Then we can define the set-theoretical measure $\mu(A)$ by the relation

$$\mu(A) = \mu(\chi_A), \ (A \in \mathcal{B}^*).$$

Let $\mathcal{B} = \sigma(\mathcal{B}^*)$ be a σ-ring generated by $\mathcal{B}^*$. Then the triplet $(\Lambda, \mathcal{B}, \mu)$ is a measure space. Then we can define $\xi(E)$ by the relation

$$\xi(E) = \xi(\chi_E), \ (E \in \mathcal{B}^*).$$

Then ξ becomes an $\mathcal{H}$-valued completely additive orthogonal measure over the measure space $(\Lambda, \mathcal{B}, \mu)$. We have the relation

$$\mathcal{B}^* = \{A \subset \Lambda; \mu(A) < \infty\}.$$

Then we have the integral relation

$$\xi(f) = \int_\Lambda f(\lambda)\xi(d\lambda), \ (f \in L_{2,\mu}(\Lambda)).$$

By the restriction of this integral to the functions in $\mathcal{K}(\Lambda)$, we obtain the former orthogonal Radon measure $\xi(\phi)$ on (Λ, μ).

By these relations we can identify an orthogonal Radon measure on (Λ, μ) and the corresponding set-theoretical $\mathcal{H}$-valued completely additive orthogonal measure over the measure space $(\Lambda, \mathcal{B}, \mu)$.

2. Bra and ket vectors and braket products

In this section we mention the definitions of bra and ket vectors and braket products.

Definition 2.1. Let Λ be an open set in $\boldsymbol{R}^n$. Then we call $\mu \in \mathcal{K}'(\Lambda)$ a **Radon probability measure** if the following two conditions are satisfied:

(i) μ is a positive Radon measure.

(ii) For the constant 1 on Λ, $\mu(1)$ is defined and $\mu(1) = 1$ holds.

Definition 2.2. Let $\mathcal{H}$ be a Hilbert space over the complex number field, and Λ an open set in $\boldsymbol{R}^n$. Assume $\mu \in \mathcal{K}'(\Lambda)$ be a Radon probability measure. Then we call an $\mathcal{H}$-valued orthogonal Radon measure ξ over (Λ, μ) a **ket vector**, or simply a **ket** over Λ.

A ket vector is an $\mathcal{H}$-valued orthogonal Radon probability measure. We also denote a ket vector ξ by a symbol $|\xi\rangle$.

Theorem 2.3. *Let $\boldsymbol{K}$ be the set of all ket vectors over Λ. Then $\boldsymbol{K}$ is a subset of $\mathcal{K}'(\Lambda; \mathcal{H})$ and satisfies the following.*

If $\xi, \eta \in \boldsymbol{K}$ and $\xi + \eta$ becomes an $\mathcal{H}$-valued orthogonal Radon measure, then there exists $\mu_{\xi\eta} \in \mathcal{K}'(\Lambda)$ such that the following conditions are satisfied:

(i) $(\xi(\phi), \eta(\psi)) = \mu_{\xi\eta}(\phi^*\psi), \ (\phi, \psi \in \mathcal{K}(\Lambda))$.

(ii) $\mu_\xi = \mu_{\xi\xi}$ *is a Radon probability measure.*

(iii) $\mu_{\xi\eta} = (\mu_{\eta\xi})^*$.

Definition 2.4. Let $\boldsymbol{K}$ be as in Theorem 2.3. For $\xi \in \boldsymbol{K}$, we define ξ^* by the relation

$$\xi^*(\phi) = \xi(\phi^*)^*, \ (\phi \in \mathcal{K}(\Lambda)).$$

Then we call the above ξ^* a **bra vector**, or simply a **bra** over Λ.

We also denote a bra vector ξ^* by a symbol $\langle\xi|$.

Theorem 2.5. *Let $\mathcal{H}^*$ be the dual space of a Hilbert space $\mathcal{H}$ and $\boldsymbol{K}^*$ the set of all bra vectors over Λ. Then $\boldsymbol{K}^*$ is a subset of $\mathcal{K}'(\Lambda; \mathcal{H}^*)$ and satisfies the following. If ξ, $\eta \in \boldsymbol{K}^*$ and $\xi^* + \eta^*$ is an $\mathcal{H}$-valued orthogonal Radon measure, then there exists $\mu_{\xi^*\eta^*} \in \mathcal{K}'(\Lambda)$ such that the following conditions are satified:*

$$(\xi^*(\phi), \eta^*(\psi)) = \mu_{\xi^*\eta^*}(\phi^*\psi), \ (\phi, \psi \in \mathcal{K}(\Lambda)).$$

Here, for $\xi \in \boldsymbol{K}^*$, we define ξ^* by the relation $\xi^*(\phi) = \xi(\phi^*)^*$, $(\phi \in \mathcal{K}(\Lambda))$. Then $\xi^* \in \boldsymbol{K}$ and $\mu_{\xi^*\eta^*}$ has the same meaning as in Theorem 2.3.

The map from $\boldsymbol{K}$ to $\boldsymbol{K}^*$ defined by $\xi \longmapsto \xi^*$ is a bijective map.

Now we define the braket product of a ket vector $\eta \in \boldsymbol{K}$ and a bra vector $\xi \in \boldsymbol{K}^*$. We also denote η and ξ by symbols $|\eta\rangle$ and $\langle\xi|$ respectively. Then we define the **braket product** $\langle\xi||\eta\rangle \equiv \langle\xi|\eta\rangle$ of $\langle\xi|$ and $|\eta\rangle$ by a sesquilinear map

$$\langle\xi|\eta\rangle : (\phi, \psi) \longmapsto (\xi^*(\phi), \eta(\psi))_{\mathcal{H}}.$$

Here $(,)_{\mathcal{H}}$ denotes the inner product of $\mathcal{H}$. We usually use $\langle\xi|\eta\rangle$ instead of $\langle\xi||\eta\rangle$ for simplicity. Therefore we have

$$\langle\xi(\phi)|\eta(\psi)\rangle = (\xi^*(\phi), \eta(\psi))_{\mathcal{H}}, \ (\phi, \psi \in \mathcal{K}(\Lambda)).$$

Thus we have the relations

$$\langle\xi|\eta\rangle = \mu_{\xi^*\eta} = (\mu_{\eta\xi^*})^*.$$

Therefore, we have the relations, for $\xi, \eta \in \boldsymbol{K}$,

$$\langle\xi^*(\phi)|\eta(\psi)\rangle = (\xi(\phi), \eta(\psi)) = \mu_{\xi\eta}(\phi^*\psi), \ (\phi, \psi \in \mathcal{K}(\Lambda))$$

and, for $\xi, \eta \in \boldsymbol{K}^*$,

$$\langle\xi(\phi)|\eta^*(\psi)\rangle = (\xi^*(\phi), \eta^*(\psi)) = \mu_{\xi^*\eta^*}(\phi^*\psi), \ (\phi, \psi \in \mathcal{K}(\Lambda)).$$

The support of $\langle\xi|\eta\rangle$, $\mathrm{supp}(\langle\xi|\eta\rangle)$, is contained in the diagonal set of $\Lambda \times \Lambda$.

3. The new axiom of quantum mechanics

The problems of quantum mechanics are as follows:

1. To clarify the quantum state of a system of microparticles and the law of its time evolution.

2. To calculate the expectations of physical quantities of a system of microparticles and the statistics.

The characteristics of quantum systems are as follows:

1. Twofoldness of particle and wave.

2. Discreteness of physical quantities.

3. Canonical commutation relations and Heisenberg's relations of uncertainty.

In the sequel, we formulate the new axiom of quantum mechanics, by which we can explain those characteristics of quantum mechanics, and obtain the solutions of the problems above.

Axiom I (quantum system). We assume that a quantum system Ω is a probability space $(\Omega, \mathcal{B}, P)$. Here Ω is an ensemble of microparticles ω, $\mathcal{B}$ is a

σ-algebra of subsets of Ω, and P is a completely additive probability measure on $\mathcal{B}$.

An ensemble of microparticles, namely a quantum system, is generally an infinite ensemble. For example, we can consider an ensemble of atoms or molecules as a quantum system.

Examples (1) One particle system(a free particle system). This is a quantum system whose elementary event is composed of only one microparticle. For example, we have an ensemble of free electrons.

(2) **Two particles system.** This is a quantum system whose elementary event is composed of two combined particles. For example, we have an ensemble of hydrogen atoms. One hydrogen atom is a combined system of one atomic nucleus and one electron. One hydrogen atom is an elementary event of this quantum system.

(3) n **particles system.** This is a quantum system whose elementary event is composed of n combined particles. For example, we have an ensemble of general atoms. One atom is a combined system of one atomic nucleus and $n-1$ electrons. One atom is an elementary event of this quantum system.

Axiom II (quantum state). We assume that the quantum state of a quantum system $\Omega = \Omega(\mathcal{B}, P)(= (\Omega, \mathcal{B}, P))$ is the state of quantum probability distribution of position variables $\boldsymbol{r}(\omega)$ and momentum variables $\boldsymbol{p}(\omega)$ of microparticles composing the quantum system. Here we consider the orthogonal coordinate systems of n-dimensional Euclidean space $\boldsymbol{R}^n$ and its dual space $\boldsymbol{R}_n$, and $n = dN$. Here d is the dimension of the physical space and N is the number of particles composing one elementary event ω.

The quantum state is determined as follows.

($\mathbf{II_1}$) The quantum probability distribution of position variables $\boldsymbol{r} = \boldsymbol{r}(\omega)$ is described by a ket vector $|\psi\rangle$ or ψ, which is an orthogonal Radon probability measure corresponding to an L^2-function ψ on $\boldsymbol{R}^n$.

($\mathbf{II_2}$) The quantum probability distribution of motion variables $\boldsymbol{p} = \boldsymbol{p}(\omega)$ is described by the ket vector $|\hat{\psi}\rangle$ or $\hat{\psi}$, which is the Fourier transform of ψ:

$$\hat{\psi}(\boldsymbol{p}) = (2\pi\hbar)^{-n/2} \int \psi(\boldsymbol{r}) e^{-i(\boldsymbol{p}\cdot\boldsymbol{r})/\hbar} d\boldsymbol{r},$$

$$\psi(\boldsymbol{r}) = (2\pi\hbar)^{-n/2} \int \hat{\psi}(\boldsymbol{p}) e^{i(\boldsymbol{p}\cdot\boldsymbol{r})/\hbar} d\boldsymbol{p},$$

$$\boldsymbol{r} = (x_1, x_2, \cdots, x_n),\ \boldsymbol{p} = (p_1, p_2, \cdots, p_n),\ \boldsymbol{p}\cdot\boldsymbol{r} = p_1x_1 + p_2x_2 + \cdots + p_nx_n.$$

Here we put $\hbar = h/2\pi$ and h is Planck's constant.

We note that the above Fourier transformation is the classical one.

($\mathbf{II_3}$) We put $\mu = \langle\psi^*|\psi\rangle$. Then μ is a Radon probability measure on $\boldsymbol{R}^n$. We identify μ and the corresponding probability measure in the set-theoretical sense, which we denote by the same symbol μ. Then we have, for a μ-measurable set A in $\boldsymbol{R}^n$,

$$P(\{\omega \in \Omega; \boldsymbol{r}(\omega) \in A\}) = \mu(A).$$

This $\mu(A)$ denotes the probability with which the position variable $\boldsymbol{r}(\omega)$ belongs to the region A. Then we have a probability space $(\boldsymbol{R}^n, \mathcal{B}_n, \mu)$, where $\mathcal{B}_n$ is a family of μ-measurable events.

(II$_4$) We put $\hat{\mu} = \langle \hat{\psi}^* | \hat{\psi} \rangle$. Here $\hat{\mu}$ is only the new symbol which does not mean the Fourier transform of μ. Then $\hat{\mu}$ is a Radon probability measure on $\boldsymbol{R}_n$. We also identify $\hat{\mu}$ and the corresponding probability measure in the set-theoretical sense. Then we have, for a $\hat{\mu}$-measurable set B in $\boldsymbol{R}_n$,

$$P(\{\omega \in \Omega; \boldsymbol{p}(\omega) \in B\}) = \hat{\mu}(B).$$

This $\hat{\mu}(B)$ denote the probability with which the momentum variable $\boldsymbol{p}(\omega)$ belongs to the region B. Then we have a probability space $(\boldsymbol{R}_n, \hat{\mathcal{B}}_n, \hat{\mu})$, where $\hat{\mathcal{B}}_n$ is a family of $\hat{\mu}$- measurable events.

Axiom III (motion of a quantum system). We call the time evolution of the state of a quantum system the motion of the quantum system. The law of the motion of a quantum system is described by a Schrödinger equation. We call the Schrödinger equation the equation of motion of the quantum system.

A Schrödinger equation is defined by an equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi, \ \psi = \psi(t, \boldsymbol{r}).$$

We call the operator H a Hamiltonian, which has a various form corresponding to each quantum system. H is assumed to be a self-adjoint operator on the image space $\mathcal{H}$ of the ket vector ψ. $\mathcal{H}$ is a Hilbert space.

Theorem 3.1(the principle of superposition of quantum states). *Let ψ, ψ_1, ψ_2 be three solution kets of the Schrödinger equation of a quantum system*

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi.$$

Assume we have a relation

$$\psi = \alpha_1 \psi_1 + \alpha_2 \psi_2, \ (\alpha_1, \alpha_2 \in \boldsymbol{C}),$$

where $\boldsymbol{C}$ is the complex number field. Put

$$\langle \psi_1^* | \psi_1 \rangle = \mu_1, \ \langle \psi_2^* | \psi_2 \rangle = \mu_2, \ \langle \psi^* | \psi \rangle = \mu, \ \langle \psi_1^* | \psi_2 \rangle = \mu_{12}.$$

Then we have the relation

$$\mu = |\alpha_1|^2 \mu_1 + \alpha_1^* \alpha_2 \mu_{12} + \alpha_1 \alpha_2^* \mu_{12}^* + |\alpha_2|^2 \mu_2.$$

Using this principle, we can explain the twofoldness of particle and wave.

Now we consider stationary states and boundary conditions. The equation of motion of a quantum system is given by a Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi.$$

If this equation does not involve the time variable explicitly, this system corresponds to a conservative system. In this case the solution of this equation is represented as

$$\psi = \phi e^{-iEt/\hbar},$$

$$(*) \qquad H\phi = E\phi.$$

Here E is a real number. The equation (*) is a Schrödinger equation independent of the time variable. Then ψ is said to be the stationary state with energy E and ϕ is said to be the wave function of the stationary state. The eigenvalue problem of the equation (*) is to obtain its eigenvlues and eigenfunctions. Under the special conditions, this state is said to be a **restraint state**. In this case, eigenvalues are discrete. This explain the discreteness of physical quantities.

Definition 3.2 (expectation of dynamical variables). Dynamical variables M and N are assumed to be functions $M = M(\boldsymbol{r})$ and $N = N(\boldsymbol{p})$ of $\boldsymbol{r}$ and $\boldsymbol{p}$ respectively. The expectation values $\langle M\rangle$ and $\langle N\rangle$ of M and N, respectively, are defined by the relations

$$\langle M\rangle = \langle M(\boldsymbol{r})\rangle = \langle \psi^*|M(\boldsymbol{r})|\psi\rangle = \int \psi^*(t,\boldsymbol{r})M(\boldsymbol{r})\psi(t,\boldsymbol{r})d\boldsymbol{r},$$

$$\langle N\rangle = \langle N(\boldsymbol{p})\rangle = \langle \hat{\psi}^*|N(\boldsymbol{p})|\hat{\psi}\rangle = \int \hat{\psi}^*(t,\boldsymbol{p})N(\boldsymbol{p})\hat{\psi}(t,\boldsymbol{p})d\boldsymbol{p}.$$

Theorem 3.3. *For dynamical variables $M(\boldsymbol{r})$ and $N(\boldsymbol{p})$, we have the relations*

$$\langle N(\boldsymbol{p})\rangle = \int \psi^*(t,\boldsymbol{r})N(\frac{\hbar}{i}\nabla)\psi(t,\boldsymbol{r})d\boldsymbol{r},$$

$$\langle M(\boldsymbol{r})\rangle = \int \hat{\psi}^*(t,\boldsymbol{p})M(i\hbar\nabla_{\boldsymbol{p}})\hat{\psi}(t,\boldsymbol{p})d\boldsymbol{p}.$$

Here $\nabla = \nabla_{\boldsymbol{r}}$ and $\nabla_{\boldsymbol{p}}$ denote the gradient operator with respect to $\boldsymbol{r}$ and $\boldsymbol{p}$, respectively.

In the integral representation above, operators $N(\frac{\hbar}{i}\nabla)$ and $M(i\hbar\nabla_{\boldsymbol{p}})$ work on those kets ψ and $\hat{\psi}$ on the right hand sides, respectively.

Examples We have the following:

$$\langle x\rangle = \int \psi^*(t,\boldsymbol{r})x\psi(t,\boldsymbol{r})d\boldsymbol{r},$$

$$\langle p_x\rangle = \int \hat{\psi}^*(t,\boldsymbol{p})p_x\hat{\psi}(t,\boldsymbol{p})d\boldsymbol{p} = \int \psi^*(t,\boldsymbol{r})(\frac{\hbar}{i}\frac{\partial}{\partial x})\psi(t,\boldsymbol{r})d\boldsymbol{r},$$

$$\Delta x^2 = \langle (x - \langle x\rangle)^2\rangle,\ \Delta p_x^2 = \langle (p_x - \langle p_x\rangle)^2\rangle,$$

etc..

Theorem 3.4. *If $A(\boldsymbol{r}, -i\hbar\nabla)$ is an Hermitian operator, we have the relation*

$$\langle\psi^*|A(\boldsymbol{r}, \frac{\hbar}{i}\nabla)|\psi\rangle = \langle\hat{\psi}^*|A(i\hbar\nabla_{\boldsymbol{p}}, \boldsymbol{p})|\hat{\psi}\rangle.$$

Now we consider canonical commutation relations and uncertainty relations. In the position representation, operators x, $(\hbar/i)(\partial/\partial x), \cdots$, etc. correspond to dynamical variables x, $p_x, \cdots$, etc., respectively. Then we have canonical commutation relations

$$[x, p_x] = xp_x - p_x x = i\hbar,$$

etc.. Then we have the following.

Theorem 3.5(Heisenberg's uncertainty relation). *We consider an one-dimensional quantum system. Let Δx and Δp be the standard deviations of the position variable and the momentum variable, respectively. Then we have the Heisenberg's uncertainty relation*

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}.$$

In the other quantum systems, we have the Heisenberg's uncertainty relation for a pair of the canonical conjugate dynamical variables which satisfy the canonical commutation relation.

In this framework of the new theory of quantum mechanics, L^2 wave functions in the old quantum mechanics can be reasonably interpreted if the interpretation is exchanged by the new one.

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