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On Maslov's Quantization Condition for Mechanics in a Magnetic Field

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Abstract

This paper considers Maslov's quantization condition for the dynamical system in a magnetic field on the basis of the theory of Fourier integral operators. As a result, it is clarified that a quasi-classical eigenvalue (energy level) according to the quantization rule provides an approximation of order $\hbar^2$ to the true eigenvalue of the Schrödinger operator.

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Introduction

Let $\pi : P \to M$ be a principal $U(1)$-bundle over a compact Riemannian manifold (M, g) $(n = \dim M)$, and suppose P is endowed with a connection $\widetilde{\nabla}$. We have a $U(1)$-invariant metric (the so-called *Kaluza-Klein metric*) $\tilde{g}$ on P induced from the metric g on M, the connection $\widetilde{\nabla}$ on P and a $U(1)$-invariant metric on the structure group $U(1)$ (cf. [5, §3]). Let us consider the Laplace-Beltrami operator Δ_P on P defined by the metric $\tilde{g}$.

Let H_P be the principal symbol of Δ_P, which is a smooth, $U(1)$-invariant function on the cotangent bundle T^*P of P. The Hamiltonian system (T_0^*P, Ω_P, H_P) describes the geodesic flow on T_0^*P with the standard symplectic form Ω_P. (The subscript 0 means that the zero section has been deleted.) The natural action of $U(1)$ on (T_0^*P, Ω_P, H_P) conserves its flow, and we obtain according to

Marsden-Weinstein reduction procedure the family of reduced dynamical systems $(P_\mu, \Omega_\mu, H_\mu)$ $(P_\mu := J^{-1}(\mu)/U(1))$ parameterized by values $\mu \in \mathfrak{u}(1)^*$ of the associated momentum map $J : T_0^*P \to \mathfrak{u}(1)^*$. Moreover, $(P_\mu, \Omega_\mu, H_\mu)$ is isomorphic with the dynamical system $(T^*M, \Omega_\mu^M, H_\mu^M)$ on T^*M, where

$$\Omega_\mu^M = \Omega_M + \langle \mu, \pi_M^* \Theta \rangle, \quad (\pi_M : T^*M \to M),$$

with Θ being the $\mathfrak{u}(1)$-valued curvature form on M induced from $\widetilde{\nabla}$, and

$$H_\mu^M = H_M + |\mu|^2,$$

with $H_M(x,\xi) = \|\xi\|_x^2$ $((x,\xi) \in T^*M)$. The dynamical system $(T^*M, \Omega_\mu^M, H_\mu^M)$ describes the motion of a charged particle (the *magnetic flow*) with the "charge" μ under the magnetic field Θ. (See e.g. [5] for details of the mechanics in a magnetic field.) Let μ_0 be the element of $\mathfrak{u}(1)^*$ satisfying $\langle \mu_0, \partial/\partial t \rangle = 1$ for the invariant vector field $\partial/\partial t$ of $U(1) = \{e^{it} |\, 0 \le t < 2\pi\}$. In this paper we pay attention to the system $(T^*M, \Omega_{\mu_0}^M, H_{\mu_0}^M)$ describing the motion of the particle with the *unit* charge.

Considering the self-adjoint differential operator $D_t = -i\partial/\partial t$ on P associated to the action of $U(1)$, we have the eigenspace decomposition

$$L^2(P) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_m,$$

where the space $\mathcal{H}_m$ consists of functions f satisfying $f(p \cdot e^{it}) = e^{imt} f(p)$ $(p \in P)$. Since Δ_P commutes with D_t, we can define the restriction of Δ_P to the subspace $\mathcal{H}_m$, which we denote by $\hat{H}_m$. We call $\hat{H}_m$ the *magnetic Schrödinger operator* with charge m.

Let $E_m \to M$ $(m \in \mathbb{Z})$ be the associated line bundle of P defined by the unitary representation, $e^{it} \mapsto e^{-imt}$, of $U(1)$. Then, through the unitary relation $L^2(E_m) \cong \mathcal{H}_m$ the operator $\hat{H}_m$ is identified with the second order, self-adjoint elliptic differential operator on $L^2(E_m)$, which is locally written as

$$-\sum_{j,k} g^{jk}(\nabla_j - imA_j)(\nabla_k - imA_k) + m^2|\mu_0|^2, \tag{0.1}$$

where ∇ is the Levi-Civita connection of (M,g), and $\Theta = d(\sum_j A_j dx^j) \otimes \partial/\partial t$.

The spectrum of the operator $\hat{H}_m$ consists of non-negative eigenvalues

$$\lambda_1^{(m)} \le \lambda_2^{(m)} \le \ldots \le \lambda_j^{(m)} \le \ldots \uparrow \infty.$$

Note that the spectrum of Δ_P is the union of the spectra of $\hat{H}_m$'s:

$$\bigcup_{m \in \mathbb{Z}} \{\lambda_j^{(m)} |\, j = 1, 2, \ldots\}.$$

The analysis on the relation between the spectrum $\{\lambda_j^{(m)}\}$ (or associated eigenfunctions) and the classical system $(T^*M, \Omega_\mu^M, H_\mu^M)$ has been developed in various aspects. Among others, V. Guillemin and A. Uribe [1], [2] have obtained a formula which gives a relation between the asymptotic distribution of $\{(\lambda_j^{(m)})^{1/2}\}$ along the line $\nu = Em$ $(\nu := \lambda^{1/2})$ and the periodic trajectories lying on the energy level: $\sqrt{H_{\mu_0}^M} = E$ of the system $(T^*M, \Omega_{\mu_0}^M, H_{\mu_0}^M)$. On the other hand, R. Schrader and M. Taylor [7], and S. Zelditch [13] presented some results concerning the asymptotic distribution of eigenfunctions when the classical system is ergodic. We also refer [8] by T. Tate as a research in the same direction. The recent work [9] presented a generalization of Helton's theorem concerning cluster points of $\{(\lambda_j^{(m)})^{1/2}\}$ in the case of magnetic flow with "few" periodic trajectories. Our interest in this paper is an opposite case, namely the case where the classical system is completely integrable.

In [12] A. Yoshioka defined a quantization condition for the Hamiltonian system of magnetic flow, $(T^*M, \Omega_\mu^M, H_\mu^M)$, as a generalization of Maslov's quantization condition [6], and searched the Lagrangian tori (energy levels) satisfying the quantization condition for the system on $(\mathbb{C}P^n, can)$ with the harmonic magnetic field. On the other hand, in the case of the geodesic flow (the system of free particle) on a Riemannian manifold A. Weinstein [11] clarified a relationship between Maslov's quantization condition and the (asymptotic) distribution of eigenvalues of the Laplace-Beltrami operator within the theory of Fourier integral operators.

We consider in the present paper the relationship between the quantization condition by Yoshioka and the spectrum $\{\lambda_j^{(m)}\}$ (or $\{(\lambda_j^{(m)})^{1/2}\}$) along Weinstein's idea. The main theorem is the following.

Eigenvalue Theorem *Let L be a compact Lagrangian submanifold of $(T^*M, \Omega_{\mu_0}^M)$ and E a real number satisfying $E > |\mu_0|$. Suppose L and E satisfy the following conditions:*

(i) $\sqrt{H_{\mu_0}^M} \equiv E$ on L,

(ii) L is invariant under the magnetic flow φ_s, and the restricted flow $\varphi_s|_L$ leaves invariant a non-zero half-density,

(iii) L satisfies the Maslov-Yoshioka quantization condition (Q) (formulated in §1).

Let d be the smallest element of the set $\{1, 2, 4\}$ for which $dm_L([\gamma]) \equiv 0 \pmod 4$ for all $[\gamma] \in \pi_1(L)$, where $m_L \in H^1(L, \mathbb{Z})$ is the Maslov class of L.

Then, there is a sequence $\{\lambda_{j_k}^{(dk+1)}\}_{k=0}^\infty$ of eigenvalues of Δ_P such that

$$\left|\sqrt{\lambda_{j_k}^{(dk+1)}} - (dk+1)E\right| < R/(dk+1) \tag{0.2}$$

for a positive constant R (see Fig.1).

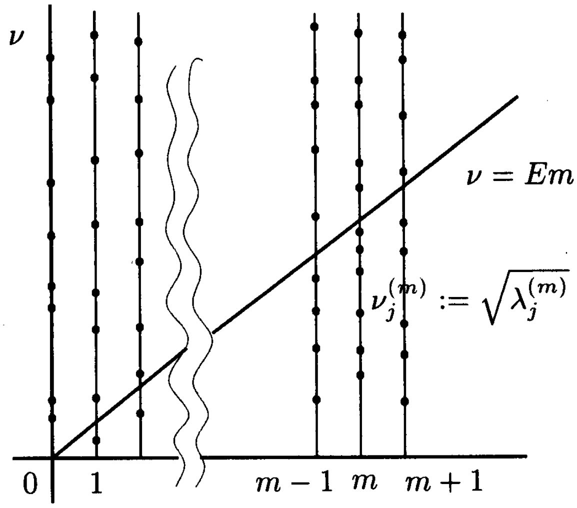

Figure 1

We notice that this theorem is understood in a *semi-classical* sense as follows: Consider the Schrödinger operator

$$\hat{H} = -\sum_{j,k} g^{jk}(\hbar\nabla_j - iA_j)(\hbar\nabla_k - iA_k) + |\mu_0|^2$$

corresponding to the operator (0.1) with $\hbar := 1/m$ being regarded as Planck's constant. The eigenvalue problem

$$\hat{H}\psi = \lambda\psi$$

is equivalent to

$$\hat{H}_m\psi = m^2\lambda\psi.$$

Put $\lambda(\hbar) := \lambda_{j_k}^{(dk+1)}/(dk+1)^2$, which is a eigenvalue of $\hat{H}$. Then, the asymptotic property (0.2) is written as

$$|\sqrt{\lambda(\hbar)} - E| < R\hbar^2, \quad i.e., \quad |\lambda(\hbar) - E^2| < R'\hbar^2.$$

Thus, E^2 is an approximate eigenvalue of $\hat{H}$ of order 2 in $\hbar$.

Remark Suppose the dynamical system $(T^*M, \Omega_{\mu_0}^M, H_{\mu_0}^M)$ is completely integrable. Then, for commutative first integrals, $f_1 \equiv H_{\mu_0}^M$, $f_2, \dots, f_n$, each torus

$$L_c := \{m \in T^*M |\ f_j(m) = c_j\ (1 \le j \le n)\}$$

is a Lagrangian submanifold, which satisfies automatically the condition (i),(ii) in Eigenvalue Theorem.

In Section 1 we define the quantization condition for the dynamical system of magnetic flow, which is essentially same as the formulation by Yoshioka [12]. In subsequent sections (§§2-7) we prove Eigenvalue Theorem by modifying the proof in the case of geodesic flow by Weinstein [11] (see also [10, Ch.XII, §4]). In our case, different from [11], the dynamical system has the $U(1)$-symmetry, and we have to perform the analysis of Lagrangian manifolds and Fourier integral operators under the $U(1)$ action.

1 Quantization condition for magnetic flow

The Hamiltonian system of magnetic flow, $(T^*M, \Omega^M_{\mu_0}, H^M_{\mu_0})$, is obtained along the reduction procedure from (T^*_0P, Ω_P, H_P) as follows:

$$T^*_0P \xleftarrow[\supset]{i_{\mu_0}} J^{-1}(\mu_0) \xrightarrow{\pi_{\mu_0}} P_{\mu_0}(= J^{-1}(\mu_0)/U(1)) \xrightarrow[\cong]{\Psi_{\mu_0}} T^*M$$

where Ψ_{μ_0} is a symplectic diffeomorphism, i.e., $\Psi^*_{\mu_0}\Omega^M_{\mu_0} = \Omega_{\mu_0}$ and $\pi^*_{\mu_0}\Omega_{\mu_0} = i^*_{\mu_0}\Omega_P$ hold. Let L be a compact Lagrangian submanifold of $(T^*M, \Omega^M_{\mu_0})$. Put

$$L_P := (\Psi_{\mu_0} \circ \pi_{\mu_0})^{-1}(L),$$

which is a compact submanifold of T^*_0P. Obviously we have

Lemma 1.1 *L_P is a Lagrangian submanifold of (T^*_0P, Ω_P).*

Let $m_{L_P} \in H^1(L_P, \mathbb{Z})$ be the Maslov class of the Lagrangian submanifold L_P. We can regard m_{L_P} as a map $m_{L_P} : \pi_1(L_P) \to \mathbb{Z}$. Let ω_P be the canonical 1-form on T^*_0P, i.e., $d\omega_P = \Omega_P$. We define a quantization condition for the system of magnetic flow essentially following Yoshioka [12].

Definition We say that a compact Lagrangian submanifold L of $(T^*M, \Omega^M_{\mu_0})$ satisfies the *Maslov-Yoshioka quantization condition* if

$$(Q) \qquad \frac{1}{2\pi}\int_\gamma \omega_P - \frac{1}{4}m_{L_P}([\gamma]) \in \mathbb{Z}$$

holds for every closed curve γ on L_P, where $[\gamma]$ denotes the equivalent class of γ.

We see the relation between two Lagrangian manifolds L and L_P by using local coordinates. Let $(x^1, \dots, x^n)$ be a local coordinate of $U \subset M$ ($n = \dim M$). We take $(x,t) = (x^1, \dots, x^n, t)$ $(0 \le t < 2\pi)$ as a local coordinate of $U \times U(1) \cong \pi^{-1}(U) \subset P$, and let (x, t, η, τ) be the canonical coordinate of $T^*(U \times U(1))$. The action of $e^{is} \in U(1)$ on T^*_0P is given by $(x,t,\eta,\tau) \mapsto (x, t+s, \eta, \tau)$ and the momentum map $J : T^*_0P \to \mathfrak{u}(1)^*$ associated to this action is written as $J(x,t,\eta,\tau) = \tau\mu_0$. Hence, the submanifold $J^{-1}(\mu_0)$ is locally equal to the set $\{(x,t,\eta,1)\}$. Thus (x,η) is regarded as a local coordinate of the reduced phase space P_{μ_0}. Let $(x,\xi) = (x^1,\dots,x^n,\xi_1,\dots,\xi_n)$ be the canonical coordinate of T^*M. The diffeomorphism Ψ_{μ_0} is written with respect to these coordinates as

$$\Psi_{\mu_0}(x^j,\eta_j) = (x^j,\xi_j) \quad \text{with} \quad \xi_j = \eta_j - A_j(x),$$

where $\Theta = d(\sum_j A_j(x)dx^j) \otimes \partial/\partial t$. Thus, we have locally

$$L_P = \{(x,t,\xi + A(x),1) \in T_0^*P \mid (x,\xi) \in L,\ 0 \leq t < 2\pi\}.$$

By means of the local expressions above we easily obtain the following lemmas.

Lemma 1.2 *Let γ be a closed curve on L_P, and put $\check{\gamma} := \Psi_{\mu_0} \circ \pi_{\mu_0}(\gamma) \subset L$. Then,*

$$m_{L_P}([\gamma]) = m_L([\check{\gamma}]).$$

Lemma 1.3 *Let $\ell = (x,t_0,\eta,1) \in L_P$, and let $c = c(t) = e^{it}\cdot\ell = (x,t_0+t,\eta,1)$ $(0 \leq t \leq 2\pi)$, which is the closed orbit of ℓ by the $U(1)$ action on L_P. Then,*

$$\frac{1}{2\pi}\int_c \omega_P = 1, \quad \textit{and} \quad m_{L_P}([c]) = 0.$$

Remark If the magnetic field (the curvature form) Θ on M is exact, i.e., $\Theta = d\theta$, then we have

$$\Omega^M_{\mu_0} = d\omega^M_{\mu_0} \quad \text{with} \quad \omega^M_{\mu_0} = \omega_M + \langle \mu_0, \pi_M^*\theta\rangle = \sum_{j=1}^n (\xi_j + A_j(x))dx^j$$

on whole T^*M, and it follows from Lemmas 1.2 and 1.3 that the quantization condition (Q) for L $(\subset T^*M)$ is equivalent to

$$\frac{1}{2\pi}\int_{\check{\gamma}} \omega^M_{\mu_0} - \frac{1}{4}m_L([\check{\gamma}]) \in \mathbb{Z}$$

for every closed curve $\check{\gamma}$ on L.

2 Strategy to prove Eigenvalue Theorem

Let

$$\mathbb{T}^2 := S^1 \times S^1 = \{(e^{ir},e^{is}) \mid 0 \leq r,s < 2\pi\}$$

be the two dimensional torus. The strategy to prove Eigenvalue Theorem is to construct a suitable operator $A : \mathcal{D}'(\mathbb{T}^2) \to \mathcal{D}'(P)$ (where $\mathcal{D}'(\cdot)$ denotes the space of distributions). The idea is essentially same as [11] by Weinstein, in which A is an operator from $\mathcal{D}'(S^1)$ to $\mathcal{D}'(P)$. We take $\mathbb{T}^2$ instead of S^1 in our case in order to consider the $U(1)$-symmetries in the systems.

Each element $u(r,s)$ in $L^2(\mathbb{T}^2)$ is written as the Fourier series:

$$u(r,s) = \sum_{\ell,m\in\mathbb{Z}} \hat{u}_{\ell,m} e^{i\ell r} e^{ims}. \tag{2.1}$$

Put $m_k := dk+1$ ($k \in \mathbb{Z}$). For the sequence $\{m_k\}_{k=0}^{\infty}$ we define the subspace $L^2(\mathbb{T}^2;\{m_k\})$ of $L^2(\mathbb{T}^2)$ as follows: A function $u \in L^2(\mathbb{T}^2)$ written as (2.1) belongs to $L^2(\mathbb{T}^2;\{m_k\})$ if and only if $\hat{u}_{\ell,m} = 0$ holds for every $(\ell,m) \notin \{(m_k,m_k)\}_{k=0}^{\infty}$.

Now, let us consider a continuous linear operator $A : \mathcal{D}'(\mathbb{T}^2) \to \mathcal{D}'(P)$ which satisfies the following conditions:

(A-i) $E^{-2}\Delta_P A - AD_{\mathbb{T}^2}$ induces a bounded operator from $L^2(\mathbb{T}^2)$ to $L^2(P)$, where, $D_{\mathbb{T}^2} := (-1/4)(\partial/\partial r + \partial/\partial s)^2$.

(A-ii) $A : L^2(\mathbb{T}^2;\{m_k\}) \to L^2(P)$ is an isometry,

(A-iii) $A(e^{im_k(r+s)})$ belongs to $\mathcal{H}_{m_k}$ for $k \geq k_0$, where k_0 is some non-zero integer.

Suppose we have the above operator A. Put $w_k := A(e^{im_k(r+s)}) \in \mathcal{H}_{m_k}$ ($k \geq k_0$). Then, by virtue of (A-i) we have

$$\begin{aligned} \|(E^{-2}\Delta_P - m_k^2)w_k\|_{L^2(P)} &= \|(E^{-2}\Delta_P A - AD_{\mathbb{T}^2})e^{im_k(r+s)}\|_{L^2(P)} \\ &\leq M\|e^{im_k(r+s)}\|_{L^2(\mathbb{T}^2)} = M, \end{aligned}$$

M being a constant. Let $\{\varphi_j^{(m_k)}\}$ be the orthonormal basis of eigenfunction of $\hat{H}_{m_k}$. Using the expansion: $w_k = \sum_j \hat{w}_{k,j}\varphi_j^{(m_k)}$, we have

$$\begin{aligned} &\|(E^{-2}\Delta_P - m_k^2)w_k\|_{L^2(P)}^2 \\ &= \|E^{-2}\sum_j \hat{w}_{k,j}(\nu_j^{(m_k)})^2\varphi_j^{(m_k)} - \sum_j m_k^2\hat{w}_{k,j}\varphi_j^{(m_k)}\|_{L^2(P)}^2 \\ &= \frac{1}{E^4}\sum_j\{(\nu_j^{(m_k)})^2 - E^2m_k^2\}^2|\hat{w}_{k,j}|^2 \\ &\geq \frac{1}{E^4}\inf_j\{(\nu_j^{(m_k)})^2 - E^2m_k^2\}^2\sum_j|\hat{w}_{k,j}|^2 \\ &= \frac{1}{E^4}\inf_j\{(\nu_j^{(m_k)})^2 - E^2m_k^2\}^2. \end{aligned}$$

Note $\sum_j |\hat{w}_{k,j}|^2 = 1$ by means of (A-ii). Combining two inequalities above, we have

$$\inf_j\{(\nu_j^{(m_k)})^2 - E^2m_k^2\}^2 \leq E^4M,$$

hence,

$$\inf_j|(\nu_j^{(m_k)})^2 - E^2m_k^2| = \inf_j|\nu_j^{(m_k)} - Em_k||\nu_j^{(m_k)} + Em_k| \leq Const.$$

This implies

$$\inf_j |\nu_j^{(m_k)} - Em_k| \le Rm_k^{-1},$$

that means the conclusion of Eigenvalue Theorem.

In the subsequent sections we construct the operator A as a Fourier integral operator (§§3 and 4) and check the properties (A-i)-(A-iii) (§§5-7).

3 Canonical relation of A

Let d be the number in Eigenvalue Theorem. Then, it follows from Lemma 1.2 that d is the smallest element of $\{1,2,4\}$ satisfying $dm_{L_P}([\gamma]) \equiv 0 \pmod 4$ for every closed curve γ on L_P. The Maslov class m_{L_P} induces a homomorphism $[m_{L_P}]$ from $\pi_1(L_P)$ to $\mathbb{Z}_4 := \mathbb{Z}/4\mathbb{Z} = \{0,1,2,3\}$. Then, the flat line bundle M_{L_P} over L_P defined by $[m_{L_P}] \in H^1(L_P, \mathbb{Z}_4)$ is so-called the *Keller-Maslov line bundle.* Note that the image of $[m_{L_P}]$ is the subgroup $\mathbb{Z}_d$ of $\mathbb{Z}_4$, where $\mathbb{Z}_1 = \{0\}$, $\mathbb{Z}_2 = \{0,2\}$.

Let $p : \overline{L}_P \to L_P$ be the covering of L_P whose deck transformation group is given by $\pi_1(L_P)/\ker([m_{L_P}]) \cong \mathbb{Z}_d$. For a closed curve $\bar\gamma$ on $\overline{L}_P$, we have $m_{L_P}([p(\bar\gamma)]) \in 4\mathbb{Z}$. Hence, the quantization condition (Q) means that

$$\int_{\bar\gamma} p^*\omega_P \in 2\pi\mathbb{Z}$$

for every closed curve $\bar\gamma$ on $\overline{L}_P$. Fixing a point $\bar\ell_0$ on $\overline{L}_P$, we define $\alpha : \overline{L}_P \to S^1 (= U(1))$ by

$$\alpha(\bar\ell) = \exp\big(i \int_{\bar c} p^*\omega_P\big),$$

where $\bar c$ is a curve on $\overline{L}_P$ from $\bar\ell_0$ to $\bar\ell$.

An element $q \in \mathbb{Z}_d$ acts on $\overline{L}_P$ as the deck transformation and on S^1 as the multiplication by $e^{i\pi q/2}$. By virtue of the quantization condition we have the following (see [11, Lemma 1.2]).

Lemma 3.1 *$\alpha : \overline{L}_P \to S^1$ is $\mathbb{Z}_d$-equivariant, i.e.,* $\alpha(q\cdot\bar\ell) = e^{i\pi q/2}\alpha(\bar\ell)$.

Let $c = c(t) = e^{it}\cdot\ell\ (0 \le t \le 1, \ell \in L_P)$ be the closed orbit of ℓ by the $U(1)$-action on L_P. Then, by virtue of Lemma 1.3 the lift of c to $\overline{L}_P$ is a closed curve. Hence, the action of $U(1)$ on L_P is naturally lifted to $\overline{L}_P$, which is commutative with the action of $\mathbb{Z}_d$. Moreover, we easily see the following.

Lemma 3.2 *$\alpha : \overline{L}_P \to S^1$ is $U(1)$-equivariant, i.e.,* $\alpha(e^{it}\cdot\bar\ell) = e^{it}\alpha(\bar\ell)$.

Now, we define a conic Lagrangian submanifold $\Lambda \subset T_0^*P \times T_0^*\mathbb{T}^2$. Let the map

$$j : \overline{L}_P \times \mathbb{R}^+ \times S^1 \to T_0^*P \times T_0^*\mathbb{T}^2 \cong T_0^*P \times \{(S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R})\}_0$$

be defined by

$$j(\bar\ell, \tau, z) = (\tau\ell; (\alpha(z^{-1}\cdot\bar\ell), -\tau), (z, -\tau)),$$

where $\ell = p(\bar\ell)$, and $\tau\ell$ is the scalar multiplication in T_0^*P.

Lemma 3.3 *j is a Lagrangian embedding.*

Proof. The injectivity of j is easy to see by recalling Lemma 3.1. The differential Tj of j is computed for $\bar{v} \in T_{\bar{\ell}}\overline{L}_P$, $a \in T_\tau\mathbb{R}^+ \cong \mathbb{R}$ and $b \in T_z S^1 \cong \mathbb{R}$ as

$$(3.1) \qquad [T_{(\bar{\ell},\tau,z)}j](\bar{v},a,b) = (\tau v + a\ell; (\langle\omega, v\rangle - b, -a), (b, -a)),$$

where $v = [T_{\bar{\ell}}p](\bar{v}) \in T_\ell L_P$ and $a\ell$ denotes the vertical vector at $\ell \in T_0^*P$. By this formula we see Tj to be injective. Thus the image of j is an $n+3$ dimensional submanifold. Moreover, the vector (3.1) is annihilated by the canonical 1-form $\omega_P + \omega_{S^1} + \omega_{S^1}$ of $T_0^*P \times T_0^*\mathbb{T}^2$, where $\omega_{S^1}(t,\tau) = \tau dt$. □

Let Λ be the image of the map j, and Λ is a conic Lagrangian submanifold of $T_0^*P \times T_0^*\mathbb{T}^2$. The manifold

$$(3.2) \qquad C := \Lambda' = \{(\tau\ell; (\alpha(z^{-1}\cdot\bar{\ell}), \tau), (z, \tau))\}$$

corresponding to Λ is called a homogeneous canonical relation, which is a Lagrangian submanifold with respect to the symplectic form $\Omega_P - \Omega_{\mathbb{T}^2} = \Omega_P - (\Omega_{S^1} + \Omega_{S^1})$.

Let the actions of $q \in \mathbb{Z}_d$ and $e^{it} \in U(1)$ on $\overline{L}_P \times \mathbb{R}^+ \times S^1$ be defined by

$$(\bar{\ell}, \tau, z) \mapsto (q\cdot\bar{\ell}, \tau, z),$$

and

$$(\bar{\ell}, \tau, z) \mapsto (e^{it}\cdot\bar{\ell}, \tau, e^{it}z),$$

respectively. On the other hand, their actions on $T_0^*P \times T_0^*\mathbb{T}^2$ are defined by

$$(\ell; (z,\rho), (w,\sigma)) \mapsto (\ell; (e^{i\pi q/2}z, \rho), (w, \sigma)),$$

and

$$(\ell; (z,\rho), (w,\sigma)) \mapsto (e^{it}\cdot\ell; (z,\rho), (e^{it}w, \sigma)),$$

respectively. Then, the following is easy to see.

Lemma 3.4 *The map j is $\mathbb{Z}_d$ and $U(1)$-equivariant, hence, Λ (or C) is invariant under the actions of $\mathbb{Z}_d$ and $U(1)$.*

4 Principal symbol of A

The principal symbol of the Fourier integral operator $A : \mathcal{D}'(\mathbb{T}^2) \to \mathcal{D}'(P)$ is defined as a section of the bundle $|TC|^{1/2} \otimes M_C$ over the canonical relation C, where $|TC|^{1/2}$ denotes the bundle of half-densities on C, and M_C is the Keller-Maslov line bundle over C.

Let $M_{\overline{L}_P} := p^* M_{L_P}$ be the line bundle over $\overline{L}_P$ induced from the Keller-Maslov line bundle M_{L_P} over L_P, and we have the covering $\tilde{p} : M_{\overline{L}_P} \to M_{L_P}$. Then, $M_{\overline{L}_P}$ is a trivial bundle with a natural global section s_0 constructed by the lift of locally constant sections on M_{L_P}. We have the following lemma concerning s_0 under the group actions.

Lemma 4.1 *Let $q \in \mathbb{Z}_d$, $e^{it} \in U(1)$. Then, we have*

$$\tilde{p}(s_0(q\cdot\bar{\ell})) = e^{i\pi q/2}\tilde{p}(s_0(\bar{\ell})), \tag{4.1}$$

$$\tilde{p}(s_0(e^{it}\cdot\bar{\ell})) = \tilde{p}(s_0(\bar{\ell})). \tag{4.2}$$

Proof. The formula (4.1) is just lemma 1.3 in [11], and (4.2) follows from Lemma 1.3. □

Recall the following diagram:

$$\begin{array}{ccccccc}
M_C & & & & M_{\overline{L}_P} & \xrightarrow{\tilde{p}} & M_{L_P} \\
\downarrow & & & & \downarrow & & \downarrow \\
C & \underset{\cong}{\xleftarrow{j'}} & \overline{L}_P \times \mathbb{R}^+ \times S^1 & \xrightarrow{Pr} & \overline{L}_P & \xrightarrow{p} & L_P
\end{array}$$

Lemma 4.2 *The Keller-Maslov line bundle M_C over C is equal to the pull back of $M_{\overline{L}_P}$ by the map $Pr \circ (j')^{-1}$.*

Proof. We describe the Lagrangian submanifolds in terms of phase functions. Let the Lagrangian submanifold $L \subset T_0^*M$ be locally defined by the phase function $\tilde{\phi}$: $U \times V \to \mathbb{R}$ $(U \subset M,\ V \subset \mathbb{R}^N)$ as L being (locally) given by $\{(x, \tilde{\phi}'_x)|\ \tilde{\phi}'_\theta(x,\theta) = 0\}$. Then, the Lagrangian submanifold L_P is locally given by the function ϕ on $U \times (\mathbb{R}/2\pi\mathbb{Z}) \times V$ defined by $\phi(x,t,\theta) = \tilde{\phi}(x,\theta) + t$. Namely, put $\Sigma := \{(x,t,\theta)|\ \phi'_\theta(x,t,\theta) = 0\}$. Then, L_P is locally given by

$$\{(x,t,\phi'_x,\phi'_t)|\ \phi'_\theta(x,t,\theta) = 0\} = \{(x,t,\tilde{\phi}'_x,1)|\ \tilde{\phi}'_\theta(x,\theta) = 0\},$$

and there is a diffeomorphism from Σ onto an open subset of L_P. By lifting this diffeomorphism to $\overline{L}_P$, we get $\lambda : \Sigma \to \overline{L}_P$. Then, we have

$$d(\phi|_\Sigma) = (\phi'_x dx^j + \phi'_t dt)\big|_\Sigma = (p\circ\lambda)^*\omega_P = \lambda^*(p^*\omega_P).$$

Hence, $e^{i\phi}|_\Sigma = \alpha\circ\lambda$ holds (by adding a constant to ϕ). We recall that $\phi'_t \equiv 1$ and $\phi'_x(x,t,\theta) = \phi'_x(x,t',\theta)$ for $\forall t,t' \in \mathbb{R}/2\pi\mathbb{Z}$. Let us define a phase function ψ by

$$\psi(x,t,r,s,\theta,\tau) := \tau\{\phi(x,t-s,\theta) - r\} = \tau\{\tilde{\phi}(x,\theta) + t - s - r\},$$

where $r \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$. Consider the critical set

$$\begin{aligned}
\Xi &:= \{(x,t,r,s,\theta,\tau)|\ \psi'_\theta = \psi'_\tau = 0\} \\
&= \{(x,t,r,s,\theta,\tau)|\ (x,t-s,\theta) \in \Sigma,\ r = \phi(x,t-s,\theta)\}.
\end{aligned}$$

The point of the Lagrangian submanifold of $T_0^*P \times T_0^*\mathbb{T}^2$ corresponding to $(x,t,r,s,\theta,\tau) \in \Xi$ is given by

$$\begin{aligned}
&(x,t,\psi'_x,\psi'_t,\phi(x,t-s,\theta),\psi'_r,s,\psi'_s)\\
&\quad= (x,t,\tau\phi'_x(x,t-s,\theta),\tau\phi'_t(x,t-s,\theta),\phi(x,t-s,\theta),-\tau,s,-\tau\phi'_t(x,t-s,\theta))\\
&\quad= (x,t,\tau\phi'_x(x,t,\theta),\tau\phi'_t(x,t,\theta),\phi(x,t-s,\theta),-\tau,s,-\tau)\\
&\quad= j(\lambda(x,t,\theta),\tau,e^{is}).
\end{aligned}$$

Thus, Λ is defined by the phase function ψ. Note that the signature of the matrix

$$\begin{pmatrix} \psi''_{\theta\theta}(x,t,r,s,\theta,\tau) & \psi''_{\theta\tau}(x,t,r,s,\theta,\tau) \\ \psi''_{\tau\theta}(x,t,r,s,\theta,\tau) & \psi''_{\tau\tau}(x,t,r,s,\theta,\tau) \end{pmatrix} = \begin{pmatrix} \tau\phi''_{\theta\theta}(x,t-s,\theta) & 0 \\ 0 & 0 \end{pmatrix}$$

is equal to that of $(\tau\phi''_{\theta\theta}(x,t,\theta))$ $(\tau>0)$, which means that the transition function for M_Λ (or M_C) is same as that of $M_{\overline{L}_P}$ (cf. [3, Ch.XXV]). □

Noticing the diagram above, we take for $k \in \mathbb{R}$ a half-density a on C written as

$$\text{(4.3)} \qquad a = (j'^{-1})^* b \quad \text{with} \quad b(\bar{\ell},\tau,z) = \beta(\ell)\tau^{k-1/2}|d\tau \wedge ds|^{1/2},$$

where $\bar{\ell} \in \overline{L}_P$, $\ell = p(\bar{\ell}) \in L_P$, $z = e^{is} \in S^1$, and β is a half-density on L_P. By virtue of Lemma 4.2 the Keller-Maslov bundle M_C is trivial, and we have a global section $\sigma_0 := (Pr \circ j'^{-1})^* s_0$ of M_C. Let a be a half-density given by (4.3). Then, the section $a \otimes \sigma_0$ of the bundle $|TC|^{1/2} \otimes M_C$ belongs to the symbol class $S^k(C;|TC|^{1/2} \otimes M_C)$. We have a Fourier integral operator $A \in I^{k-\frac{1}{4}(n+3)}(P \times \mathbb{T}^2; C)$ defined modulo $I^{k-\frac{1}{4}(n+3)-1}(P \times \mathbb{T}^2; C)$ by the principal symbol $a \otimes \sigma_0 \in S^k(C;|TC|^{1/2} \otimes M_C)$ (see [3, Theorem 25.1.9]).

5 Actions of $\mathbb{Z}_d$ and $U(1)$

We see in this section the properties of A in relation with the action of $\mathbb{Z}_d$ and $U(1)$. For $q \in \mathbb{Z}_d$ we define the operator $\hat{q} : C^\infty(\mathbb{T}^2) \to C^\infty(\mathbb{T}^2)$ by

$$(\hat{q}u)(r,s) := u(r - \frac{\pi q}{2}, s),$$

which is a Fourier integral operator of order zero associated with the canonical relation $C_q = \{((r+\frac{\pi q}{2},\rho),(s,\sigma);(r,\rho),(s,\sigma))\} \subset T_0^*\mathbb{T}^2 \times T_0^*\mathbb{T}^2$ and with the principal symbol (without the M_{C_q}-part) $|dr \wedge ds \wedge d\rho \wedge d\sigma|^{1/2}$. From (3.2) we have the composition of the canonical relations:

$$\text{(5.1)} \qquad C \circ C_q = \{(\tau\ell;(e^{ir},\tau),(z,\tau)) \text{ with } \alpha(z^{-1}\cdot\bar{\ell}) = e^{ir}e^{i\pi q/2}\}.$$

Note that $C \times C_q$ intersects $T_0^*P \times (\mathrm{diag}T_0^*\mathbb{T}^2) \times T_0^*\mathbb{T}^2$ transversally. Replace $\bar{\ell}$ with $e^{i\pi q/2}\cdot\bar{\ell}$. Then, we have $\alpha(z^{-1}\cdot\bar{\ell}) = e^{ir}$ from Lemma 3.1, and accordingly

$C = C \circ C_q$ concerning the canonical relation of the composition $A \circ \hat{q}$. On the other hand, the principal symbol $\sigma(A \circ \hat{q})$ is found to be $e^{i\pi q/2}\sigma(A)$ from (4.1), (4.3) and (5.1). Therefore, by considering

$$\frac{1}{d}\sum_{h\in\mathbb{Z}_d} e^{-i\pi h/2} A \circ \hat{h},$$

we get a Fourier integral operator $A \in I^{k-\frac{1}{4}(n+3)}(P \times \mathbb{T}^2; C)$ such that $\sigma(A) = a \otimes \sigma_0$ with a given by (4.3) and that

$$A \circ \hat{q} = e^{i\pi q/2} A \tag{5.2}$$

holds for $\forall q \in \mathbb{Z}_d$.

Next, for $\zeta = e^{it} \in U(1)$ we define

$$\hat{\zeta} : C^\infty(\mathbb{T}^2) \to C^\infty(\mathbb{T}^2); \quad (\hat{\zeta}u)(r,s) := u(r, s-t),$$

$$\hat{\zeta}_P : C^\infty(P) \to C^\infty(P); \quad (\hat{\zeta}_P f)(p) := f(p\cdot e^{-it}).$$

They are Fourier integral operators of order zero associated with the canonical relations C_ζ and C_ζ^P respectively induced from the transformations on the manifolds. We see the transversality condition between canonical relations and get

$$C_\zeta^P \circ C = \{(\tau(\zeta\cdot\ell); (\alpha(z^{-1}\cdot\bar{\ell}), \tau), (z, \tau))\} = \{(\tau\ell; (\alpha((\zeta z)^{-1}\cdot\bar{\ell}), \tau), (z, \tau))\} = C \circ C_\zeta$$

by noticing Lemma 3.2. For the half-density a given by (4.3) we assume that the half-density β on L_P is invariant under the $U(1)$ action, i.e.,

$$\hat{\zeta}\beta = \beta \quad \text{for } \forall\zeta \in U(1). \tag{5.3}$$

Note that the half-density $\beta := [(\Psi_{\mu_0} \circ \pi_{\mu_0})^*\breve{\beta}]|dt|^{1/2}$ induced from a half-density $\breve{\beta}$ on L is $U(1)$-invariant. It follows from (4.2) that the principal symbols of $\zeta_P^* \circ A$ and $A \circ \hat{\zeta}$ are the same. By considering the modified operator

$$\frac{1}{2\pi}\int_{U(1)} \hat{\zeta}_P^{-1} \circ A \circ \hat{\zeta} d\zeta$$

of A satisfying (5.2), we have the following.

Lemma 5.1 *Given $a \otimes \sigma_0 \in S^k(C; |TC|^{1/2} \otimes M_C)$, where a is given by (4.3) with β being $U(1)$-invariant. Then, there exists $A \in I^{k-\frac{1}{4}(n+3)}(P \times \mathbb{T}^2; C)$ with $\sigma(A) = a \otimes \sigma_0$ which satisfies (5.2) for $\forall q \in \mathbb{Z}_d$ and*

$$\hat{\zeta}_P \circ A = A \circ \hat{\zeta} \tag{5.4}$$

for $\forall\zeta \in U(1)$.

We see that the operator A in Lemma 5.1 has the property (A-iii) mentioned in §2. In fact, put $w_k := A(e^{im_k(r+s)})$, and it turns out from (5.4) that

$$\begin{aligned} w_k(p\cdot e^{it}) &= [\hat{\zeta}_P^{-1} w_k](p) = [\hat{\zeta}_P^{-1} \circ A e^{im_k(r+s)}](p) \qquad (\zeta = e^{it}) \\ &= [A \circ \hat{\zeta}^{-1} e^{im_k(r+s)}](p) = [A(e^{im_k t} e^{im_k(r+s)})](p) \\ &= e^{im_k t}[A e^{im_k(r+s)}](p) = e^{im_k t} w_k(p). \end{aligned}$$

6 The property (A-ii)

From now on we assume that $A \in I^{k-\frac{1}{4}(n+3)}(P \times \mathbb{T}^2; C)$ satisfies the properties (5.2) and (5.4) for the group actions.

We introduce some subspaces of $\mathcal{D}'(\mathbb{T}^2)$. For $q \in \mathbb{Z}_d$, let

$$\mathcal{D}'_{d,q}(\mathbb{T}^2) := \{u \in \mathcal{D}'(\mathbb{T}^2) | \widehat{(4/d)} u = e^{-i\pi q/2} u\},$$

where $4/d \in \mathbb{Z}_d$ and $[\widehat{(4/d)} u](r,s) = u(r - \frac{2\pi}{d}, s)$. Then, by the theory of Fourier series we see that

$$\mathcal{D}'(\mathbb{T}^2) = \bigoplus_{q \in \mathbb{Z}_d} \mathcal{D}'_{d,q}(\mathbb{T}^2).$$

Let q_0 be the element of $\mathbb{Z}_d$ such that $dq_0 = 4$ (if $d \neq 1$) or $q_0 = 0$ (if $d = 1$), and put $\mathcal{D}'_d(\mathbb{T}^2) := \mathcal{D}'_{d,q_0}(\mathbb{T}^2)$. Note that $L^2_d(\mathbb{T}^2) := L^2(\mathbb{T}^2) \cap \mathcal{D}'_d(\mathbb{T}^2)$ consists of functions $u(r,s)$ in the form:

$$u(r,s) = \sum_{k \in \mathbb{Z}} u_k(s) e^{i(dk+1)r}. \tag{6.1}$$

Lemma 6.1 *The orthogonal projection Π_d from $L^2(\mathbb{T}^2)$ to $L^2_d(\mathbb{T}^2)$ is written as*

$$\Pi_d = \frac{1}{d} \sum_{q \in \mathbb{Z}_d} e^{i\frac{\pi}{2}q} \hat{q}. \tag{6.2}$$

Proof. For $u(r,s) = \sum_{m \in \mathbb{Z}} u_m(s) e^{imr}$ the formula (6.2) gives

$$(\Pi_d u)(r,s) = \frac{1}{d} \sum_{q \in \mathbb{Z}_d} e^{i\frac{\pi}{2}q} \sum_{m \in \mathbb{Z}} u_m(s) e^{im(r - \frac{\pi}{2}q)} = \sum_m \Big(\frac{1}{d} \sum_q e^{-i\frac{\pi}{2}q(m-1)}\Big) u_m(s) e^{imr}.$$

Noticing that

$$\frac{1}{d} \sum_q e^{-i\frac{\pi}{2}q(m-1)} = \begin{cases} 1 & (m \equiv 1 \mod d) \\ 0 & (\text{other}) \end{cases},$$

we get the assertion. □

Lemma 6.2 *If $d \neq 1$ and u belongs to $\mathcal{D}'_{d,q}(\mathbb{T}^2)$ with $q \neq q_0$, then $Au = 0$.*

Proof. Let $u \in \mathcal{D}'_{d,q}(\mathbb{T}^2)$ $(q \neq q_0)$. Then, by means of (5.2) we have

$$e^{i\pi q/2}Au = Ae^{i\pi q/2}u = (A \circ \widehat{(4/d)})u = e^{2\pi i/d}Au.$$

Since $e^{i\pi q/2} \neq e^{2\pi i/d}$, we get $Au = 0$. □

Corollary 6.3 *$A = A\Pi_d$ holds.*

The adjoint operator A^* of $A \in I^{k-\frac{1}{4}(n+3)}(P \times \mathbb{T}^2; C)$ is a Fourier integral operator belonging to $I^{k-\frac{1}{4}(n+3)}(\mathbb{T}^2 \times P; C^{-1})$, where

$$C^{-1} = \{((\alpha(z^{-1}\cdot\bar{\ell}), \tau), (z, \tau); \tau\ell)\}.$$

Lemma 6.4 *$C^{-1} \times C$ and $T_0^*\mathbb{T}^2 \times (\mathrm{diag}T_0^*P) \times T_0^*\mathbb{T}^2$ intersect cleanly with excess n, and*

$$\begin{aligned} C^{-1} \circ C &= \{((\alpha(z_1^{-1}\cdot\bar{\ell}_1), \tau), (z_1, \tau); (\alpha(z_2^{-1}\cdot\bar{\ell}_2), \tau), (z_2, \tau)) \quad \textit{with } \ell_1 = \ell_2 = \ell\} \\ &= \{((\alpha(z_1^{-1}\cdot q\cdot\bar{\ell}), \tau), (z_1, \tau); (\alpha(z_2^{-1}\cdot\bar{\ell}), \tau), (z_2, \tau)) |\ \bar{\ell} \in \bar{L}_P, q \in \mathbb{Z}_d\}. \end{aligned}$$

Proof. An element of $(C^{-1} \times C) \cap (T_0^*\mathbb{T}^2 \times (\mathrm{diag}T_0^*P) \times T_0^*\mathbb{T}^2)$ is given by

$$(((\alpha(z_1^{-1}\cdot\bar{\ell}_1), \tau), (z_1, \tau), \tau\ell); (\tau\ell, (\alpha(z_2^{-1}\cdot\bar{\ell}_2), \tau), (z_2, \tau))).$$

Each tangent vector to $C^{-1} \times C$ at this point is given by

$$(((\langle\omega, v_1\rangle - b_1, a_1), (b_1, a_1), \tau v_1 + a_1\ell); (\tau v_2 + a_2\ell, (\langle\omega, v_2\rangle - b_2, a_2), (b_2, a_2))$$

(see (3.1)), and it belongs to $T[(C^{-1} \times C) \cap (T_0^*\mathbb{T}^2 \times (\mathrm{diag}T_0^*P) \times T_0^*\mathbb{T}^2)]$ if and only if

$$\tau v_1 + a_1\ell = \tau v_2 + a_2\ell, \quad \textit{i.e.,} \quad \tau(v_1 - v_2) = (a_2 - a_1)\ell. \tag{6.3}$$

The vertical vector $(a_2 - a_1)\ell$ is transversal to L_P because the Hamiltonian H_P is constant on L_P. Therefore, (6.3) means that $v_1 = v_2$, $a_1 = a_2$, and accordingly

$$\begin{aligned} &T[(C^{-1} \times C) \cap (T_0^*\mathbb{T}^2 \times (\mathrm{diag}T_0^*P) \times T_0^*\mathbb{T}^2)] \\ &\quad = T(C^{-1} \times C) \cap T[T_0^*\mathbb{T}^2 \times (\mathrm{diag}T_0^*P) \times T_0^*\mathbb{T}^2], \end{aligned}$$

which is just the clean intersection condition. The expression of $C^{-1} \circ C$ in the lemma is easy to see. The excess is the dimension of the fiber for the fiber space:

$$(C^{-1} \times C) \cap (T_0^*\mathbb{T}^2 \times (\mathrm{diag}T_0^*P) \times T_0^*\mathbb{T}^2) \to C^{-1} \circ C, \tag{6.4}$$

which is equal to the dimension of the space of $\bar{\ell} \in \bar{L}_P$ satisfying $\alpha(z^{-1}\bar{\ell}) = c$ for fixed z, $c \in U(1)$. Since the differential $T\alpha$ of $\alpha : \bar{L}_P \to S^1$ is surjective, we have $\dim(\mathrm{Ker}\alpha) = n$, and get the assertion. □

By virtue of Lemma 6.4 the product A^*A is a Fourier integral operator of order

$$2\Big(k - \frac{1}{4}(n+3)\Big) + \frac{n}{2} = 2k - \frac{3}{2}$$

associated to the canonical relation $C^{-1} \circ C \subset T_0^*\mathbb{T}^2 \times T_0^*\mathbb{T}^2$. Note that $C^{-1} \circ C$ consists of d connected components $\{M_q\}_{q \in \mathbb{Z}_d}$ given by

$$\begin{aligned} M_q &= \{((\alpha(z_1^{-1} \cdot q \cdot \bar{\ell}), \tau), (z_1, \tau); (\alpha(z_2^{-1} \cdot \bar{\ell}), \tau), (z_2, \tau)) |\ \bar{\ell} \in \overline{L}_P, q \in \mathbb{Z}_d\} \\ &= \{((e^{i\pi q/2} z_1^{-1} e^{iv}, \tau), (z_1, \tau); (z_2^{-1} e^{iv}, \tau), (z_2, \tau))\}. \end{aligned}$$

Moreover, we have $M_q = M_0 \circ C_q$ with C_q being the canonical relation for the operator $\hat{q}$ seen in §6. We have the decomposition

$$A^*A = \sum_{q \in \mathbb{Z}_d} K_q$$

corresponding to $C^{-1} \circ C = \cup_q M_q$. The principal symbol (without the Keller-Maslov bundle part) of K_q is given by

$$(6.5) \qquad (2\pi)^{-n/2} \int_{N_m} \bar{a} \times a,$$

where N_m denotes the fiber over $m \in M_q \subset C^{-1} \circ C$ for the fiber space (6.4). For $m = ((e^{i\pi q/2} z_1^{-1} e^{iv}, \tau), (z_1, \tau); (z_2^{-1} e^{iv}, \tau), (z_2, \tau)) \in M_q$ the fiber N_m is identified with the manifold

$$\overline{L}_P(v) := \{\bar{\ell} \in \overline{L}_P |\ \alpha(\bar{\ell}) = e^{iv}\},$$

which is, moreover diffeomorphic to $L_P(v) := p(\overline{L}_P(v)) \subset L_P$. The integral (6.5) for a given by (4.3) is calculated as

$$(6.6) \qquad \Big((2\pi)^{-n/2} \int_{L_P(v)} \overline{\beta}\beta\Big) \tau^{k-1/2} |dr \wedge ds \wedge ds' \wedge d\tau|^{1/2},$$

where (r, s, s', τ) is a local coordinate of M_q given by $z_1 = e^{is}$, $z_2 = e^{is'}$, and $e^{ir} = e^{i(v-s)}$. Concerning two manifolds $L_P(v)$ and $L_P(v')$ we have $L_P(v) = e^{i(v-v')} \cdot L_P(v')$. In fact, let $\bar{\ell}' \in \overline{L}_P(v')$ and let $\bar{\ell} := e^{i(v-v')} \cdot \bar{\ell}'$. Then, we have $\alpha(\bar{\ell}) = e^{i(v-v')} \alpha(\bar{\ell}') = e^{iv}$, and accordingly $\bar{\ell} \in \overline{L}_P(v)$. Therefore, $\int_{L_P(v)} \overline{\beta}\beta$ is a constant (independent on v) if β is $U(1)$-invariant.

Noticing that $M_q = M_0 \circ C_q$, we have $K_q = B_q \circ \hat{q}$, where B_q is a Fourier integral operator associated to M_0. Then, by means of (4.1) and (6.6) the principal symbol $\sigma(B_q)$ is equal to $e^{i\pi q/2} \sigma(B_0)$. Hence, we have $B_q = e^{i\pi q/2} B_0 + R_q$ with R_q being a smoothing operator. As a consequence, it follows from (6.2) and Lemma 6.2 that

$$A^*A = \sum_{q \in \mathbb{Z}_d} (B_0 \circ e^{i\pi q/2} \hat{q} + R_q \circ \hat{q}) = (dB_0 + R)\Pi_d,$$

where R is a smoothing operator.

Now, let $\Pi_\Delta : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ be the projection operator defined by

$$(\Pi_\Delta u)(r,s) := \sum_{m \geq 0} \hat{u}_{m,m} e^{imr} e^{ims}$$

for an element $u(r,s) = \sum_{\ell,m} \hat{u}_{\ell,m} e^{i\ell r} e^{ims}$ of $L^2(\mathbb{T}^2)$.

Lemma 6.5 *The operator Π_Δ is a Fourier integral operator of order $-1/2$ associated with the canonical relation*

$$C_\Delta = \{((r,\tau),(s,\tau);(r',\tau),(s',\tau)) | \; r+s = r'+s'\}. \tag{6.7}$$

Moreover, the principal symbol (without the M_{C_Δ} part) of Π_Δ is given by

$$\frac{1}{\sqrt{2\pi}} |dr \wedge ds \wedge ds' \wedge d\tau|^{1/2} \tag{6.8}$$

on C_Δ^+ and zero on C_Δ^-, where (r,r',v,τ) with $v := r+s = r'+s'$ is taken as a coordinate of C_Δ, and

$$C_\Delta^\pm := \{((r,\tau),(s,\tau);(r',\tau),(s',\tau)) \in C_\Delta \quad \textit{with } \tau \gtrless 0\}.$$

Proof. We have (formally)

$$\begin{aligned}
(\Pi_\Delta u)(r,s) &= \sum_{m\geq 0} \hat{u}_{m,m} e^{im(r+s)} \\
&= \sum_{m\geq 0} \Big((2\pi)^{-2} \int_0^{2\pi}\int_0^{2\pi} e^{-imr'} e^{-ims'} u(r',s') dr' ds' \Big) e^{im(r+s)} \\
&= (2\pi)^{-2} \iint \Big(\sum_{m\geq 0} e^{im(r-r'+s-s')} \Big) u(r',s') dr' ds' \\
&= (2\pi)^{-2} \iint [1 - e^{i(r-r'+s-s')}]^{-1} u(r',s') dr' ds'.
\end{aligned}$$

We can justify the above calculation by replacing $r+s$ by $r+s+i\varepsilon$ and go to the limit as $\varepsilon \to +0$. Suppose u belongs to $C_0^\infty(\mathbb{T}^2)$ and $\operatorname{supp} u \subset V$. Put

$$p(r,s,r',s') := (r-r'+s-s')[1 - e^{i(r-r'+s-s')}]^{-1} g(r',s').$$

where $g \in C_0^\infty(\mathbb{T}^2)$ with $g \equiv 1$ on V. Then, we have

$$\begin{aligned}
(\Pi_\Delta u)(r,s) &= (2\pi)^{-2} \iint \frac{p(r,s,r',s')}{r-r'+s-s'} u(r',s') dr' ds' \\
&= (2\pi)^{-2} \lim_{\varepsilon \to +0} \iint \frac{p(r,s,r',s')}{r-r'+s-s'+i\varepsilon} u(r',s') dr' ds'.
\end{aligned}$$

Note that

$$(r - r' + s - s' + i\varepsilon)^{-1} = \frac{1}{i}\int_0^{+\infty} e^{i\tau(r-r'+s-s'+i\varepsilon)}d\tau,$$

and we have

$$\begin{aligned}(\Pi_\Delta u)(r,s) &= \frac{1}{4\pi^2 i}\iint_{\tau>0} e^{i\tau(r-r'+s-s')}p(r,s,r',s')u(r',s')dr'ds'd\tau \\ &= (2\pi)^{-3/2}\iint_{\tau>0} e^{i\tau(r-r'+s-s')}\frac{1}{\sqrt{2\pi}i}p(r,s,r',s')u(r',s')dr'ds'd\tau.\end{aligned}$$

Thus Π_Δ is a Fourier integral operator with the non-degenerate linear phase function

$$\varphi(r,s,r',s',\tau) = \tau(r - r' + s - s').$$

The canonical relation is given by

$$C_\Delta = \{((r,\varphi'_r),(s,\varphi'_s);(r',-\varphi'_{r'}),(s',-\varphi'_{s'}))|\ \varphi'_\tau(r,s,r',s',\tau) = 0\},$$

which is just (6.7). By noticing that $p(r,s,r',s') = i$ when $r + s = r' + s'$, we obtain (6.8) as the principal symbol, which belongs to $S^{1/2}(|TC_\Delta|^{1/2})$. Thus the order of Π_Δ is equal to

$$\frac{1}{2} - \frac{4}{4} = -\frac{1}{2}.$$

□

Note that C_Δ^+ is equal to the canonical relation M_0 of the operator B_0. Set $k = 1/2$ in (4.3). Then, A (and B_0) belongs to $I^{-\frac{1}{4}(n+1)}(P \times \mathbb{T}^2; C)$, and the order of A^*A (and B_0) is equal to $-1/2$. Moreover, by virtue of (6.6) we can take such β (by multiplying a suitable constant) that the principal symbol of A^*A is equal to that of Π_Δ. As a consequence, there exists $A \in I^{-\frac{1}{4}(n+1)}(P \times \mathbb{T}^2; C)$ such that $dB_0 - \Pi_\Delta \in I^{-3/2}(\mathbb{T}^2 \times \mathbb{T}^2; C_\Delta)$. Thus we have

$$A^*A = (\Pi_\Delta + R')\Pi_d = \Pi + R'\Pi_d$$

where $\Pi := \Pi_\Delta\Pi_d = \Pi_d\Pi_\Delta$ is the orthogonal projection on $L^2(\mathbb{T}^2; \{m_k\})$, and $R' \in I^{-3/2}(\mathbb{T}^2 \times \mathbb{T}^2; C_\Delta)$. Here we notice the following.

Lemma 6.6 *Let $A' := A\Pi_\Delta$. Then, A' belongs to $I^{-\frac{1}{4}(n+1)}(P \times \mathbb{T}^2; C)$, and the principal symbol $\sigma(A')$ is equal to $\sigma(A)$.*

Proof. We can easily (similarly to the proof of Lemma 6.4) check that the composition $C \circ C_\Delta$ is clean with excess 1, and $C \circ C_\Delta = C$ holds. Thus, A' is associated with the canonical relation C and the order of A' is equal to

$$-\frac{1}{4}(n+1) - \frac{1}{2} + \frac{1}{2} = -\frac{1}{4}(n+1).$$

The fiber of $(\tau\ell; (z^{-1}\alpha(\bar{\ell}), \tau), (z, \tau)) \in C$ for the fiber space:

$$(C \times C_\Delta) \cap (T_0^*P \times (\mathrm{diag} T_0^*\mathbb{T}^2) \times T_0^*\mathbb{T}^2) \to C \circ C_\Delta = C$$

is diffeomorphic to $\{(z_1, z_2) \in S^1 \times S^1 | \, z_1 z_2 = \alpha(\bar{\ell})\} \cong S^1$, and we get the principal symbol of the product $A\Pi_\Delta$ from (4.3) and (6.8), which turns out to be equal to $\sigma(A)$. □

We recall that A' satisfies the property (5.4) in Lemma 5.1 and that $A' = A'\Pi_d$. By noticing the above lemma, we replace A by $A'(= A\Pi_\Delta)$. Then, we have

$$A^*A = \Pi + R'\Pi = (I + R_{-1})\Pi, \quad \text{with } R_{-1}\Pi = \Pi R_{-1} = \Pi R_{-1}\Pi \tag{6.9}$$

where R_{-1} is a pseudo-differential operator of order -1 on $\mathbb{T}^2$. In fact, it is easy to see that there exists a pseudo-differential operator R_{-1} such that $R'\Pi_\Delta = R_{-1}\Pi_\Delta$.

The operator $I + R_{-1}$ in (6.9) is a self-adjoint, elliptic pseudo-differential operator of order zero, and commutes with Π. It may not be bijective, so let N be its kernel. Then, N is a finite dimensional subspace of $L^2(\mathbb{T}^2)$ consisting of C^∞ functions. Let S be the orthogonal compliment of N, i.e.,

$$L^2(\mathbb{T}^2) = N \oplus S. \tag{6.10}$$

Let A_1 be an isometry of N onto a finite dimensional subspace of $C^\infty(P)$ orthogonal to the range of A. Define the linear operator $A^\# : L^2(\mathbb{T}^2) \to L^2(P)$ as being equal to A on S and to A_1 on N. Note that $A^\#$ differs from A by a smoothing operator. For $A^\#$ we have

$$(A^\#)^*A^\# = (I + R_{-1}^\#)\Pi \tag{6.9\#}$$

with $I + R_{-1}^\#$ being bijective (see [10, Ch.XII,p.648] for details). Finally, put $A := A^\#(I + R_{-1}^\#)^{-1/2}$, and we get

$$A^*A = \Pi. \tag{6.11}$$

Let u, v be functions in $C^\infty(\mathbb{T}^2) \cap L^2(\mathbb{T}^2; \{m_k\})$. Then, by virtue of (6.11) we have

$$(Au, Av)_{L^2(P)} = (A^*Au, v)_{L^2(\mathbb{T}^2)} = (\Pi u, v)_{L^2(\mathbb{T}^2)} = (u, v)_{L^2(\mathbb{T}^2)}.$$

Thus we see that A satisfies the property (A-ii).

7 The properties (A-i) and (A-iii)

We first see the property (A-iii). As already seen in the last part of §6 the property (A-iii) is derived from the commutativity (5.4) between A and the action of $U(1)$. However in the proof of (A-ii) in §7 we replace the original A (which satisfy (5.4)) by $A^{\#}(I+R^{\#}_{-1})^{-1/2}$, hence we have to check the revised A. It follows from (6.9) that

$$\hat{\zeta}(I+R_{-1}) = (I+R_{-1})\hat{\zeta} \tag{7.1}$$

holds for $\forall\zeta \in U(1)$ if A satisfies (5.4). From this fact we have

$$(I+R_{-1})(e^{im_k(r+s)}) = \lambda_k e^{im_k(r+s)} \quad (\lambda_k \in \mathbb{C}).$$

In fact, obviously $(I+R_{-1})(e^{im_k(r+s)}) \in L^2(\mathbb{T}^2;\{m_k\})$, hence we let

$$(I+R_{-1})(e^{im_k(r+s)}) = \sum_{\ell} a_\ell e^{im_\ell(r+s)}.$$

Then, (7.1) for $\zeta = e^{it}$ derives

$$\sum_{\ell} a_\ell e^{im_\ell t} e^{im_\ell(r+s)} = e^{im_k t}\sum_{\ell} a_\ell e^{im_\ell(r+s)},$$

and accordingly

$$a_\ell e^{im_\ell t} = a_\ell e^{im_k t} \quad \text{for } \forall\ell.$$

Thus we have $a_\ell = 0$ for $\ell \neq k$. Note that S = range of $(I+R_{-1})$ in (6.10), and we see that $e^{im_k(r+s)}$ belongs to N or S. Since N is finite dimensional, $e^{im_k(r+s)}$ belongs to S for $k \geq \exists k_0$. Recall that $A^{\#}$ is equal to A on S, we see $A^{\#}$ satisfies (5.4), and moreover (7.1) for $I+R^{\#}_{-1}$ on S by virtue of (6.9#). As a consequence, the property (A-iii) is checked to be satisfied. □

Next we see the property (A-i) for the operator $T := E^{-2}\Delta_P A - AD_{\mathbb{T}^2}$. It belongs to $I^{2-\frac{1}{4}(n+1)}(P\times\mathbb{T}^2;C)$ with its principal symbol vanishing on C seen as $(E^{-2}(\tau E)^2 - \tau^2)a = 0$. The subprincipal symbol of T is given by

$$i^{-1}\mathcal{L}_{H_D} a + \sigma_{sub}(D)a,$$

where $\mathcal{L}_{H_D}$ is the Lie derivative along the Hamiltonian flow H_D on $T_0^*P \times T_0^*\mathbb{T}^2$ defined by the principal symbol of the differential operator $D := E^{-2}\Delta_P \otimes I - I \otimes D_{\mathbb{T}^2}$ on $P\times\mathbb{T}^2$, and $\sigma_{sub}(D)$ is the subprincipal symbol of D. Here, note that the vector field H_D is tangent to C, and note that the $\sigma_{sub}(D) = 0$. Let $\check{\beta}$ be (as mentioned in Eigenvalue theorem) the non-zero half-density on $L \subset T^*M$ invariant under the magnetic flow. Put $\beta := [(\Psi_{\mu_0}\circ\pi_{\mu_0})^*\check{\beta}]|dt|^{1/2}$. Then, β is a $U(1)$-invariant half-density on L_P, and is invariant under the geodesic flow on T_0^*P (Liouville's theorem). Hence, we have $\mathcal{L}_{H_D} a = 0$. As a consequence, T belongs to $I^{-\frac{1}{4}(n+1)}(P\times\mathbb{T}^2;C)$.

The final step is to see that $T \in I^{-\frac{1}{4}(n+1)}(P\times\mathbb{T}^2;C)$ is a bounded operator from $L^2(\mathbb{T}^2)$ to $L^2(P)$. The theory of L^2 continuity of a Fourier integral operator associated to a homogeneous canonical relation (which is not the graph of a homogeneous canonical transformation) is developed in [3, Ch.XXV, Lemma 25.3.6

- Theorem 25.3.8]. As in §2 let (x, t, η, τ) be a local coordinate of T_0^*P, and let $m = ((x, t, \tau\eta, \tau); (r, \tau), (s, \tau))$ be a point in the canonical relation C. Then, from (3.1) the tangent space of C at m is given by

$$T_m C = \{((v_x, w, \tau v_\eta + a\eta, a); (\langle \omega_M, v_x\rangle + \langle \omega_{S^1}, w\rangle - b, a), (b, a))\},$$

Let $\pi_P : C \to T_0^*P$ and $\pi_{\mathbb{T}^2} : C \to T_0^*\mathbb{T}^2$ be the projections. The image of the differential $T_m\pi_P$ of π_P is decomposed as

$$\lambda_1 \oplus C_1 := \{(v_x, 0, \tau v_\eta, 0)\} \oplus \{(0, w, a\eta, a)\},$$

and that of $T_m\pi_{\mathbb{T}^2}$ is decomposed as

$$C_2 \oplus \lambda_2 := \{((w, a), (0, a))\} \oplus \{((-b, 0), (b, 0))\}.$$

On the other hand, we have the symplectically orthogonal decompositions

$$T_p(T_0^*P) = S_{11} \oplus S_{12} := \{(v_x, 0, v_\eta, 0)\} \oplus \{(0, w, 0, a)\} \cong T_{p'}(T^*M) \oplus T_{p''}(T^*S^1)$$

$(p := \pi_P(m))$, and

$$T_q(T_0^*\mathbb{T}^2) = S_{21} \oplus S_{22} := \{((w, a), (w, a))\} \oplus \{((-b, c), (b, -c))\},$$

$(q := \pi_{\mathbb{T}^2}(m))$. The space λ_1 is symplectically isomorphic to the tangent space of the Lagrangian submanifold L of $(T^*M, \Omega^M_{\mu_0})$ through the map $\Psi_{\mu_0} \circ \pi_{\mu_0} : J^{-1}(\mu_0) \to T^*M$, hence is a Lagrangian subspace of S_{11}. Obviously, λ_2 is a Lagrangian subspace of S_{22}. Moreover, we have $T_m C = \lambda_1 \oplus \tilde{C} \oplus \lambda_2$, where $\tilde{C}$ is the graph of a linear symplectic transformation from S_{21} to S_{12}. Let $\Omega_C := \pi_P^*\Omega_P (= \pi_{\mathbb{T}^2}^*\Omega_{\mathbb{T}^2})$. Then,

$$\text{corank of } \Omega_C = \dim \lambda_1 + \dim \lambda_2 = n + 1.$$

Therefore, by virtue of Theorem 25.3.8 in [3] $T \in I^{-\frac{1}{4}(n+1)}(P \times \mathbb{T}^2; C)$ is a bounded operator from $L^2(\mathbb{T}^2)$ to $L^2(P)$. Thus, the proof of Eigenvalue Theorem is completed. □

Finally, we refer to an example.

Example(cf. [4], [12]) Let S^{2n+1} be the sphere with radius 2 in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$. Then we have the principal $U(1)$-bundle (called the Hopf bundle):

$$\pi : S^{2n+1} \to \mathbb{C}P^n = S^{2n+1}/U(1). \tag{7.2}$$

This is the Riemannian submersion with $\mathbb{C}P^n$ being equipped with the so-called Fubini-Study metric of holomorphic sectional curvature 1, and induces the connection $\tilde{\nabla}$ on S^{2n+1}, whose curvature form Θ on $\mathbb{C}P^n$ is harmonic. The eigenvalues of $\sqrt{\hat{H}_m}$ $(m \in \mathbb{Z})$ are

$$\sqrt{\lambda_j^{(m)}} = \sqrt{(j + \frac{|m|}{2})(j + \frac{|m|}{2} + n)} = (j + \frac{|m| + n}{2}) + O\Big((j + \frac{|m| + n}{2})^{-1}\Big)$$

($j = 0, 1, 2, \dots$) (see [4]). On the other hand, due to Yoshioka [12] the classical dynamical system $(T^*M, \Omega^M_{\mu_0}, H^M_{\mu_0})$ ($M = \mathbb{C}P^n$) of magnetic flow is completely integrable, and the energies $\sqrt{H^M_{\mu_0}}$ of the Lagrangian tori satisfying the quantization condition (Q) are

$$E_\ell = \ell + \frac{n+1}{2} \quad (\ell = 0, 1, 2, \dots).$$

Moreover, $d = 2$ holds for the Maslov class of these Lagrangian torus. Hence, we have

$$(2k+1)E_\ell = (2k+1)\ell + (n+1)k + \frac{1}{2}(n+1) \quad (k = 0, 1, 2, \dots),$$

and

$$\sqrt{\lambda_j^{(2k+1)}} = (j + k + \frac{n+1}{2}) + O\Big((j + k + \frac{n+1}{2})^{-1}\Big).$$

Hence, by putting $j_k = (2k+1)\ell + nk$ we see the assertion of Eigenvalue Theorem.

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