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New Definition of Products of Distributions

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Abstract

In this article we give a new definition of products of distributions. Thereby we understand the product δ^2 of Dirac's δ reasonably. Further we obtain a new class of generalized functions, namely, a new class of continuous nonlinear distributions. Thus we found out new concrete examples of functions defined on an infinite dimensional locally convex space.

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Introduction

In the quantum field theory, we often have to calculate the square T^2 of a distribution T. Many authors tried to make sense out of the products of distributions. But, by the difficulty of divergence, nobody did not succeed in making sense out of them reasonably. For example, we refer the reader to B. Simon(1974), p.134. In this article, we give a new definition of products of distributions. Thereby, we can remove the difficulty of divergence.

Until now, to give the meaning to the square δ^2 of δ-function, several methods of defining products of distributions were considered. Then it is considered that the product ST of two distributions S and T is also a distribution. Namely it is considered that the product is also a linear functional. But, it should be considered that products of linear functionals are nonlinear functionals. It is not reasonable to consider them as linear functionals. This is a reason why we cannot define δ^2 reasonably.

The product fg of two functions f and g is defined to be a function which assign the product of values of f and g at a point x to the point $x \in \Omega$. Namely, we put

$$(fg)(x) = f(x)g(x), \ (x \in \Omega).$$

Here Ω is the domain of definition of the functions f and g, which is a domain in $\mathbf{R}^n$.

Let Ω be a domain in $\mathbf{R}^n$ and $\mathcal{D}(\Omega)$ the Schwartz's space of test functions on Ω. Schwartz's distributions S and T are considered as continuous linear functionals on $\mathcal{D}(\Omega)$ and they are, namely, considered as functions of an element φ of $\mathcal{D}(\Omega)$. Therefore, it is natural to define the product ST by the relation

$$(ST)(\varphi) = S(\varphi)T(\varphi), \ (\varphi \in \mathcal{D}(\Omega)).$$

Then ST is a nonlinear functional.

The product of an ordinary function and a distribution is not considered as a product of two distributions, the ordinary function being condsidered as a distribution. We consider the function as a differential operator of order zero and the product as an operation of this differential operator on the distribution. As far as we consider distributions as linear functionals, in general only the operations of addition and scalar multiplication are meaningful for distributions.

1. Definition of products of distributions

We know that $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$ is dense in $\mathcal{D}' = \mathcal{D}'(\mathbf{R}^n)$. Therefore, for the Dirac's delta function $\delta \in \mathcal{D}'$, there exists a sequence $f_j \in \mathcal{D}, (j = 1, 2, \cdots)$ such that we have $f_j \longrightarrow \delta$ in $\mathcal{D}'$. Then someone says that δ^2 can be defined by the relation

$$\lim_{j \to \infty} f_j^2 = \delta^2$$

(cf. B. Simon(1974), p.134). But, the convergence $f_j \longrightarrow \delta$ has a meaning because it is considered in the topology of $\mathcal{D}'$. In this time, it is not evident with respect to which topology $f_j^2 \longrightarrow \delta^2$ converges.

In general, it is a customary idea in the definition of products of distributions that ST is considered as an element in $\mathcal{D}'$ for two elements S and T in $\mathcal{D}'$.

But it is unreasonable that a product of linear functionals on $\mathcal{D}$ is considered as a linear functional on $\mathcal{D}$. Really, the product ST of $S, T \in \mathcal{D}'$ becomes a nonlinear functional on $\mathcal{D}$. It is accurate to define the product by the relation

$$(ST)(\varphi) = S(\varphi)T(\varphi), (\varphi \in \mathcal{D}).$$

2. Polynomial distributions of one variable

Now we define polynomial distributions of one variable. Let $P(X)$ be a polynomial of degree m of an indeterminate X. Then, we define the polynomial $P(T)$ of $T \in \mathcal{D}'$ by the relation

$$P(T)(\varphi) = P(T(\varphi)), \ (\varphi \in \mathcal{D}).$$

Then we define that two polynomial $P(T)$ and $Q(T)$ of T are equal if the equality

$$P(T)(\varphi) = Q(T)(\varphi)$$

holds for any $\varphi \in \mathcal{D}$. We call the polynomial $P(T)$ of a distribution T a polynomial distribution.

If $\varphi_j \longrightarrow \varphi$ holds in $\mathcal{D}$, $P(T)(\varphi_j) = P(T(\varphi_j)) \longrightarrow P(T(\varphi)) = P(T)(\varphi)$ holds. Therefore, $P(T)$ is a continuous nonlinear functional on $\mathcal{D}$.

Then, since a locally integrable function f on $\mathbf{R}^n$ is considered as an element T_f in $\mathcal{D}'$, $P(f)$ is considered as $P(T_f)$. Then, for $f \in C(\mathbf{R}^n)$, we have

$$T_f^2(\varphi) = f^2(\varphi) = f(\varphi)^2, \ (\varphi \in \mathcal{D}).$$

Here, if $\varphi \longrightarrow \delta_x$ in $\mathcal{D}'$, then $f^2(\delta_x)$ and $f(\delta_x)$ have a meaning, and we have

$$\begin{aligned} T_f^2 = T_f^2(\delta_x) = f^2(\delta_x) &= (f(\delta_x))^2 \\ &= f(x)^2 = f^2(x). \end{aligned}$$

Namely, f being considered as a distribution, the product f^2, as a polynomial distribution, coincides the product f^2 as an ordinary function. Therefore, polynomial distributions are a genelarization of polynomials of an ordinary function.

Now we consider δ^2. For $\varphi \in \mathcal{D}$, we have

$$\delta^2(\varphi) = (\delta(\varphi))^2 = \varphi(0)^2.$$

Namely, δ^2 is an nonlinear functional which assigns $\varphi(0)^2$ to every φ. Thereby we can give δ^2 a reasonable meaning. Here there is no difficulty of divergence.

More generally, we can consider nonlinear distributions defined by series of linear distributions. In the following, the domain of each nonlinear distributions is determined appropriately corresponding to the domain of convergence of each Taylor series.

Example 2.1. We have the following:

(1) $e^{\delta} = \Sigma_{n=1}^{\infty} \dfrac{\delta^n}{n!}$.

(2) $\log(1+\delta) = \Sigma_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} \delta^n$.

(3) $\sin\delta = \Sigma_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\delta^{2n+1}$.

(4) $\cos\delta = \Sigma_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\delta^{2n}$.

(5) $(1+\delta)^\alpha = \Sigma_{n=0}^{\infty}\binom{\alpha}{n}\delta^n$. Here α is a nonzero real number.

Here, as an example, the example 2.1(1) is defined as follows. We define e^δ to be, for every $\varphi \in \mathcal{D}$,

$$e^\delta(\varphi) = \Sigma_{n=0}^{\infty}\frac{1}{n!}\delta(\varphi)^n = \Sigma_{n=0}^{\infty}\frac{\varphi(0)^n}{n!} = e^{\varphi(0)}.$$

Example 2.2. We have the following for $p = 1, 2, \cdots$:

(1) $e^{\delta^{(p)}} = \Sigma_{n=1}^{\infty}\frac{(\delta^{(p)})^n}{n!}$.

(2) $\log(1+\delta^{(p)}) = \Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(\delta^{(p)})^n$.

(3) $\sin\delta^{(p)} = \Sigma_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\delta^{(p)})^{2n+1}$.

(4) $\cos\delta^{(p)} = \Sigma_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\delta^{(p)})^{2n}$.

(5) $(1+\delta^{(p)})^\alpha = \Sigma_{n=0}^{\infty}\binom{\alpha}{n}(\delta^{(p)})^n$. Here α is a nonzero real number.

Here, as an example, the example 2.2(1) is defined as follows. We define $e^{\delta^{(p)}}$ to be, for every $\varphi \in \mathcal{D}$,

$$e^{\delta^{(p)}}(\varphi) = \Sigma_{n=0}^{\infty}\frac{1}{n!}(\delta^{(p)}(\varphi))^n = \Sigma_{n=0}^{\infty}\frac{((-1)^p\varphi^{(p)}(0))^n}{n!} = e^{(-1)^p\varphi^{(p)}(0)}.$$

3. Polynomial distributions of several variables

Now we define polynomial distributions of several variables. Let $P(X_1, \cdots, X_d)$ be a polynomial of degree m of indeterminates $X_1, \cdots, X_d$. Then we define a polynomial $P(T_1, \cdots, T_d)$ of $T_1, \cdots T_d \in \mathcal{D}'$ by the relation

$$P(T_1, \cdots, T_d)(\varphi) = P(T_1(\varphi), \cdots, T_d(\varphi)), \ (\varphi \in \mathcal{D}).$$

Then we say that two polynomials $P(T_1, \cdots, T_d)$ and $Q(T_1, \cdots, T_d)$ of $T_1, \cdots, T_d$ are equal if we have, for every $\varphi \in \mathcal{D}$,

$$P(T_1, \cdots, T_d)(\varphi) = Q(T_1, \cdots, T_d)(\varphi).$$

Then we call $P(T_1, \cdots, T_d)$ a polynomial distribution of $T_1, \cdots, T_d$.

If $\varphi_j \longrightarrow \varphi$ in $\mathcal{D}$, we have $P(T_1, \cdots, T_d)(\varphi_j) \longrightarrow P(T_1, \cdots, T_d)(\varphi)$. Therefore, $P(T_1, \cdots, T_d)$ is a continuous nonlinear functional on $\mathcal{D}$.

Then, since locally integrable functions $f_1, \cdots, f_d$ on $\mathbf{R}^n$ are considered as elements $T_{f_1}, \cdots, T_{f_d}$ in $\mathcal{D}'$, $P(f_1, \cdots, f_d)$ can be considered as $P(T_{f_1}, \cdots, T_{f_d})$. Then the product fg of ordinary functions f and g can be considered as a product of corresponding distributions.

4. Cauchy-Hilbert transformation

Now we consider the Cauchy-Hilbert transformation of polynomial distributions. Here we put the Cauchy-Hilbert transformation of a distribution $T \in \mathcal{D}'$ with compact support as

$$F(z) = T_t((2\pi i)^{-n}\Pi_j(t_j - z_j)^{-1}).$$

Then, as the Cauchy-Hilbert transformation T^2, we have

$$\begin{aligned} T_t^2((2\pi i)^{-n}\Pi_j(t_j - z_j)^{-1}) &= \{T_t((2\pi i)^{-n}\Pi_j(t_j - z_j)^{-1})\}^2 \\ &= F(z)^2 = F^2(z). \end{aligned}$$

In general, the Cauchy-Hilbert transformation of a polynomial distribution $P(T)$ is

$$P(T)((2\pi i)^{-n}\Pi_j(t_j - z_j)^{-1}) = P(F(z)).$$

Let S and T be two distributions in $\mathcal{D}'$ with compact support. Then, if we put the Cauchy-Hilbert transformations S amd T as

$$F(z) = S_t((2\pi i)^{-n}\Pi_j(t_j - z_j)^{-1})$$

and

$$G(z) = T_t((2\pi i)^{-n}\Pi_j(t_j - z_j)^{-1}),$$

the Cauchy-Hilbert transformation ST is

$$(ST)_t((2\pi i)^{-n}\Pi_j(t_j - z_j)^{-1}) = F(z)G(z) = (FG)(z).$$

Let $T_i \in \mathcal{D}'(i = 1, \cdots, d)$ have the compact support. Then, putting the Cauchy-Hilbert transformation of T_i as

$$F_i(z) = T_{i,t}((2\pi i)^{-n}\Pi_j(t_j - z_j)^{-1}), \ (i = 1, \cdots, d),$$

the Cauchy-Hilbert transformation of a polynomial distribution $P(T_1, \cdots, T_d)$ is

$$\begin{aligned} &P(T_1, \cdots, T_d)((2\pi i)^{-n}\Pi_j(t_j - z_j)^{-1}) \\ &= P(F_1(z), \cdots, F_d(z)) = P(F_1, \cdots, F_d)(z). \end{aligned}$$

Here, $P(F_1, \cdots, F_d)$ is a holomorphic function defined on the intersection of the domains of definition of holomorphic functions $F_1, \cdots, F_d$.

At last we note that Imai's formal products in the theory of Sato hyperfunctions should be understood in this sense. (cf. Imai[1]).

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