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L_2-estimates and Existence Theorems for the Exterior Differential Operators

By

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Abstract

In this article, we obtain the notion of real-pseudoconvex domains in $\boldsymbol{R}^n$, and we prove L_2-estimates and existence theorems for the exterior differential operators. For a real-pseudoconvex domain Ω, we have the vanishing of the de Rham cohomology $H^p(\Omega, \boldsymbol{R}) = 0$, $p > 0$. Thereby we can prove the global solvability of exterior differential equations in several spaces of functions and generalized functions in a real-pseudoconvex domain in $\boldsymbol{R}^n$. These are analogs of Poincaré's Lemma in several categories.

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Introduction

At the Conference of Mathematical Society of Japan, October 1992 at Nagoya University, Gromov lectured about manifolds determined by functional inequalities. Among them there exist pseudoconvex domains in $\boldsymbol{C}^n$. For a pseudoconvex domain Ω in $\boldsymbol{C}^n$, we have $H^p(\Omega, \mathcal{O}) = 0$, $p > 0$, where $\mathcal{O}$ denotes the sheaf of holomorphic functions over $\boldsymbol{C}^n$. When I heard his lecture, I had a question. Namely what are real analogs of pseudoconvex domains? Using a similar method to that of Hörmander for $\bar{\partial}$-operators[5], [6], we have L_2-estimates and existence theorems for the exterior differential operators. As a result we obtain the notion of real-pseudoconvex domains in $\boldsymbol{R}^n$. For a real-pseudoconvex domain Ω, we have the vanishing of the de Rham cohomology $H^p(\Omega, \boldsymbol{R}) = 0$, $p > 0$. Thereby we can prove the global solvability of exterior differential equations in several spaces of functions and generalized functions in a real-pseudoconvex domain in $\boldsymbol{R}^n$. These are analogs of Poincaré's Lemma in several categories.

Pseudoconvex domains in $\boldsymbol{C}^n$ are not in general real-pseudoconvex domains in $\boldsymbol{R}^{2n}$. Convex sets in $\boldsymbol{C}^n$ and disjoint unions of several convex sets in $\boldsymbol{C}^n$ are not only pseudoconvex but also real-pseudoconvex.

In general, it is hard to calculate the de Rham cohomologies and to prove their vanishing. We here succeeded in characterizing general regions in $\boldsymbol{R}^n$ where the de Rham cohomologies vanish.

It is known that the de Rham cohomologies do not vanish in general for pseudoconvex domains in $\boldsymbol{C}^n$ (cf. Hörmander[6], p.59).

In this article, we always consider real-valued functions without explicit mention of the contrary. When we quote the equation (1) in the section 1 for example, we denote it (1.1).

Here I wish to express my hearty thanks to Professors H. Komatsu and T. Kori for many valuable advices and discussions during the preparations of this work.

1. Subharmonic functions

In this section we remember the well-known facts on subharmonic functions. As for subharmonic functions, we refer to Hörmander[6] and Radó[18].

Let $\boldsymbol{R}^n$ be the n-dimensional, real, Euclidean spaces and D a domain in $\boldsymbol{R}^n$. Let $u(x)$ be a real C^2-function defined in D. If u satisfies the Laplace equation

$$\Delta u = \partial^2 u/\partial x_1^2 + \cdots + \partial^2 u/\partial x_n^2 = 0, \ (x = (x_1, \cdots, x_n) \in D),$$

we say that u is harmonic in D or a harmonic function in D. Harmonic functions are by definition 2-times continuously differentiable. In fact, they are real-analytic in D.

As for subharmonic functions, we give the following definition.

Definition 1.1. Let $u(x)$ be a real function defined in a domain D of $\boldsymbol{R}^n$. Then we say u to be subharmonic if it satisfies the following three conditions:

(i) $-\infty \leq u < +\infty$, $u \not\equiv -\infty$.

(ii) u is upper semicontinuous.

(iii) For each $x_0 \in D$ and every closed ball $B = B(x_0, r)$ with center at x_0 and of radius r contained in D, $u(x)$ satisfies the following inequality

$$u(x_0) \leq \frac{1}{\omega_n r^n} \int_B u d\lambda \equiv A(x_0, r). \tag{1}$$

Here $d\lambda$ denotes the Lebesgue measure in $\boldsymbol{R}^n$ and ω_n denotes the volume of the unit ball in $\boldsymbol{R}^n$.

For the case $n \geq 2$, we can replace the condition (iii) in the above definition by the following condition (iii)$'$:

(iii)$'$ In the notation in (iii), $u(x)$ satisfies the inequality

$$u(x_0) \leq \frac{1}{\sigma_n r^{n-1}} \int_{\partial B} u d\sigma \equiv L(x_0, r). \tag{2}$$

Here $d\sigma$ denotes the measure on the sphere ∂B and σ_n denotes the surface area of the unit ball in $\boldsymbol{R}^n$.

In general, we say that the maximum principle holds for a family of functions defined in a common domain D if none of them attains its maximum in a interior point of the domain D unless it is constant.

It is known that an upper semicontinuous function u in D is subharmonic if and only if, for every subdomain $D' \subset D$ and every harmonic function h in D', (the family of) $u - h$ satisfies the maximum principle.

In case where u is of class C^2 in D, u is subharmonic in D if and only if u satisfies the inequality

$$\Delta u = \partial^2 u/\partial x_1^2 + \cdots + \partial^2 u/\partial x_n^2 \geq 0,$$

$$(x = (x_1, \cdots, x_n) \in D).$$

Even if u is not differentiable in the ordinary sense, an upper semicontinuous function u in D is subharmonic if and only if Δu is a positive measure in D where the differentiation is taken in the sense of distribution in D.

Now we mention the properties of subharmonic functions.

Theorem 1.2. *Let D be a domain in $\boldsymbol{R}^n$. Then we have the following:*

(1) *If $u_1, \cdots, u_k$ are subharmonic in D and $a_1, \cdots, a_k$ are positive constants, then $a_1u_1 + \cdots + a_ku_k$ and $\max(u_1(x), \cdots, u_k(x))$ are subharmonic in D.*

(2) *For a subharmonic function u in D, the function obtained by replacing u in the interior of a closed ball B contained in a domain D of $\boldsymbol{R}^n (n \geq 2)$ by the Poisson integral with the boundary value u is subharmonic in D.*

(3) *If u is subharmonic in D and $f(t)$ is monotone-increasing convex function in $\boldsymbol{R}$, then $f(u)$ is subharmonic in D. In particular, if $v \geq 0$ and $\log v$ is subharmonic in D, then v is subharmonic in D.*

(4) *If $f(z)$ is holomorphic in a domain D of $\boldsymbol{C}$, then $\log|f(z)|$ and $|f(z)|$ are subharmonic in D.*

(5) *If h is harmonic in D, then $|h|$ is subharmonic in D.*

Now we remark that, in the conditions (iii) in Definition 1.1 and (iii)$'$ below it, it is sufficient that, for every $x_0 \in D$, there exists $r(x_0) > 0$ such that (1.1) or (1.2) holds for every $r(0 < r < r(x_0))$. Further we remark that we have $-\infty < A(x_0, r) \leq L(x_0, r)$ and A and L both $\downarrow u(x_0)$ when $r \downarrow 0$. If u is subharmonic in D, then u is integrable on each compact set in D. A and L are both monotone-increasing with respect to r and convex functions of $-\log r (n = 2)$ or $r^{2-n} (n \geq 3)$, hence they are continuous functions of r. Assuming $D' \cup \partial D' \subset D$, let r_0 be the distance from ∂D and fix r arbitrary as $0 < r < r_0$. Then $A(x, r)$ is a continuous subharmonic function of x in D'. Let $A_k(x, r)$ be the k-times average of type $A(x, r)$ of u. Then $A_k(x, r)$ is a subharmonic C^{k-1}-function of x and decreasingly converges to $u(x)$ when $r \downarrow 0$. If we choose a function $\varphi_r((x_1^2 + \cdots + x_n^2)^{1/2})$ suitably, the convolution $u * \varphi_r$ becomes a subharmonic C^∞-function and decreasingly converges to u when $r \downarrow 0$.

Now we consider the sequence of subharmonic functions. We have the following theorem.

Theorem 1.3. *Let D be a domain in $\boldsymbol{R}^n$. Then we have the following:*

(1) *The limit of a monotone-decreasing sequence of subharmonic functions in D(or lower-bounded directed family in D) is subharmonic in D or equal to $-\infty$ identically.*

(2) *The limit of a uniform convergence sequence of subharmonic functions in D is subharmonic in D.*

(3) *If $u_1, u_2, \cdots$ are subharmonic in D, then* $\sup(u_1, u_2, \cdots)$ *is not necessarily subharmonic. But if the function $u(x) = \sup\{u_1(x), u_2(x), \cdots\}$, $(x \in D)$ is bounded on every compact set in D, then it is equal to a certain subharmonic function in D except on a set of capacity 0.*

2. Real-plurisubharmonic functions

In this section we define the notion of real-plurisubharmonic functions and mention some of their properties.

Definition 2.1. A real function u defined in an open set $\Omega \subset \boldsymbol{R}^n$ with values in $[-\infty, +\infty)$ is called real-plurisubharmonic if

(a) u is upper semicontinuous.

(b) For arbitrary x and $w \in \boldsymbol{R}^n$, the function $t \to u(x + tw)$ is subharmonic in the part of $\boldsymbol{R}$ where it is defined.

We denote the set of all such functions by $P(\Omega)$.

We remark that the concept of real-plurisubharmonic functions is different from that of plurisubharmonic functions.

Here we mention the properties of real-plurisubharmonic functions. Some properties are immediate consequences of their analogues of subharmonic functions. We mention several other properties.

Theorem 2.2. *We use the notation in Definition 2.1. A function $u \in C^2(\Omega)$ is real-plurisubharmonic if and only if*

$$\Sigma_{j,k=1}^{n} \partial^2 u(x)/\partial x_j \partial x_k w_j w_k \geq 0, \ (x \in \Omega, \ w \in \boldsymbol{R}^n).$$

Proof. The theorem follows from the fact that the function $t \rightarrow u(x + tw)$ is convex. Q.E.D.

Theorem 2.3. *Let Ω be an open set in $\boldsymbol{R}^n$. Let $0 \leq \varphi \in C_0^\infty(\boldsymbol{R}^n)$ be equal to 0 when $|x| > 1$, let φ depend only on $|x_1|, \cdots, |x_n|$, and assume that $\int \varphi(x) d\lambda(x) = 1$ where $d\lambda$ is the Lebesgue measure in $\boldsymbol{R}^n$. If u is real-plurisubharmonic in Ω, it follows that*

$$u_\varepsilon(x) = \int u(x - \varepsilon\xi)\varphi(\xi) d\lambda(\xi)$$

is real-plurisubharmonic, that $u_\varepsilon \downarrow u$ when $\varepsilon \downarrow 0$. (We assume that u is not equal to $-\infty$ identically).

Proof. In section 1, we proved that u_ε decreases when $\varepsilon \downarrow 0$ in the case $n = 1$. Iteration of this result shows that u_ε is also decreasing if $n > 1$, and from the case $n = 1$ we also immediately find that $u \leq u_\varepsilon$. Since $\limsup_{\varepsilon \to 0} u_\varepsilon \leq u$ in view of the upper semicontinuity of u, we conclude that $u_\varepsilon \searrow u$ when $\varepsilon \downarrow 0$. That u_ε is real-plurisubharmonic follows from Theorem 1.3. Q.E.D.

Theorem 2.4. *Let $\Omega \subset \boldsymbol{R}^n$ and $\Omega' \subset \boldsymbol{R}^m$, let f be a C^2-map of Ω into Ω', and let $u \in P(\Omega')$. Then $f^*u \in P(\Omega)$.*

Proof. First asuume that $u \in C^2(\Omega')$. Then we have

$$\Sigma_{j,k=1}^{n} \partial^2 u(f(x))/\partial x_j \partial x_k w_j w_k$$

$$= \Sigma_{j,k=1}^{n} \partial^2 u/\partial f_j \partial f_k v_j v_k \geq 0, \ w \in \boldsymbol{R}^n$$

where we write $v_j = \Sigma_{i=1}^{n} w_i \partial f_j/\partial x_i$. Hence $f^*u \in P(\Omega)$. For a general $u \in P(\Omega')$, we only have to use Theorem 2.3 to choose a sequence of real-plurisubharmonic C^∞-functions which decrease toward u. Then Theorem 1.3 shows immediately that the limit of a decreasing sequence of real-plurisubharmonic functions is real-plurisubharmonic. This completes the proof. Q.E.D.

3. Real-pseudoconvex domains

Definition 3.1. Let Ω be an open set in $\boldsymbol{R}^n$. Then Ω is said to be a real-pseudoconvex domain if there exists a continuous and real-plurisubharmonic function u in Ω such that $\Omega_c = \{x; x \in \Omega, \ u(x) < c\} \subset\subset \Omega$ for every $c \in \boldsymbol{R}$.

Theorem 3.2. *If Ω_α is a real-pseudoconvex open set for every α in an index set A, then the interior Ω of $\bigcap_{\alpha\in A}\Omega_\alpha$ is also real-pseudoconvex.*

Proposition 3.3. *A convex open set in $\boldsymbol{R}^n$ is a real-pseudoconvex domain. A disjoint union of several number of convex open sets is also a real-pseudoconvex domain.*

Theorem 3.4. *Let Ω, Ω' be two open sets in $\boldsymbol{R}^n$. Assume that Ω is a real-pseudoconvex domain. Then if there exists a C^2-diffeomorphism of Ω onto Ω', Ω' is a real-pseudoconvex domain.*

4. L_2-estimates for the exterior differential operators

Let Ω be an open set in $\boldsymbol{R}^n$. If φ is a continuous function in Ω, we denote by $L_2(\Omega,\varphi)$ the space of functions in Ω which are square integrable with respect to the measure $e^{-\varphi}d\lambda$, where $d\lambda$ is the Lebesgue measure. This is a subspace of the space $L_{2,\mathrm{loc}}(\Omega)$ of functions in Ω which are locally square integrable with respect to the Lebesgue measure, and it is clear that every function in $L_{2,\mathrm{loc}}(\Omega)$ belongs to $L_2(\Omega,\varphi)$ for some φ. By $L_2^p(\Omega,\varphi)$ we denote the space of forms of degree p with coefficients $L_2(\Omega,\varphi)$,

$$f=\Sigma'_{|I|=p}f_I dx^I,\ dx^I=dx^{i_1}\wedge\cdots\wedge dx^{i_p},$$

where Σ' means that the summation is performed only over strictly increasing multi-indices. We set

$$|f|^2=\Sigma'_I|f_I|^2$$

and

$$\| f \|'_\varphi=\int|f|^2e^{-\varphi}d\lambda.$$

It is clear that $L_2(\Omega,\varphi)$ is a Hilbert space with this norm. Similarly we define $L^p_{2,\mathrm{loc}}(\Omega)$ and $\mathcal{D}^p(\Omega)$ where $\mathcal{D}(\Omega)$ is a symbol for $C_0^\infty(\Omega)$. The space $\mathcal{D}^p(\Omega)$ is of course dense in $L_2^p(\Omega,\varphi)$ for every φ.

If φ_1 and φ_2 are two continuous functions in Ω, then the operator d defines a linear, closed, densely defined operator

$$T:L_2^p(\Omega,\varphi_1)\longrightarrow L_2^{p+1}(\Omega,\varphi_2).$$

Namely, an element $u\in L_2^p(\Omega,\varphi_1)$ is in D_T if du, defined in the sense of distribution theory, belongs to $L_2^{p+1}(\Omega,\varphi_2)$, and then we set $Tu=du$. That T is closed follows from the fact that the differentiation is a continuous operation in distribution theory, and the domain is dense since it contains $\mathcal{D}^p(\Omega)$.

For suitable densities, we want to prove that the range of T consists of all $f\in L_2^{p+1}(\Omega,\varphi_2)$ such that $df=0$ (which is of course a necessary condition for f

to be in the range of T). The following lemma reduces this question to the study of an estimate.

Lemma 4.1. *Let T be a linear, closed, densely defined operator from one Hilbert space H_1 to another H_2, and let F be a closed subspace of H_2 containing the range R_T of T. Then $F = R_T$ if and only if for some constant C*

$$\| f \|_{H_2} \leq C \| T^* f \|_{H_1}, \quad f \in F \cap D_{T^*}. \tag{1}$$

Proof. See Hörmander[6], Lemma 4.1.1, p.78. Q.E.D.

In proving approximation theorems, we shall need the information concerning the operator T^* which follows from (4. 1).

Lemma 4.2. *Let T be a linear, closed, densely defined operator from one Hilbert space H_1 to another H_2, and let F be a closed subspace of H_2 containing the range R_T of T. Assume that (4.1) is valid. For every $v \in H_1$ which is orthogonal to the null space of T, one can then find $f \in D_{T^*}$ such that $T^* f = v$ and*

$$\| f \|_{H_2} \leq C \| v \|_{H_1} . \tag{2}$$

Proof. See Hörmander[6], Lemma 4.1.2, p.79. Q.E.D.

In our application of Lemma 4.1, the spaces H_1 and H_2 will be $L_2^p(\Omega, \varphi_1)$ and $L_2^{p+1}(\Omega, \varphi_2)$, respectively, T the operator between the spaces defined as explained above by the d operator, and F the set of all $f \in L_2^{p+1}(\Omega, \varphi_2)$ with $df = 0$ (in the sense of distribution theory). Let φ_3 be another continuous function and let S be the operator from $L_2^{p+1}(\Omega, \varphi_2)$ to $L_2^{p+2}(\Omega, \varphi_3)$ defined by d. Then F is the null space of S and to prove (4.1) it is sufficient to show that

$$\| f \|_{\varphi_2}^2 \leq C^2 (\| T^* f \|_{\varphi_1}^2 + \| Sf \|_{\varphi_3}^2), \quad f \in D_{T^*} \cap D_S, \tag{3}$$

for the last term drops out when f is in the null space of S. If the densities are suitably chosen, it is enough to prove (4.3) when $f \in \mathcal{D}^{p+2}(\Omega)$, for we have the following lemma.

Lemma 4.3. *Let η_ν, $\nu = 1, 2, \cdots$ be a sequence of functions in $C_0^\infty(\Omega)$ such that $0 \leq \eta_\nu \leq 1$ and $\eta_\nu = 1$ on any compact subset of Ω when ν is large. Suppose that $\varphi_2 \in C^1(\Omega)$ and that*

$$e^{-\varphi_{j+1}} \Sigma_{k=1}^n |\partial \eta_\nu / \partial x_k|^2 \leq e^{-\varphi_j}, \quad j = 1, 2, \ \nu = 1, 2, \cdots. \tag{4}$$

Then $\mathcal{D}^{p+1}(\Omega)$ is dense in $D_{T^*} \cap D_S$ for the graph norm

$$f \longrightarrow \| f \|_{\varphi_2} + \| T^* f \|_{\varphi_1} + \| Sf \|_{\varphi_3} .$$

Note that (4.4) means only a finite number of bounds for $\varphi_j - \varphi_{j+1}$ on each compact subset of Ω, so one can always find continuous functions φ_1, φ_2, φ_3 satisfying (4.4).

Proof of the Lemma 4.3. Since

$$S(\eta_\nu f) - \eta_\nu Sf = d\eta_\nu \wedge f,\ f \in D_S,$$

it follows from (4.4) that

$$|S(\eta_\nu f) - \eta_\nu Sf|^2 e^{-\varphi_3} \leq |f|^2 e^{-\varphi_2}.$$

Hence the dominated convergence theorem gives

$$\| S(\eta_\nu f) - \eta_\nu Sf \|_{\varphi_3} \longrightarrow 0 \text{ when } \nu \longrightarrow \infty,\ f \in D_S. \tag{5}$$

If $f \in D_{T^*}$ and $\eta \in C_0^\infty(\Omega)$, it follows that $\eta f \in D_{T^*}$. In fact,

$$\begin{aligned}(\eta f, Tu)_{\varphi_2} &= (f, \eta Tu)_{\varphi_2} = (f, T(\eta u))_{\varphi_2} + (f, \eta Tu - T(\eta u))_{\varphi_2} \\ &= (\eta T^* f, u)_{\varphi_1} + (f, \eta Tu - T(\eta u))_{\varphi_2},\ u \in D_T.\end{aligned}$$

Since no derivative of u occurs in the last term, it follows that $(\eta f, Tu)_{\varphi_2}$ is continuous for the norm $\| u \|_{\varphi_1}$, so there is an element $v \in L_2^p(\Omega, \varphi_1)$ with

$$(v, u)_{\varphi_1} = (\eta f, Tu)_{\varphi_2},\ u \in D_T.$$

This means that $\eta f \in D_{T^*}$ and that $T^*(\eta f) = v$. When $\eta = \eta_\nu$, we obtain by estimating $\eta_\nu Tu - T(\eta_\nu u)$ as in the proof of (4.5) that

$$|(T^*(\eta_\nu f) - \eta_\nu T^* f, u)_{\varphi_1}| \leq \int |f| e^{-\varphi_2/2} |u| e^{-\varphi_1/2} d\lambda,$$

which implies the bound

$$|T^*(\eta_\nu f) - \eta_\nu T^* f|^2 e^{-\varphi_1} \leq |f|^2 e^{-\varphi_2}.$$

As above, we can therefore conclude by dominated convergence that

$$\| T^*(\eta_\nu f) - \eta_\nu T^* f \|_{\varphi_1} \longrightarrow 0 \text{ when } \nu \longrightarrow \infty \text{ if } f \in D_{T^*}. \tag{6}$$

Hence $\eta_\nu f \longrightarrow f$ in the graph norm if $f \in D_{T^*} \cap D_S$.

To complete the proof we only have to approximate elements $f \in D_{T^*} \cap D_S$ with compact support in Ω by elements in $\mathcal{D}^{p+1}(\Omega)$. This requires an elementary lemma:

Lemma 4.4. *Let χ be a function in $C_0^\infty(\boldsymbol{R}^n)$ with $\int \chi dx = 1$, and set $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$, $x \in \boldsymbol{R}^n$. If $g \in L_2(\boldsymbol{R}^n)$, it follows that*

$$g * \chi_\varepsilon(x) = \int g(y)\chi_\varepsilon(x-y)dy = \int g(x - \varepsilon y)\chi(y)dy$$

is a C^∞-function such that $\| g * \chi_\varepsilon - g \|_{L_2} \longrightarrow 0$ when $\varepsilon \longrightarrow 0$. The support of $g * \chi_\varepsilon$ has no points at distance $> \varepsilon$ from the support of g if the suport of χ lies in the unit ball.

Proof. See Hörmander[6], Lemma 4.1.4, p.81. Q.E.D.

Using the above lemma we complete the proof of Lemma 4.3. If $f \in D_{T^*} \cap D_S$ has a compact support, we define $f * \chi_\varepsilon$ by choosing χ as in Lemma 4.4 and letting the convolution act on each coefficient of f. The support of $f * \chi_\varepsilon$ is then contained in a fixed compact subset of Ω when $\varepsilon \longrightarrow 0$, and the lemma gives that $\| f - f * \chi_\varepsilon \|_{\varphi_1} \longrightarrow 0$. Since $S(f * \chi_\varepsilon) - (Sf) * \chi_\varepsilon$, we also obtain from the lemma that $\| Sf - S(f * \chi_\varepsilon) \|_{\varphi_3} \longrightarrow 0$. The operator T^* does not have constant coefficients but we can write $e^{\varphi_2 - \varphi_1} T^* = \vartheta + a$ where ϑ is a constant coefficient differential operator and a is of degree 0. Since

$$(\vartheta + a)(f * \chi_\varepsilon) = ((\vartheta + a)f) * \chi_\varepsilon + a(f * \chi_\varepsilon) - (af) * \chi_\varepsilon$$

and the right-hand side is L_2-convergence to the limit $(\vartheta + a)f + af - af$ according to Lemma 4.4, it follows that $\| T^*(f * \chi_\varepsilon) - T^* f \|_{\varphi_1} \longrightarrow 0$, which completes the proof of Lemma 4.3. Q.E.D.

We now actually compute T^*, which also gives another proof of (4.6). Thus choose

$$u = \Sigma'_{|I|=p} u_I dx^I \in \mathcal{D}^p(\Omega),$$

$$f = \Sigma'_{|J|=p+1} f_J dx^J \in L_2^{p+1}(\Omega, \varphi_2).$$

Since f_J is defined for all J as an antisymmetric function of the indices in J, and

$$du = \Sigma'_{|I|=p} \Sigma_{j=1}^n \partial u_I / \partial x^j dx^j \wedge dx^I,$$

we obtain, if $f \in D_{T^*}$,

$$\int \Sigma'_I (T^* f)_I u_I e^{-\varphi_1} d\lambda = (T^* f, u)_{\varphi_1} = (f, Tu)_{\varphi_2}$$

$$= (-1)^p \int \Sigma'_{|I|=p} \Sigma_{j=1}^n f_{jI} \partial u_I / \partial x_j e^{-\varphi_2} d\lambda,$$

which means that

$$T^* f = (-1)^{p-1} \Sigma'_I \Sigma_{j=1}^n e^{\varphi_1} \partial (e^{-\varphi_2} f_{jI}) / \partial x_j dx^I. \tag{7}$$

5. Existence theorems in real-pseudoconvex domains

Choose a function $\psi \in C^\infty(\Omega)$ such that

$$\Sigma_{k=1}^n |\partial \eta_\nu / \partial x_k|^2 \leq e^\psi \text{ in } \Omega, \ \nu = 1, 2, \cdots.$$

If we set

$$\varphi_1 = \varphi - 2\psi,\ \varphi_2 = \varphi - \psi,\ \varphi_3 = \varphi, \qquad (*)$$

the condition (4.4) is satisfied for any choice of φ. We shall now study $\| T^* f \|_{\varphi_1}$ and $\| Sf \|_{\varphi_3}$ when $f \in \mathcal{D}^{p+1}(\Omega)$, keeping ψ fixed in all that follows but making all estimates uniform in φ so that we can make a suitable choice of φ at the end of the discussion. We assume that $f \in C^2(\Omega)$.

First note that, since

$$df = \Sigma'_{|I|=p}\Sigma_{j=1}^n \partial f_I/\partial x_j dx_j \wedge dx^I,$$

we obtain

$$|df|^2 = \Sigma'_{I,J}\Sigma_{j,l=1}^n \partial f_I/\partial x_j \partial f_J/\partial x_l \varepsilon_{lJ}^{jI}$$

where $\varepsilon_{lJ}^{jI} = 0$ unless $j \notin I$, $l \notin J$, and $\{j\} \cup I = \{l\} \cup J$, in which case ε_{lJ}^{jI} is the sign of permutation $\begin{pmatrix} j & I \\ l & J \end{pmatrix}$. We rearrange the terms in this sum. First consider the terms with $j = l$. Then we must have $I = J$ and $j \notin I$ if $\varepsilon_{lJ}^{jI} \neq 0$, so the sum of these terms is

$$\Sigma'_I \Sigma_{j \notin I} |\partial f_I/\partial x_j|^2.$$

Next consider the terms with $j \neq l$. If $\varepsilon_{lJ}^{jI} \neq 0$, we must then have $l \in I$ and $j \in J$, and deletion of l from I or j from J gives the same multi-index K. Since

$$\varepsilon_{lJ}^{jI} = \varepsilon_{jlK}^{jI}\varepsilon_{ljK}^{jlK}\varepsilon_{lJ}^{ljK} = -\varepsilon_{lK}^{I}\varepsilon_{J}^{jK},$$

the sum of the terms in question is

$$-\Sigma'_K \Sigma_{j \neq l} \partial f_{lK}/\partial x_j \partial f_{jK}/\partial x_l.$$

Hence we obtain

$$|df|^2 = \Sigma'_I \Sigma_j |\partial f_I/\partial x_j|^2 - \Sigma'_K \Sigma_{j,l} \partial f_{jK}/\partial x_l \partial f_{lK}/\partial x_j. \qquad (1)$$

(When $p = 1$, this follows from the fact that $|df|^2 = \Sigma|\partial f_j/\partial x_k - \partial f_k/\partial x_j|^2/2$.)

Next we consider $T^* f$. With the notation

$$\delta_j w = e^{\varphi}\partial(we^{-\varphi})/\partial x_j = \partial w/\partial x_j - w\partial\varphi/\partial x_j,$$

we obtain from (4.7)

$$e^{\psi}T^* f = (-1)^{p-1}\Sigma'_I\Sigma_{j=1}^n \delta_j f_{jI} dx^I + (-1)^{p-1}\Sigma'_I\Sigma_{j=1}^n f_{jI}\partial\psi/\partial x_j dx^I.$$

Hence

$$\int \Sigma'_K \Sigma_{j,k=1}^n \delta_j f_{jK} \overline{\delta_k f_{kK}} e^{-\varphi} d\lambda \leq 2 \| T^* f \|_{\varphi_1}^2 + 2\int |f|^2 |d\psi|^2 e^{-\varphi} d\lambda.$$

Combining this estimate with (5.1), we obtain

$$\int \Sigma'_K \Sigma^n_{j,k=1}(\delta_j f_{jK}\delta_k f_{kK} - \partial f_{jK}/\partial x_k \partial f_{kK}/\partial x_j)e^{-\varphi}d\lambda$$

$$+\int \Sigma'_I \Sigma^n_{j=1}|\partial f_I/\partial x_j|^2 e^{-\varphi}d\lambda$$

$$\leq 2 \parallel T^* f \parallel^2_{\varphi_1} + \parallel Sf \parallel^2_{\varphi_3} + 2\int |f|^2|d\psi|^2 e^{-\varphi}d\lambda. \tag{2}$$

Now the operators $\partial/\partial x_k$ and $-\delta_k$ are adjoint in the sense that

$$\int w_1 \partial w_2/\partial x_k e^{-\varphi}d\lambda = -\int \delta_k w_1 w_2 e^{-\varphi}d\lambda,$$

$$w_1,\ w_2 \in C_0^\infty(\Omega);$$

and we have the commutation relations

$$\delta_j \partial/\partial x_k - \partial/\partial x_k \delta_j = \partial^2\varphi/\partial x_k \partial x_j.$$

Shifting the differentiations to the left in the first sum in (5.2) therefore gives

$$\Sigma'_K \int \Sigma^n_{j,k=1} f_{jK} f_{kK} \partial^2\varphi/\partial x_j \partial x_k e^{-\varphi}d\lambda$$

$$+\Sigma'_I \Sigma^n_{j=1}\int |\partial f_I/\partial x_j|^2 e^{-\varphi}d\lambda$$

$$\leq 2 \parallel T^* f \parallel^2_{\varphi_1} + \parallel Sf \parallel^2_{\varphi_3} + 2\int |f|^2|d\psi|^2 e^{-\varphi}d\lambda,$$

$$f \in \mathcal{D}^{p+1}(\Omega). \tag{3}$$

Now assume that the function φ is strictly real-plurisubharmonic, namely we have

$$\Sigma^n_{j,k=1}\partial^2\varphi/\partial x_j \partial x_k w_j w_k \geq c\Sigma^n_{j=1} w_j^2,\ w \in \boldsymbol{R}^n,$$

where c is a positive continuous function in Ω. Then it follows from (5.3) that

$$\int (c - 2|d\psi|^2)|f|^2 e^{-\varphi}d\lambda \leq 2 \parallel T^* f \parallel^2_{\varphi_1} + \parallel Sf \parallel^2_{\varphi_3},$$

$$f \in \mathcal{D}^{p+1}(\Omega). \tag{4}$$

Recalling Lemma 4.3, we have now proved the following.

Lemma 5.1. *With φ_1, φ_2, φ_3 defined by (*), where φ, $\psi \in C^2(\Omega)$, we have*

$$\parallel f \parallel^2_{\varphi_2} \leq \parallel T^* f \parallel^2_{\varphi_1} + \parallel Sf \parallel^2_{\varphi_3},\ f \in D_{T^*} \cap D_S, \tag{5}$$

provided that

$$\Sigma_{j,k=1}^{n}\partial^2\varphi/\partial x_j\partial x_k w_j w_k \geq 2(|d\psi|^2+e^{\psi})\Sigma_{j=1}^{n}w_j^2,\ w\in \boldsymbol{R}^n. \tag{6}$$

We can now easily prove an existence theorem.

Theorem 5.2. *Let Ω be a real-pseudoconvex open set in $\boldsymbol{R}^n$. Then the equation $du=f$ has (in the sense of distribution theory) a solution $u\in L_{2,\mathrm{loc}}^{p}(\Omega)$ for every $f\in L_{2,\mathrm{loc}}^{p+1}(\Omega)$ such that $df=0$.*

Proof. From the property of real-plurisubharmonic functions in Ω, we can choose a strictly real-plurisubharmonic function $p\in C^{\infty}(\Omega)$ such that

$$K_c=\{x;x\in\Omega,\ p(x)\leq c\}\subset\subset\Omega,\ \text{for every } c\in\boldsymbol{R}.$$

Let

$$\Sigma_{j,k=1}^{n}\partial^2 p/\partial x_j\partial x_k w_j w_k\geq m\Sigma_{j=1}^{n}w_j^2,$$

where $0<m\in C^0(\Omega)$. If χ is a convex increasing C^{∞}-function and $\varphi=\chi(p)$, we obtain

$$\Sigma_{j,k=1}^{n}\partial^2\varphi/\partial x_j\partial x_k w_j w_k\geq\chi'(p)m\Sigma_{j=1}^{n}w_j^2.$$

Hence φ satisfies (5.6) if

$$\chi'(p)m\geq 2(|d\psi|^2+e^{\psi}),$$

that is, if

$$\chi'(t)\geq\sup_{K_t}2(|d\psi|^2+e^{\psi})/m. \tag{7}$$

The right-hand side of (5.7) is a finite increasing function of t, which is defined when $t\geq\min p$. Hence there exists a increasing C^{∞}-function χ' satisfying (5. 7). It is clear that we can choose χ so that, in addition, any given $f\in L_{2,\mathrm{loc}}^{p+1}(\Omega)$ belongs to $L_2^{p+1}(\Omega,\varphi-\psi)$. But then it follows from Lemma 4.1 that the equation $du=f$ has a solution $u\in L_2^p(\Omega,\varphi-2\psi)$. This proves the theorem. Q.E.D.

We shall now examine the regularity properties of the solution u of the equation $du=f$ which we have obtained. In doing so, it is important to note that the solution of the equation $Tu=f$ given by Lemma 4.1 can be chosen orthogonal to the null space of T, that is, in (the closure of) the range of T^*. This will yield an additional differential equation for u which is essential in proving the smoothness of u.

Let W^s, where s is a non-negative integer, denote the space of functions in $\boldsymbol{R}^n$ whose derivatives of order $\leq s$ are in L_2. By $W_{\mathrm{loc}}^s(\Omega)$ we denote the set of functions in Ω satisfying the same condition on compact subsets of Ω. The space of forms of degree p with coefficients in this space is accordingly denoted by $W_{\mathrm{loc}}^{s,p}(\Omega)$. If f is of degree $p+1 (p\geq 0)$, we set

$$\vartheta f=\Sigma_I'\Sigma_{j=1}^{n}\partial f_{jI}/\partial x_j dx^I.$$

This is essentially the principal part of the differential operator in (4.7).

Lemma 5.3. *If $f \in L_2^{p+1}(\boldsymbol{R}^n)$ has the compact support, and if $df \in L_2^{p+2}(\boldsymbol{R}^n)$ and $\vartheta f \in L_2^p(\boldsymbol{R}^n)$, then $f \in W^{1,p+1}(\boldsymbol{R}^n)$.*

Proof. First note that if $f \in \mathcal{D}^{p+1}(\Omega)$, then (5.3) with $\varphi = \psi = 0$ gives

$$\Sigma_I' \Sigma_{j=1}^n \int |\partial f_I/\partial x_j|^2 d\lambda \leq 2 \parallel \vartheta f \parallel_0^2 + \parallel df \parallel_0^2 . \tag{8}$$

If f only satisfies the hypotheses in the lemma, we can form a regularization $f * \chi_\varepsilon$ of f as defined in Lemma 4.4. If we apply (5.8) to $f * \chi_\varepsilon - f * \chi_\delta$, noting that $\vartheta(f * \chi_\varepsilon) = (\vartheta f) * \chi_\varepsilon \longrightarrow \vartheta f$ in $L_2^p(\boldsymbol{R}^n)$ and the corresponding fact for $d(f * \chi_\varepsilon)$, it follows that $\chi_\varepsilon * \partial f_I/\partial x_j$ converges in L_2 for all I, j when $\varepsilon \longrightarrow 0$. Hence $\partial f_I/\partial x_j \in L_2$. The proof is completed. Q.E.D.

We can now give an improvement of Theorem 5.2.

Theorem 5.4. *Let Ω be a real-pseudoconvex open set in $\boldsymbol{R}^n$, and let $0 \leq s \leq \infty$. Then the equation $du = f$ has a solution $u \in W_{\text{loc}}^{s+1,p}(\Omega)$ for every $f \in W_{\text{loc}}^{s,p+1}(\Omega)$ such that $df = 0$. Every solution of the equation $du = f$ has this property when $p = 0$.*

Proof. (a) First assume that $p = 0$. We know from Theorem 5.2 that the equation $du = f$ has a solution $u \in L_{2,\text{loc}}^0(\Omega)$. The equation $du = f$ means

$$\partial u/\partial x_j = f_j \in W_{\text{loc}}^s(\Omega)$$

for all j. Suppose that $u \in W_{\text{loc}}^\sigma(\Omega)$ for a certain finite σ with $0 \leq \sigma \leq s$. We know that this is true if $\sigma = 0$. If $\chi \in C_0^\infty(\Omega)$, we then obtain

$$\partial(\chi u)/\partial x_j = \chi f_j + \partial\chi/\partial x_j u \in W^\sigma.$$

If v is a derivative of order σ of χu, it follows that $\partial v/\partial x_j \in L_2$ for every j. Hence $v \in W^1$, that is, all derivatives of χu of order $\sigma + 1$ are in L_2. This means that $u \in W_{\text{loc}}^{\sigma+1}(\Omega)$. Repeating the argument, we conclude that $u \in W_{\text{loc}}^{s+1}(\Omega)$.

(b) Next assume that $p > 0$. As pointed out after the proof of Theorem 5.2, the solution of the equation $du = f$ given in that theorem can be chosen in (the closure of) the range of T^*. In view of (4.7) and the fact that $\vartheta^2 = 0$, we have

$$\vartheta(e^{-\varphi_1}u) = 0, \ du = f.$$

This can also be written

$$du = f, \ \vartheta u = au,$$

where a is a differential operator of order 0 with C^∞-coefficients acting on u. Assume that we have already proved that $u \in W_{\text{loc}}^{\sigma,p}(\Omega)$ for a certain finite σ with $0 \leq \sigma \leq s$. If $\chi \in C_0^\infty(\Omega)$, we obtain

$$d(\chi u) \in W^{\sigma,p+1}, \ \vartheta(\chi u) \in W^{\sigma,p-1}.$$

If D is a differentiation of order σ, the form $D(\chi u)$ satisfies the hypotheses of Lemma 5.3, which proves that $D(\chi u) \in W^{1,p}$. Hence $\chi u \in W^{\sigma+1,p}$, that is, $u \in W_{\mathrm{loc}}^{\sigma+1,p}(\Omega)$. This completes the proof. Q.E.D.

Corollary 5.5. *If Ω is a real-pseudoconvex open set in $\boldsymbol{R}^n$, the equation $du = f$ has a solution $u \in C^{\infty,p}(\Omega)$ for every $f \in C^{\infty,p+1}(\Omega)$ such that $df = 0$.*

Proof. By the well-known Sobolev lemma, we have

$$W_{\mathrm{loc}}^{s+n,p}(\Omega) \subset C^{s,p}(\Omega)$$

so that Corollary follows from Theorem 5.4. Q.E.D.

6. The de Rham complexes and the de Rham theorems. Vanishing of the de Rham cohomologies

In general, it is hard to caluculate the de Rham cohomology directly. The following two facts for the de Rham cohomology are well-known(cf. Matsushima[15], p.135).

1. Let M be an n-dimensional differentiable manifold. If M has k connected components, then $H^0(M, \boldsymbol{R})$ is a k-dimensional vector space.

2. For the n-dimensional Euclidean space $\boldsymbol{R}^n$, we have $H^p(\boldsymbol{R}^n, \boldsymbol{R}) = 0$ for $p > 0$.

The second fact follows from Poincaré's Lemma below.

Lemma 6.1(Poincaré's Lemma). *For every differential form ω of degree $p(p > 0)$ with C^∞-coefficients on $\boldsymbol{R}^n$ such that $d\omega = 0$, there exists some differential form θ of degree $(p-1)$ with C^∞-coefficients on $\boldsymbol{R}^n$ so that $\omega = d\theta$.*

Proof. See Matsushima[15], Lemma(Poincaré's Lemma), p.135. Q.E.D.

As for Poincaré's Lemma, we also refer to Akizuki[1], Komatsu[13] and Murakami[16].

We now prove several de Rham complexes. Then, using them, we caluculate the de Rham cohomologies. After that, we prove the vanishing of the de Rham cohomologies.

Definition 6.2. We define the sheaf $\mathcal{L}^p$ over $\boldsymbol{R}^n$ to be the sheaf $\{\mathcal{L}^p(\Omega); \Omega$ is an open set in $\boldsymbol{R}^n\}$, where the section module $\mathcal{L}^p(\Omega)$ on an open set Ω in $\boldsymbol{R}^n$ is the space

$$\mathcal{L}^p(\Omega) = \{f; f \in L_{2,\mathrm{loc}}^p(\Omega),\ df \in L_{2,\mathrm{loc}}^{p+1}(\Omega)\}.$$

Here df is defined in the distribution sense. Especially we put $\mathcal{L} = \mathcal{L}^0$.

Then $\mathcal{L}^p$ constitutes a soft sheaf. Equipped with a graph topology with respect to the d-operator, $\mathcal{L}^p(\Omega)$ becomes an FS*-space for an open set Ω in $\boldsymbol{R}^n$.

Theorem 6.3(the de Rham complex). *We use the notation in Definition 6.2. Then the sequence of sheaves over $\boldsymbol{R}^n$*

$$0 \longrightarrow \boldsymbol{R} \longrightarrow \mathcal{L}^0 \xrightarrow{d^0} \mathcal{L}^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathcal{L}^n \longrightarrow 0$$

is exact. Here d^p denotes the exterior differential operator.

Proof. It follows from Theorem 5.2. Q.E.D.

Definition 6.4. We define the sheaf $\mathcal{E}^p$ over $\boldsymbol{R}^n$ to be the sheaf of differential p-forms with C^∞- coefficients. Especially we put $\mathcal{E} = \mathcal{E}^0$.

Then $\mathcal{E}^p$ constitutes a soft sheaf. The section module $\mathcal{E}^p(\Omega)$ is a Fréchet space for an open set Ω in $\boldsymbol{R}^n$.

Theorem 6.5(the de Rham complex). *We use the notation in Definition 6.4. Then the sequence of sheaves over $\boldsymbol{R}^n$*

$$0 \longrightarrow \boldsymbol{R} \longrightarrow \mathcal{E}^0 \xrightarrow{d^0} \mathcal{E}^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathcal{E}^n \longrightarrow 0$$

is exact. Here d^p denotes the exterior differential operator.

Proof. It follows from Corollary 5.5. See also Kaneko[7], Example 5.1.11, p.244. Q.E.D.

Definition 6.6. We define the sheaf $\mathcal{D}'^{,p}$ over $\boldsymbol{R}^n$ to be the sheaf of differential p-forms with distribution coefficients. Especially we put $\mathcal{D} = \mathcal{D}'^{,0}$.

Then $\mathcal{D}'^{,p}$ constitutes a soft sheaf. The section module $\mathcal{D}'^{,p}(\Omega)$ is a DLFS-space for an open set Ω in $\boldsymbol{R}^n$.

Theorem 6.7(the de Rham complex). *We use the notation in Definition 6.6. Then the sequence of sheaves over $\boldsymbol{R}^n$*

$$0 \longrightarrow \boldsymbol{R} \longrightarrow \mathcal{D}'^{,0} \xrightarrow{d^0} \mathcal{D}'^{,1} \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathcal{D}'^{,n} \longrightarrow 0$$

is exact. Here d^p denotes the exterior differential operator.

Proof. See Kaneko[7], Example 5.1.11, p.244. Q.E.D.

Definition 6.8. We define the sheaf $\mathcal{O}^p$ over $\boldsymbol{C}^n$ to be the sheaf of differential p-forms with holomorphic coefficients. Especially we put $\mathcal{O} = \mathcal{O}^0$.

The section module $\mathcal{O}^p(\Omega)$ is an FS-space for an open set Ω in $\boldsymbol{C}^n$.

Theorem 6.9(the de Rham complex). *We use the notation in Definition 6.8. Then the sequence of sheaves over $\boldsymbol{C}^n$*

$$0 \longrightarrow \boldsymbol{C} \longrightarrow \mathcal{O}^0 \xrightarrow{d^0} \mathcal{O}^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathcal{O}^n \longrightarrow 0$$

is exact. Here d^p denotes the exterior differential operator acting on real and imaginary parts separately.

Proof. See Kaneko[7], Example 5.1.11, p.244. Q.E.D.

Definition 6.10. We define the sheaf $\mathcal{A}^p$ over $\boldsymbol{R}^n$ to be the sheaf of differential p-forms with real-analytic coefficients. Especially we put $\mathcal{A} = \mathcal{A}^0$.

The section module $\mathcal{A}^p(\Omega)$ is an FS-space for an open set Ω in $\boldsymbol{R}^n$.

Theorem 6.11(the de Rham complex). *We use the notation in Definition 6.10. Then the sequence of sheaves over $\boldsymbol{R}^n$*

$$0 \longrightarrow \boldsymbol{C} \longrightarrow \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathcal{A}^n \longrightarrow 0$$

is exact. Here d^p denotes the exterior differential operator acting on real and imaginary parts separately.

Proof. See Komatsu[9], Theorem 11, p.85 and Kaneko[7], Example 5.1.11, p.244. Q.E.D.

Definition 6.12. We define the sheaf $\mathcal{B}^p$ over $\boldsymbol{R}^n$ to be the sheaf of differential p-forms with hyperfunction coefficients. Especially we put $\mathcal{B} = \mathcal{B}^0$.

Then $\mathcal{B}^p$ constitutes a flabby sheaf.

Theorem 6.13(the de Rham complex). *We use the notation in Definition 6.12. Then the sequence of sheaves over* $\boldsymbol{R}^n$

$$0 \longrightarrow \boldsymbol{C} \longrightarrow \mathcal{B}^0 \xrightarrow{d^0} \mathcal{B}^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathcal{B}^n \longrightarrow 0$$

is exact. Here d^p *denotes the exterior differential operator acting on real and imaginary parts separately.*

Proof. See Komatsu[9], Theorem 11, p.85 and Kaneko[7], Theorem 7.3.3, p.354. Q.E.D.

Theorem 6.14(the de Rham Theorem). *We use the notation in Theorems 6.3, 6.5 and 6.7. Let* Ω *be an open set in* $\boldsymbol{R}^n$. *Then we have the following isomorphisms for every* $p > 0$;

$$\begin{aligned} H^p(\Omega, \boldsymbol{R}) &\cong \{f; f \in \mathcal{L}^p(\Omega),\ df = 0\}/\{dg; g \in \mathcal{L}^{p-1}(\Omega)\} \\ &\cong \{f; f \in \mathcal{E}^p(\Omega),\ df = 0\}/\{dg; g \in \mathcal{E}^{p-1}(\Omega)\} \\ &\cong \{f; f \in \mathcal{D}'^{,p}(\Omega),\ df = 0\}/\{dg; g \in \mathcal{D}'^{,p-1}(\Omega)\} \end{aligned}$$

Proof. It follows from the cohomology theory in Godement[4]. Q.E.D.

Theorem 6.15(the de Rham Theorem). *We use the notation in Theorem 6.9. Let* Ω *be a pseudoconvex open set in* $\boldsymbol{C}^n$. *Then we have the following isomorphisms for every* $p > 0$;

$$H^p(\Omega, \boldsymbol{C}) \cong \{f; f \in \mathcal{O}^p(\Omega),\ df = 0\}/\{dg; g \in \mathcal{O}^{p-1}(\Omega)\}$$

Proof. Hörmander[6], Theorem 2.7.10, p.58. Q.E.D.

Theorem 6.16(the de Rham Theorem). *We use the notation in Theorems 6.11, and 6.13. Let* Ω *be an open set in* $\boldsymbol{R}^n$. *Then we have the following isomorphisms for every* $p > 0$;

$$\begin{aligned} H^p(\Omega, \boldsymbol{C}) &\cong \{f; f \in \mathcal{A}^p(\Omega),\ df = 0\}/\{dg; g \in \mathcal{A}^{p-1}(\Omega)\} \\ &\cong \{f; f \in \mathcal{B}^p(\Omega),\ df = 0\}/\{dg; g \in \mathcal{B}^{p-1}(\Omega)\} \end{aligned}$$

Proof. It follows from the cohomology theory in Godement[4]. Q.E.D.

Theorem 6.17. *If* Ω *is a real-pseudoconvex open set in* $\boldsymbol{R}^n$, *then we have*

(1) $H^p(\Omega, \boldsymbol{R}) = 0$ *for every* $p > 0$.

(2) $H^p(\Omega, \boldsymbol{C}) = 0$ *for every* $p > 0$.

Proof. It follows from Theorem 6.2 or Corollary 6.5. Q.E.D.

By virtue of the vanishing theorems above of the de Rham cohomologies, we have the existence theorems for the exterior differential operator d in several categories of functions and generalized functions. Namely we have the analogs of Poincaré's Lemma in several categories. Thus we have the following.

Theorem 6.18. *Let* $\mathcal{F}$ *be one of the sheaves* $\mathcal{L}$, $\mathcal{E}$, $\mathcal{D}'$, $\mathcal{A}$ *and* $\mathcal{B}$, *and* $\mathcal{F}^p$ *be the p-forms with coefficients in the sheaf* $\mathcal{F}$. *Let* Ω *be a real-pseudoconvex open set in* $\boldsymbol{R}^n$. *Then for every differential form* f *of degree* $p(p > 0)$ *with coefficients in* $\mathcal{F}(\Omega)$ *such that* $df = 0$, *there exists some differential form* g *of degree* $(p-1)$ *with coefficients in* $\mathcal{F}(\Omega)$ *so that* $dg = f$.

Theorem 6.19. *If* Ω *is a pseudoconvex open set in* $\boldsymbol{C}^n$, *then we have* $H^p(\Omega, \boldsymbol{C}) = 0$ *for every* $p > n$.

Proof. See Hörmander[6], Theorem 2.7.10, p.58. Q.E.D.

It is known that $H^p(\Omega, \boldsymbol{C})$ does not vanish for all Runge domains in $\boldsymbol{C}^n$ when $p < n$(cf. Hörmander[6], p.59). This shows that the concept of real-pseudoconvex domains is different from that of pseudoconvex domains.

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