J. Math. Tokushima Univ.
Vol. 34 (2000), 23-50

Theory of Quantum Probability

By

Yoshifumi ITO

Department of Mathematical and Natural Sciences
Faculty of Integrated Arts and Sciences
The University of Tokushima
Tokushima 770-8502, JAPAN
e-mail address: y-ito@ias.tokushima-u.ac.jp
(Received September 14, 2000)

Abstract

In this article, we give the axiom of the theory of quantum probability as a specialization of Masani's theory of orthogonally scattered measures[11]. Then we mention the basic notions and properties of quantum probability.

2000 Mathematics Subject Classification. Primary 81P10; Secondary 81S25, 60A05, 60A10

Introduction

In the general interpretation of a wave function $\psi(x,t)$ in old quantum theory, we understand the situation as follows: the time evolution of a quantum theoretical particle is described by a wave function and the particle is found in a certain space region D with the probability $\int_D |\psi(x,t)|^2 dx$. But this interpretation seems to be expedient and does not have the logically and mathematically decisive foundation. Therefore I had continued to look for such a decisive foundation. Recently I found a mathematical substance which seems to be useful for a logical and mathematical foundation of the interpretation of a wave function in old quantum theory. Thus, in this article, we wish to axiomatize the theory of the new mathematical substance. This is the theory of quantum probability as a specialization of Masani's theory of orthogonally scattered measures [11]. This is the counterpart in this case of the fact that the theory of probability has been obtained as a specialization of the theory of measures. Thus in this article we mention the basic notions and properties of quantum probability.

This article is the full presentation of the lecture of Y. Ito[6].

We note that the term "quantum probability" is the renaming of "hypoprobability measure" used in Y. Ito[6].

The author wishes to thank Professors H. Komatsu and M. Morimoto for many valuable discussions.

1. The notion of quantum probabilities and their basic properties

In this section we mention the notion of quantum probability and their basic properties .

Definition 1.1. Let $(\Omega, \mathcal{B})$ be a measurable space which is composed of a set Ω and a σ-algebra $\mathcal{B}$ of subsets of Ω. Let $\mathcal{H}$ be a complex Hilbert space. Then we say that ξ is an ($\mathcal{H}$-valued) quantum probability over $(\Omega, \mathcal{B})$ if the following are satisfied:

(i) ξ is a set function on $\mathcal{B}$ valued in $\mathcal{H}$.

(ii) If $A = \Sigma_{n=1}^{\infty} A_n$ (countable disjoint sum of $A_n \in \mathcal{B}$), then $\xi(A) = \Sigma_{n=1}^{\infty} \xi(A_n)$ (in the norm topology of $\mathcal{H}$).

(iii) $A, B \in \mathcal{B},\ A \cap B = \emptyset \Longrightarrow \xi(A) \perp \xi(B)$.

(iv) $\| \xi(\Omega) \| = 1$, where $\| \cdot \|$ is the norm on $\mathcal{H}$.

In this article, the final space of a set function ξ is always assumed to be a complex Hilbert space $\mathcal{H}$ except for the explicit expression of any specification of $\mathcal{H}$. So we call an $\mathcal{H}$-valued quantum probability a quantum probability if we need not to specify a Hilbert space $\mathcal{H}$.

By virtue of the definition of quantum probability over $(\Omega, \mathcal{B})$, we have the following immediately.

Corollary. *If ξ is a quantum probability over $(\Omega, \mathcal{B})$, then $\| \xi(\cdot) \|$ is a probability measure over $(\Omega, \mathcal{B})$.*

Definition 1.2. $\| \xi(\cdot) \|^2$ is called the probability measure associated with the quantum probability ξ, and denoted by $P_\xi(\cdot)$.

Then $P_\xi(A) = \| \xi(A) \|^2$ holds for $A \in \mathcal{B}$.

Example 1.3. (a) Let E be a projection-valued measure for $\mathcal{H}$ on $(\Omega, \mathcal{B})$, and, for any $x \in \mathcal{H}$ such as $\| x \| = 1$, let $\xi_x(A) = E(A)x$ for $A \in \mathcal{B}$. Then each ξ_x is a quantum probability over $(\Omega, \mathcal{B})$.

(b) Let $(\Omega, \mathcal{B}, Q)$ be a probability sapce and let $\mathcal{H} = L_2(\Omega, \mathcal{B}, Q)$. Let $\xi(A) = \chi_A$, the indicator function of A, $A \in \mathcal{B}$. Then, obviously, ξ is an $\mathcal{H}$-valued quantum probability over $(\Omega, \mathcal{B})$, and $P_\xi = Q$ holds.

(c) Let $\mathbf{A} = \{x_\omega; \omega \in \Omega\}$ be an orthonormal subset of $\mathcal{H}$, Ω being an index-set. Let $\mathcal{B}$ be the family of all subsets of Ω. Let

$$\xi(A) = \Sigma_{\omega \in A} a_\omega x_\omega, \ A \in \mathcal{B},$$

where the coefficients $\{a_\omega\}$ form a sequence of complex numbers such that

$$\Sigma_{\omega \in \Omega} |a_\omega|^2 = 1.$$

Then, obviously, ξ is an $\mathcal{H}$-valued quantum probability over $(\Omega, \mathcal{B})$.

Remark. The theory of quantum probability comes within the scope of the general theory of orthogonally scattered measures and the scope of an interpretation of the theory of old quantum theory. As for this fact, we refer the reader to Ito[5], Masani[11], Dirac[1] and Neumann[12].

By the connection between ξ and P_ξ in Definition 1.2, we have the monotonicity properties of the quantum probability ξ, reminiscent of probability measures, as the following.

Proposition 1.4. *Let ξ be a quantum probability over a measurable space $(\Omega, \mathcal{B})$. Then, for any $A, B \in \mathcal{B}$, we have the following:*

(1) $\| \xi(A) - \xi(B) \|^2 = P_\xi(A) + P_\xi(B) - 2P_\xi(A \cap B)$.

(2) $B \subset A \Longrightarrow \| \xi(A) - \xi(B) \|^2 = P_\xi(A) - P_\xi(B)$.

(3) $B \subset A \Longrightarrow \| \xi(B) \| \leq \| \xi(A) \|$.

(4) $B \subset A$ *and* $\xi(A) = 0 \Longrightarrow \xi(B) = 0$.

In many applications of our theory, a probability space $(\Omega, \mathcal{B}, P)$ is given initially, and an $\mathcal{H}$-valued set function ξ is defined on $\mathcal{B}$ initially. In verifying whether such a ξ is a quantum probability, the following result is useful:

Theorem 1.5(on equivalence). *Assume that (i) $(\Omega, \mathcal{B}, P)$ is a probability space and (ii) ξ is an $\mathcal{H}$-valued set function on $\mathcal{B}$. Then the following conditions are equivalent:*

(1) *For any $A, B \in \mathcal{B}$, $(\xi(A), \xi(B)) = P(A \cap B)$.*

(2) *ξ is an $\mathcal{H}$-valued quantum probability over $(\Omega, \mathcal{B})$ with $P_\xi = P$.*

In this case, ξ is called an ($\mathcal{H}$-valued) quantum probability over $(\Omega, \mathcal{B}, P)$.

Proof. Assume that (1) holds. Then obviously we have the following:

(1.1) $A \in \mathcal{H} \Longrightarrow \| \xi(A) \|^2 = P(A) \leq 1$.

(1.2) $A, \ B \in \mathcal{B}, \ A \cap B = \emptyset \Longrightarrow \xi(A) \perp \xi(B)$.

(1.3) $\| \xi(\Omega) \|^2 = P(\Omega) = 1$.

It only remains to show that ξ is countably additive. Let $A = \Sigma_n A_n$ be a disjoint sum of countable subsets $A_n \in \mathcal{B}(n = 1, 2, 3, \cdots)$. Then, by (i), we have

$$P(A) = \Sigma_{n=1}^{\infty} P(A_n) = \lim_{N \to \infty} \Sigma_{n=1}^{N} P(A_n). \tag{1.4}$$

By (1.2) we have $\xi(A_m) \perp \xi(A_n)$ for $m \neq n$. Hence, by (1) and the Pythagorean Identity, we have

$$\begin{aligned}(1.5) \qquad & \| \xi(A) - \Sigma_{n=1}^{N}\xi(A_n) \|^2 \\ &= \| \xi(A) \|^2 + \Sigma_{n=1}^{N} \| \xi(A_n) \|^2 - 2\mathrm{Re}(\xi(A), \Sigma_{n=1}^{N}\xi(A_n)) \\ &= P(A) - \Sigma_{n=1}^{N} P(A_n).\end{aligned}$$

By (1.4), the right hand side of (1.5) tends to 0, as $N \to \infty$. This shows that ξ is countably additive on $\mathcal{B}$. Thus (2) holds.

Next, assume that (2) holds. Then it easily follows (on taking only the first m of the A_n's to be nonvoid) that

$$(1.6) \qquad A = \Sigma_{n=1}^{m} A_n, \text{ (disjoint sum)}, \ A_n \in \mathcal{B}\,(n = 1, 2, \cdots m)$$
$$\xi(A) = \Sigma_{n=1}^{m}\xi(A_n).$$

Namely, ξ is finitely additive on $\mathcal{B}$. Now let $A, B \in \mathcal{B}$. Then we have

$$(1.7) \begin{cases} A = (A \cap B) + (A \backslash B), & (A \cap B) \cap (A \backslash B) = \emptyset, \\ B = (A \cap B) + (B \backslash A), & (A \cap B) \cap (B \backslash A) = \emptyset, \\ (A \backslash B) \cap (B \backslash A) = \emptyset. \end{cases}$$

Thus it follows from (1.6) that

$$\begin{aligned}(1.8) \quad (\xi(A), \xi(B)) &= (\xi(A \cap B) + \xi(A \backslash B), \xi(A \cap B) + \xi(B \backslash A)) \\ &= (\xi(A \cap B), \xi(A \cap B)) + (\xi(A \cap B), \xi(B \backslash A)) \\ &\quad + (\xi(A \backslash B), \xi(A \cap B)) + (\xi(A \backslash B), \xi(B \backslash A)) \\ &= (\xi(A \cap B), \xi(A \cap B)) = \| \xi(A \cap B) \|^2 = P(A \cap B).\end{aligned}$$

Thus (1) holds. Q.E.D.

For ease of reference, we record here some other obvious properties of our quantum probabilities:

Proposition 1.6. *Let ξ be a quantum probability over a measurable space $(\Omega, \mathcal{B})$, and let $A, B \in \mathcal{B}$. Then we have the following:*

(1) $\xi(A \backslash B) = \xi(A) - \xi(A \cap B)$.
(2) $B \subset A \Longrightarrow \xi(A \backslash B) = \xi(A) - \xi(B)$.
(3) $\xi(A \cup B) = \xi(A) + \xi(B) - \xi(A \cap B)$.
(4) $\xi(A \triangle B) = \xi(A) + \xi(B) - 2\xi(A \cap B)$, *where* $A \triangle B = (A \backslash B) + (B \backslash A)$.
(5) $P_\xi(A \triangle B) = \| \xi(A) - \xi(B) \|^2$.

By Proposition 1.6(5), we readily infer the following useful result.

Proposition 1.7. *Let ξ be a quantum probability over $(\Omega, \mathcal{B})$. Then ξ is uniformly continuous on $(\Omega, \mathcal{B})$ or on $\mathcal{B}$ under the usual metric ρ on $\mathcal{B}$ defined by*

$$\rho(A, B) = P_\xi(A \triangle B), \text{ for any } A, B \in \mathcal{B}.$$

In particular, for A, $A_n \in \mathcal{B}(n = 1, 2, 3, \cdots)$, we have

$$P_\xi(A \triangle A_n) \to 0 \Longrightarrow \xi(A) = \lim_{n \to \infty} \xi(A_n).$$

2. The range of a quantum probability

Since the values of a quantum probability ξ are vectors in the Hilbert space $\mathcal{H}$, it is natural to associate with ξ the least closed linear subspace spanned by these vectors, and to study its properties, e.g. separability.

Definition 2.1. Let ξ be a quantum probability over a measurable space $(\Omega, \mathcal{B})$. Then the least closed linear subspace spanned by $\xi(\mathcal{B})$, $\mathcal{S}\{\xi(\mathcal{B})\}$, i.e., $\mathcal{S}\{\xi(A); A \in \mathcal{B}\}$, is called the range of the quantum probability ξ and denoted by $\mathcal{S}_\xi$.

From the facts in section 3 it emerge that every x in $\mathcal{S}_\xi$ has the essentially unique integral expansion in terms of the quantum probability ξ. This suggests calling ξ a quantum probability basis for the range $\mathcal{S}_\xi$, and adopting the following definition.

Definition 2.2. A quantum probability ξ on a measurable space $(\Omega, \mathcal{B})$ for which $\mathcal{S}_\xi = \mathcal{H}$ is called an $\mathcal{H}$-basic quantum probability

Definition 2.3. Let ξ be a quantum probability on a measurable space $(\Omega, \mathcal{B})$. Then we call $\mathcal{B}$ or $(\Omega, \mathcal{B})$ separable if the σ-algebra $\mathcal{B}$ is separable as a metric space with the metric ρ in Proposition 1.7.

Then we have a simple sufficient condition for the separability of the range $\mathcal{S}_\xi$.

Corollary. *Let ξ be a quantum probability over a measurable space $(\Omega, \mathcal{B})$ such that $\mathcal{B}$ is separable. Then $\mathcal{S}_\xi$ is a separable subspace of $\mathcal{H}$.*

Proof. Let $\mathcal{B}_0$ be a dense subclass of $\mathcal{B}$ considered as in Definition 2.3.

Then it is obvious that the family $\mathcal{F}$ of finite linear combinations of vectors of $\xi(\mathcal{B}_0)$ formed with coefficients having rational real and imaginary parts is countable and everywhere dense in $\mathcal{S}_\xi$. Q.E.D.

3. The notion of quantum expectation and its basic properties

Let $(\Omega, \mathcal{B}, P)$ be a probability space and ξ a quantum probability over $(\Omega, \mathcal{B}, P)$. Now we wish to define the quantum expectation $\mathcal{E}[\phi] = \int_\Omega \phi(\omega)\xi(d\omega)$ for a suitable complex-valued functions ϕ on Ω. We do this along the line adopted by P. Masani[11] for defining the integration with respect to $\mathcal{H}$-valued, countably additive, orthogonally scattered measures.

Fundamental to the development is the following well-known result.

Theorem 3.1. *Let $(\Omega, \mathcal{B}, P)$ be a probability space and $L_{2,P} = L_2(\Omega, \mathcal{B}, P)$ the set of all complex-valued $\mathcal{B}$-measurable functions ϕ on Ω such that $E[|\phi|^2] = \int_\Omega |\phi(\omega)|^2 P(d\omega) < \infty$ holds. Then*

(1) $L_{2,P}$ *is a Hilbert space under the inner product*

$$(\phi, \psi)_P = E[\phi\bar{\psi}] = \int_\Omega \phi(\omega)\overline{\psi(\omega)}P(d\omega),$$

when functions differing on events of probability zero are identified.

(2) *The set of all* $\mathcal{B}$*-simple functions,* $\Sigma_{n=1}^r a_n\chi_{A_n}$, $A_n \in \mathcal{B}$, $a_n \in \mathbf{C}$, *is everywhere dense in* $L_{2,P}$.

Here we use the notation of expectation

$$E[\phi] = \int_\Omega \phi(\omega)P(d\omega)$$

for a random variable ϕ.

Now let ξ be a quantum probability over $(\Omega, \mathcal{B}, P)$. We wish to define for each $\phi \in L_{2,P}$ a quantum expectation of ϕ with respect to the quantum probability ξ:

$$\mathcal{E}[\phi] = \int_\Omega \phi(\omega)\xi(d\omega)$$

so that it has the following properties:

$$(\mathrm{QE}) \begin{cases} (1)\mathcal{E}[\phi] \in \mathcal{H}, \\ (2)(\mathcal{E}[\phi], \mathcal{E}[\psi]) = (\phi, \psi)_P = E[\phi\bar{\psi}]. \end{cases}$$

We single out these properties because they entail all the other properties we want our quantum expectation to have, as the next lemma shows.

Lemma 3.2. *Every quantum expectation* $\mathcal{E}[\phi]$, *defined for a random variable* ϕ *in* $L_{2,P}$, *and having the properties (QE), has the following properties:*

For any $\phi, \psi, \phi_n \in L_{2,P}(n = 1, 2, 3, \cdots)$ *and any complex numbers* a, b,

(1) $\| \mathcal{E}[\phi] \|^2 = \| \phi \|_P^2 \equiv E[|\phi|^2]$.

(2) *If we set*

$$\mathcal{E}[\phi; A] = \int_\Omega \phi(\omega)\chi_A(\omega)\xi(d\omega),$$

where $\chi_A(\omega)$ *denotes the indicator function of an event* $A \in \mathcal{B}$, *then, for any* $A, B \in \mathcal{B}$, *we have*

$$(\mathcal{E}[\phi; A], \mathcal{E}[\psi; B]) = (\mathcal{E}[\phi; A \cap B], \mathcal{E}[\psi; A \cap B]).$$

(3) $\mathcal{E}[a\phi + b\psi] = a\mathcal{E}[\phi] + b\mathcal{E}[\psi]$.

(4) $\| \mathcal{E}[\phi_m] - \mathcal{E}[\phi_n] \|^2 = E[|\phi_m - \phi_n|^2]$.

(5) $\phi_n \longrightarrow \phi \,\mathrm{in}\, L_{2,P} \Longleftrightarrow \mathcal{E}[\phi_n] \longrightarrow \mathcal{E}[\phi] \,\mathrm{in}\, \mathcal{H}$.

Proof. (1) Take $\psi = \phi$ in QE(2).

(2) We have, by the definition and QE(2),

$$(\mathcal{E}[\phi; A], \mathcal{E}[\psi; B]) = (\mathcal{E}[\phi\chi_A], \mathcal{E}[\psi\chi_B]) = (\phi\chi_A, \psi\chi_B)_P$$

$$= (\phi\chi_{A\cap B}, \psi\chi_{A\cap B})_P = (\mathcal{E}[\phi; A\cap B], \mathcal{E}[\psi; A\cap B]).$$

(3) We have by QE(2), for any $f \in L_{2,P}$,

$$(\mathcal{E}[a\phi + b\psi], \mathcal{E}[f]) = (a\phi + b\psi, f)_P = a(\phi, f)_P + b(\psi, f)_P$$

$$= a(\mathcal{E}[\phi], \mathcal{E}[f]) + b(\mathcal{E}[\psi], \mathcal{E}[f]) = (a\mathcal{E}[\phi] + b\mathcal{E}[\psi], \mathcal{E}[f]).$$

Hence we have

(3.1) $\quad x = \mathcal{E}[a\phi + b\psi] - a\mathcal{E}[\phi] - b\mathcal{E}[\psi] \perp \mathcal{E}[f].$

Now let L be the set of all $\mathcal{E}[f]$, for $f \in L_{2,P}$. Then, by (3.1), $x \perp L$ and therefore $x \perp < L >$, where $< L >$ is the linear subspace in $\mathcal{H}$ spanned by L. But $x \in < L >$ by virtue of (3.1). Thus $x = 0$, and this yields (3).

(4) is obvious from (3) and (1).

(5) follows from (4). Q.E.D.

We now define the quantum expectation in two steps so as to ensure the properties (QE).

Step 1. For a $\mathcal{B}$-simple function ϕ, $\phi = \Sigma_{n=1}^{r} a_n \chi_{A_n}$, $A_n \in \mathcal{B}$, $a_n \in \mathbf{C}$, we put

$$\mathcal{E}[\phi] = \Sigma_{n=1}^{r} a_n \xi(A_n).$$

A simple computation shows that

(3.2) For a $\mathcal{B}$-simple ϕ, $\mathcal{E}[\phi]$ has the properties (QE).

Consequently it has all the properties (1) $\sim$ (5) in Lemma 3.2.

Step 2. Now let $\phi \in L_{2,P}$. Then, by Theorem 3.1(2), there exists a sequence $(\phi_n)_{n=1}^{\infty}$ of $\mathcal{B}$-simple functions ϕ_n such that $\phi_n \longrightarrow \phi$ in $L_{2,P}$. This sequence is Cauchy in $L_{2,P}$, and therefore, by Lemma 3.2(4), the sequence $(\mathcal{E}[\phi_n])_{n=1}^{\infty}$ is Cauchy in $\mathcal{H}$, and so has a limit $x \in \mathcal{H}$. Furthermore, if $(\psi_n)_{n=1}^{\infty}$ is another sequence of $\mathcal{B}$-simple functions converging to ϕ in $L_{2,P}$, then $\phi_n - \psi_n \longrightarrow 0$ in $L_{2,P}$, and therefore, by Lemma 3.2(4),

$$\| \mathcal{E}[\phi_n] - \mathcal{E}[\psi_n] \| \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

This shows that the limit x depends only on ϕ, and not on the approximating sequence of $\mathcal{B}$-simple functions. Thus we have the following natural definition.

Definition 3.3. (i) For a $\mathcal{B}$-simple function ϕ, $\phi = \Sigma_{n=1}^{r} a_n \chi_{A_n}$, $A_n \in \mathcal{B}$, $a_n \in \mathbf{C}$, we define $\mathcal{E}[\phi]$ by the formula

$$\mathcal{E}[\phi] = \Sigma_{n=1}^{r} a_n \xi(A_n).$$

(ii) For an arbitrary ϕ in $L_{2,P}$, which is not $\mathcal{B}$-simple, we define $\mathcal{E}[\phi]$ by the formula

$$\mathcal{E}[\phi] = \lim_{n\to\infty} \mathcal{E}[\phi_n],$$

where $(\phi_n)_{n=1}^{\infty}$ is any sequence of $\mathcal{B}$-simple functions ϕ_n converging to ϕ in $L_{2,P}$. Then we call this $\mathcal{E}[\phi]$ the quantum expectation of ϕ with respect to the quantum probability ξ.

From (3.2) and Definition 3.3, it follows easily that our quantum expectation has the properties (QE). Hence from Lemma 3.2 we conclude the following.

Theorem 3.4. *For any $\phi \in L_{2,P}$, the quantum expectation $\mathcal{E}[\phi]$ has all the properties of* (QE)*and Lemma 3.2.*

Corollary. *The set of all $\mathcal{E}[\phi]$, for $\phi \in L_{2,P}$, is the range $\mathcal{S}_\xi$ of the quantum probability ξ.*

Proof. Put $\mathcal{S} = \{\mathcal{E}[\phi]; \phi \in L_{2,P}\}$. By Lemma 3.2(3), $\mathcal{S}$ is a linear subspace in $\mathcal{H}$. Next, if $x = \lim_{n\to\infty} \mathcal{E}[\phi_n]$, where $\phi_n \in L_{2,P}$, then, from Lemma 3.2(4), we see that there exists $\phi \in L_{2,P}$ such that $x = \mathcal{E}[\phi]$. Thus,

(3.3) $\mathcal{S}$ is a closed linear subspace of $\mathcal{H}$.

Now, by the definition of the quantum expectation, every x in $\mathcal{S}$ is a linear combination $\Sigma_{n=1}^r a_n \xi(A_n)$, $A_n \in \mathcal{B}$, $a_n \in \mathbf{C}$, or a limit of such linear combinations. Thus we have

$$\mathcal{S} \subset \mathcal{S}\{\xi(\mathcal{B})\} = \mathcal{S}_\xi.$$

But, by the definition, we have $\xi(B) = \mathcal{E}(\chi_B) \in \mathcal{S}$ for any $B \in \mathcal{B}$. Hence by (3.3) we have $\mathcal{S}_\xi \subset \mathcal{S}$. Thus we have $\mathcal{S}_\xi = \mathcal{S}$. Q.E.D.

We can subsume the last theorem and corollary in the following useful result.

Theorem 3.5. *Let $(\Omega, \mathcal{B}, P)$ be a probability space and ξ a quantum probability over $(\Omega, \mathcal{B}, P)$. Then the correspondence $U : \phi \to \mathcal{E}[\phi]$ is an isometry on $L_{2,P} = L_2(\Omega, \mathcal{B}, P)$ onto $\mathcal{S}_\xi(\subset \mathcal{H})$.*

Thus, every such ξ carries with it two Hilbert spaces, $\mathcal{S}_\xi$ and $L_{2,P}$, isomorphic under the natural correspondence U.

To each x in $\mathcal{S}_\xi$ thus corresponds a unique ϕ in $L_{2,P}$ such that $x = \mathcal{E}[\phi]$. This justifies our using the term basis (in Definition 2.2 and above) in connection with the quantum probability ξ.

Let ξ be a quantum probability over a given probability space $(\Omega, \mathcal{B}, P)$. Let M_ξ be the projection on $\mathcal{H}$ onto $\mathcal{S}_\xi$. Then, by the last theorem, to each $x \in \mathcal{H}$ corresponds a $\phi_x \in L_{2,P}$ such that

$$M_\xi(x) = \mathcal{E}[\phi_x]. \tag{3.4}$$

How are x and ϕ_x related? Obviously,

$$\phi_x = U^{-1}(M_\xi(x)) = (U^{-1}M_{\mathcal{R}(U)})(x) = U^* x, \tag{3.5}$$

where U^* is the adjoint of U, U being regarded as isometry on $L_{2,P}$ into $\mathcal{H}$ and $\mathcal{R}(U)$ denotes the range of U. To find a morerevealing connection between x and ϕ_x than that in (3.5), consider the case when the probability space $(\Omega, \mathcal{B}, P)$ is given as follows:

$$\left\{ \begin{array}{l} (1)\ \Omega \text{ is a countable set } \Omega = \{1, 2, 3, \cdots\}. \\ (2)\ \mathcal{B} \text{ is a } \sigma - \text{algebra of all subsets of } \Omega. \\ (3)\ P = P_\xi \text{ for a certain quantum probability } \xi \text{ over } (\Omega, \mathcal{B}). \end{array} \right. \tag{3.6}$$

Then $(\xi(n); n = 1, 2, 3, \cdots)$ is an orthogonal sequence in $\mathcal{H}$, and in place of (3.4) we have

$$(3.4)' \qquad M_\xi(x) = \Sigma_{n=1}^{\infty}(x, \frac{\xi(n)}{\| \xi(n) \|})\frac{\xi(n)}{\| \xi(n) \|}.$$

It follows that ϕ_x is defined on Ω, and for any $n = 1, 2, 3, \cdots$,

$$(3.5)' \qquad \phi_x(n) = (x, \frac{\xi(n)}{\| \xi(n) \|})\frac{1}{\| \xi(n) \|}$$

$$= \frac{(x, \xi(n))}{P(\{n\})} = \frac{dQ_x}{dP}(n),$$

where the last is the Radon-Nikodym derivative at n of the complex measure $Q_x(\cdot) = (x, \xi(\cdot))$ with respect to P.

Theorem 3.6(Projection Theorem). *Let ξ be a quantum probability over a given probability space $(\Omega, \mathcal{B}, P)$. For any $x \in \mathcal{H}$, put $Q_x(B) = (x, \xi(B))$, $B \in \mathcal{B}$. Let m_x be the total variation $|Q_x|$ of Q_x. Then we have the following:*

(1) *For any $x \in \mathcal{H}$, Q_x is absolutely continuous with respect to P on $\mathcal{B}$.*

(2) *One determination of the Radon-Nikodym derivative dQ_x/dP is $U^*(x)$, (cf. (3.5)); thus we have*

$$dQ_x/dP \in L_{2,P} = L_2(\Omega, \mathcal{B}, P)$$

and

$$M_\xi(x) = \int_\Omega \frac{dQ_x}{dP}\xi(d\omega).$$

Proof. (1) We first prove that

$$Q_x \prec\prec P \text{ on } \mathcal{B}.$$

This is obvious for $x = 0$. If $x \neq 0$, put $\delta_\epsilon = (\epsilon / \| x \|)^2$; then clearly

$$B \in \mathcal{B} \text{ and } P(B) < \delta_\epsilon \Longrightarrow |Q_x(B)| = |(x, \xi(B))| \leq \| x \| \sqrt{\delta_\epsilon} = \epsilon.$$

(2) We know from Theorem 3.6 and the below that

$$(3.7) \qquad M_\xi(x) = \int_\Omega \phi_x(\omega)\xi(d\omega), \text{ where } \phi_x = U^*(x).$$

But since $x - M_\xi(x) \perp \xi(B)$, $B \in \mathcal{B}$, therefore for any $B \in \mathcal{B}$, we have

$$(3.8) \qquad Q_x(B) = (M_\xi(x), \xi(B))$$

$$= (\mathcal{E}[\phi_x], \mathcal{E}[\chi_B]), \text{ by (3.7)},$$

$$= E[\phi_x; B], \text{ by QE(2)}.$$

This shows that ϕ_x is a version of dQ_x/dP. The rest now follows from (3.7). Q.E.D.

Remark. As noted before Theorem 3.6, when ξ is a quantum probability over the probability space $(\Omega, \mathcal{B}, P)$ as in (3.6), Theorem 3.6(2) becomes

$$M_\xi(x) = \Sigma_{n=1}^{\infty} \frac{(x, \xi(n))}{P(\{n\})}\xi(n) = \Sigma_{n=1}^{\infty}(x, \frac{\xi(n)}{\| \xi(n) \|})\frac{\xi(n)}{\| \xi(n) \|},$$

which is the well-known expansion of $M_\xi(x)$ in terms of an orthogonal set $(\xi(n); n = 1, 2, 3, \cdots)$. Thus Theorem 3.6 is a generalization of this discrete result.

Corollary. *In the situation of Theorem 3.6, we have*

$$(x, \mathcal{E}[\phi]) = \int_\Omega \bar{\phi}(\omega) Q_x(d\omega), \text{ for any } \phi \in L_{2,P}.$$

Proof. Since $x - M_\xi(x) \perp \mathcal{E}[\phi]$, therefore we have, by the last theorem,

$$\text{the left hand side} = (\mathcal{E}[dQ_x/dP], \mathcal{E}[\phi])$$

$$= \int_\Omega \frac{dQ_x}{dP}(\omega) \cdot \bar{\phi}(\omega) P(d\omega) = \text{the right hand side},$$

the last equality being due to the substitution rule for the Radon-Nikodym derivatives. Q.E.D.

We deal next with the effect of an isometric linear transformation on our quantum probability and quantum expectation.

Theorem 3.7. *Let ξ be an $\mathcal{H}$-valued quantum probability over a given probability space $(\Omega, \mathcal{B}, P)$. Let T be an isometry on $\mathcal{S}_\xi$ into a Hilbert space $\mathcal{H}_1$, and put $\eta(B) = T\{\xi(B)\}$ for any $B \in \mathcal{B}$. Then we have the following:*

(1) *η is an $\mathcal{H}_1$-valued quantum probability over $(\Omega, \mathcal{B}, P)$.*

(2) *For any $\phi \in L_2(\Omega, \mathcal{B}, P)$,*

$$T\{\int_\Omega \phi(\omega)\xi(d\omega)\} = \int_\Omega \phi(\omega)\eta(d\omega)$$

holds.

(3) *$\mathcal{S}_\eta = T(\mathcal{S}_\xi)$.*

Proof. (1) follows easily from the assumptions, since, for $A, B \in \mathcal{B}$, we have

$$(\eta(A), \eta(B))_{\mathcal{H}_1} = (\xi(A), \xi(B))_{\mathcal{H}} = P(A \cap B).$$

(2) is easily verified for $\mathcal{B}$-simple functions ϕ, and then by the usual limiting arguments for arbitrary $\phi \in L_{2,P}$ because of the continuity of T.

(3) follows at once from (2). Q.E.D.

Conversely, a pair of $\mathcal{H}$-valued and $\mathcal{H}_1$-valued quantum probabilities ξ and η over the same probability space determines an isometry on $\mathcal{S}_\xi$ onto $\mathcal{S}_\eta$.

Theorem 3.8. *Let ξ and η be, respectively, $\mathcal{H}$-valued and $\mathcal{H}_1$-valued quantum probabilities over the same probability space $(\Omega, \mathcal{B}, P)$. Then there exists an isometry T on $\mathcal{S}_\xi$ onto $\mathcal{S}_\eta$ such that, for any $B \in \mathcal{B}$, $\eta(B) = T\{\xi(B)\}$ holds.*

Proof. By Theorem 3.5, there exist isometries U_ξ, U_η on $L_{2,P}$ onto $\mathcal{S}_\xi$, $\mathcal{S}_\eta$, respectively, such that

$$U_\xi : \phi \longrightarrow \int_\Omega \phi(\omega)\xi(d\omega),\ U_\eta : \phi \longrightarrow \int_\Omega \phi(\omega)\eta(d\omega).$$

Putting $T = U_\eta U_\xi^{-1}$, we clearly get the result. Q.E.D.

Given an $\mathcal{H}$-valued quantum probability ξ over a given probability space $(\Omega, \mathcal{B}, P)$, we can construct other $\mathcal{H}$-valued quantum probability η by indefinite integration with respect to ξ. Quantum expectations with respect to η are then related to those with respect to ξ by a rule of substitution. To prove these results we need the following.

Lemma 3.9. *Let $(\Omega, \mathcal{B}, P)$ be a probability space and ϕ a complex-valued $\mathcal{B}$-measurable function on Ω such that $\int_\Omega |\phi(\omega)|^2 P(d\omega) = 1$ and put*

$$Q(B) = \int_B |\phi(\omega)|^2 P(d\omega), \text{ for any } B \in \mathcal{B}.$$

Then $(\Omega, \mathcal{B}, Q)$ is a probability space.

Proof. It is trivial. Q.E.D.

Theorem 3.10(Indefinite integration). *Let ξ be an $\mathcal{H}$-valued quantum probability over a given probability space $(\Omega, \mathcal{B}, P)$ and ϕ a complex-valued $\mathcal{B}$-measurable function on Ω such that $\int_\Omega |\phi(\omega)|^2 P(d\omega) = 1$ and put, for any $B \in \mathcal{B}$,*

$$Q(B) = \int_B |\phi(\omega)|^2 P(d\omega)$$

and

$$\eta(B) = \int_B \phi(\omega)\xi(d\omega).$$

Then we have the following:

(1) *η is an $\mathcal{H}$-valued quantum probability over the probability space $(\Omega, \mathcal{B}, Q)$.*

(2) *For any $f \in L_2(\Omega, \mathcal{B}, Q)$, we have $f\phi \in L_2(\Omega, \mathcal{B}, P)$ and*

$$\int_\Omega f(\omega)\eta(d\omega) = \int_\Omega f(\omega)\phi(\omega)\xi(d\omega).$$

Proof. (1) By Lemma 3.2(2) and the assumptions of this theorem, we have, for any $A, B \in \mathcal{B}$,

$$(\eta(A), \eta(B)) = \int_{A\cap B} |\phi(\omega)|^2 P(d\omega) = Q(A \cap B).$$

Thus, by Theorem 1.5, we have (1).

(2) By the definition of Q, we have $dQ/dP = |\phi|^2$, a.s.(P). Hence, for $f \in L_2(\Omega, \mathcal{B}, Q)$, we have

$$\int_\Omega |f(\omega)|^2 Q(d\omega) = \int_\Omega |f(\omega)|^2 |\phi(\omega)|^2 P(d\omega),$$

whence $f\phi \in L_2(\Omega, \mathcal{B}, P)$. The equality in (2) is easily verified for $\mathcal{B}$-simple functions f. For an arbitrary f in $L_2(\Omega, \mathcal{B}, P)$, we consider a sequence of $\mathcal{B}$-simple functions f_n converging to f in the $L_{2,Q}$ norm. The corresponding sequence $(f_n\phi)_{n=1}^\infty$ then converges to $f\phi$ in $L_2(\Omega, \mathcal{B}, P)$, and the desired equality easily follows. Q.E.D.

We turn next to $\mathcal{H}$-valued quantum probabilities induced by measurable transformations. The following result is obvious.

Theorem 3.11. *Let $(\Omega, \mathcal{B}, P)$ and $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$ be two probability spaces and θ a measurable transformation of Ω into $\tilde{\Omega}$. Assume that $\tilde{P} = P \circ \theta^{-1}$. Let ξ be an $\mathcal{H}$-valued quantum probability over $(\Omega, \mathcal{B}, P)$. Then $\eta = \xi \circ \theta^{-1}$ is an $\mathcal{H}$-valued quantum probability over $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$.*

Theorem 3.12(General substitution rule). *Let the situations be as in Theorem 3.11. Let $\tilde{\phi} \in L_2(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$. Then we have the following:*

(1) $\tilde{\phi} \circ \theta \in L_2(\Omega, \mathcal{B}, P)$.

(2) $\displaystyle\int_\Omega \tilde{\phi}\{\theta(\omega)\}\xi(d\omega) = \int_{\tilde{\Omega}} \tilde{\phi}(\tilde{\omega})\xi(\theta^{-1}(d\tilde{\omega}))$.

Proof. (1) is obvious.

(2) First let $\tilde{\phi}$ be $\tilde{\mathcal{B}}$-simple, say $\tilde{\phi} = \Sigma_{n=1}^r a_n \chi_{\tilde{A}_n}$, $\tilde{A}_n \in \tilde{\mathcal{B}}$, $a_n \in \mathbf{C}$. Then $\theta^{-1}(\tilde{A}_n) \in \mathcal{B}$ and therefore

$$\tilde{\phi} \circ \theta = \Sigma_{n=1}^r a_n \chi_{\theta^{-1}(\tilde{A}_n)}$$

is $\mathcal{B}$-simple on Ω. Hence the equality in (2) is immediate from (i) of Definition 3.3 of quantum expectation.

Next, let $\tilde{\phi} \in L_2(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$. Then by (ii) of Definition 3.3,

$$\text{(3.9)} \quad \text{the right hand side of (2)} = \lim_{n\to\infty} \int_{\tilde{\Omega}} \tilde{\phi}_n(\tilde{\omega})\xi\{\theta^{-1}(d\tilde{\omega})\},$$

where $\tilde{\phi}_n$ is $\tilde{\mathcal{B}}$-simple and $\tilde{\phi}_n \longrightarrow \tilde{\phi}$ in $L_2(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$. But, by the ordinary substitution rule, we have

$$\int_\Omega |\tilde{\phi}_n \circ \theta(\omega) - \tilde{\phi} \circ \theta(\omega)|^2 P(d\omega)$$

$$= \int_{\tilde{\Omega}} |\tilde{\phi}_n(\tilde{\omega}) - \tilde{\phi}(\tilde{\omega})|^2 \tilde{P}(d\tilde{\omega})$$

and so $\tilde{\phi}_n \circ \theta \longrightarrow \tilde{\phi} \circ \theta$ in $L_2(\Omega, \mathcal{B}, P)$. Since the $\phi_n \circ \theta$'s are $\mathcal{B}$-simple, it follows from Definition 3.3 (ii) that

$$\text{(3.10)} \quad \text{the left hand side of (2)} = \lim_{n \to \infty} \int_\Omega \tilde{\phi}_n \circ \theta(\omega) \xi(d\omega).$$

The equality in (2) follows from (3.9) and (3.10), since, as shown first, the quantum expectations on the right hand sides of (3.9) and (3.10) are the same. Q.E.D.

The above arguments are clearly reversible, and so we have the following corollary.

Corollary. *Let the situations be as in Theorem 3.12. Let $\tilde{\phi}$ be a complex-valued $\tilde{\mathcal{B}}$-measurable function on $\tilde{\Omega}$ and assume that $\tilde{\phi} \circ \theta \in L_2(\Omega, \mathcal{B}, P)$. Then we have the following:*

(1) $\tilde{\phi} \in L_2(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$.

(2) $\displaystyle\int_\Omega \tilde{\phi}\{\theta(\omega)\}\xi(d\omega) = \int_{\tilde{\Omega}} \tilde{\phi}(\tilde{\omega})\xi(\theta^{-1}(d\tilde{\omega}))$.

When a variable ϕ depends on a parameter λ ranging over a probability space $(\Lambda, \mathcal{F}, Q)$, the quantum expectation $\displaystyle\int_\Omega \phi(\lambda, \omega)\xi(d\omega) = F(\lambda)$ defines a function F on Λ to $\mathcal{H}$. Under suitable conditions F can be integrated (in the sense of Bochner) with respect to Q, and the order of iterated integral can be changed. These conditions are stated in the next theorem, which is regarded as a special case of Theorem 5.20 of P. Masani[11].

Theorem 3.13(Iterated (quantum) expectations). *Let $(\Omega, \mathcal{B}, P)$ and $(\Lambda, \mathcal{F}, Q)$ be two probability spaces. Let ξ be an $\mathcal{H}$-valued quantum probability over $(\Omega, \mathcal{B}, P)$ and ψ be a complex-valued $\sigma(\mathcal{F} \times \mathcal{B})$-measurable function on $\Lambda \times \Omega$. Assume that*

(i)*for almost sure $\lambda(Q)$, $\psi(\lambda, \cdot) \in L_2(\Omega, \mathcal{B}, P)$ and*

$$\int_\Lambda \| \psi(\lambda, \cdot) \|_{L_{2,P}} Q(d\lambda) < \infty;$$

(ii)*for almost sure $\omega(P)$, $\psi(\cdot, \omega) \in L_1(\Lambda, \mathcal{F}, Q)$ and*

$$\int_\Omega \| \psi(\cdot, \omega) \|_{L_{1,Q}} P(d\omega) < \infty.$$

Then we have the following:

(1) *The function* $\displaystyle\int_\Lambda \psi(\lambda, \cdot) Q(d\lambda) \in L_2(\Omega, \mathcal{B}, P)$.

(2) *The function* $\displaystyle\int_\Omega \psi(\cdot, \omega)\xi(d\omega) \in L_1(\Lambda, \mathcal{F}, Q; \mathcal{H})$.

(3) $\displaystyle\int_\Lambda \{\int_\Omega \psi(\lambda, \omega)\xi(d\omega)\} Q(d\lambda) = \int_\Omega \{\int_\Lambda \psi(\lambda, \omega) Q(d\lambda)\}\xi(d\omega)$.

Here $\sigma(\mathcal{F} \times \mathcal{B})$ denotes the σ-algebra generated by $\mathcal{F} \times \mathcal{B}$ and $L_1(\Lambda, \mathcal{F}, Q; \mathcal{H})$ denotes the set of all $\mathcal{H}$-valued, Bochner integrable functions on Λ.

4. Conditional quantum probability and conditional quantum expectation

In this section, we define a conditional quantum probability with respect to a quantum probability ξ over a given probability space $(\Omega, \mathcal{B}, P)$ and the conditional quantum expectation.

Definition 4.1. Let ξ be a quantum probability over a given probability space $(\Omega, \mathcal{B}, P)$. Then we put

$$\xi_A(B) = \xi(A \cap B) / \parallel \xi(A) \parallel, \ B \in \mathcal{B}$$

under the condition that $\parallel \xi(A) \parallel \neq 0$ for a given $A \in \mathcal{B}$ and call $\xi_A(B)$ an ($\mathcal{H}$-valued) conditional quantum probability assuming the event A.

Corollary. *In the above notation, $\xi_A(B)$ is a quantum probability over $(\Omega, \mathcal{B}, P_A)$ considering $\xi_A(B)$ as a function of $B \in \mathcal{B}$, and $\parallel \xi_A(B) \parallel^2 = P_A(B)$ holds, where $P_A(B)$ is a conditional probability of B assuming the event A.*

Proposition 4.2. $\xi(A \cap B) = \parallel \xi(A) \parallel \xi_A(B)$.

Definition 4.3. Let $(\Omega, \mathcal{B}, P)$ be a probability space and ξ a quantum probability over $(\Omega, \mathcal{B}, P)$ and $\phi = \phi(\omega)$ a complex-valued random variable in $L_{2,P} = L_2(\Omega, \mathcal{B}, P)$. Then we define

$$\mathcal{E}_A[\phi] = \mathcal{E}[\phi; A] / \parallel \xi(A) \parallel,$$

and call $\mathcal{E}_A[\phi]$ the conditional quantum expectation of ϕ assuming the event A with $\parallel \xi(A) \parallel \neq 0$.

Corollary. *In the notations in Definition 4.3,*

$$\mathcal{E}_A[\phi] = \int_A \phi(\omega) \xi_A(d\omega) = \int_\Omega \phi(\omega) \chi_A(\omega) \xi_A(d\omega)$$

holds.

Proof. Since ξ_A is a quantum probability over $(\Omega, \mathcal{B}, P_A)$, we have

$$(\mathrm{QE_A}) \quad \begin{cases} (1) \mathcal{E}_A[\phi] \in \mathcal{H}, \\ (2) (\mathcal{E}_A[\phi], \mathcal{E}_A[\psi]) = (\phi, \psi)_{P_A} = E_A[\phi \bar{\psi}], \end{cases}$$

where $E_A(\cdot)$ is the conditional expectation assuming the event $A \in \mathcal{B}$ with $P(A) = \parallel \xi(A) \parallel^2 \neq 0$. Thus we have similar results to Lemma 3.2 replacing P by P_A. Thus we have only to prove the Corollary for $\mathcal{B}$-simple functions. But this is trivial. In fact, for $\phi = \Sigma_{n=1}^r a_n \chi_{A_n}$, $A_n \in \mathcal{B}$, $a_n \in \mathbf{C}$, we have

$$\mathcal{E}_A[\phi] = \frac{1}{\parallel \xi(A) \parallel} \mathcal{E}[\phi; A] = \frac{1}{\parallel \xi(A) \parallel} \mathcal{E}[\phi \chi_A]$$

$$= \Sigma_{n=1}^r a_n \frac{\xi(A \cap A_n)}{\parallel \xi(A) \parallel} = \Sigma_{n=1}^r a_n \xi_A(A_n) = \int_A \phi(\omega) \xi_A(d\omega).$$

Q.E.D.

Now assuming that a given random variable $\phi(\omega)$ takes a value $\check{\phi}$ and

$$\| \xi\{\phi = \check{\phi}\} \| = \| \xi(\{\omega; \phi(\omega) = \check{\phi}\} \| > 0,$$

then we define a conditional quantum probability $\xi_{\phi=\check{\phi}}(B) = \xi_A(B)$ for $B \in \mathcal{B}$ putting $A = \{\omega; \phi(\omega) = \check{\phi}\}$ and a conditional quantum expectation $\mathcal{E}_{\phi=\check{\phi}}[\psi] = \mathcal{E}_A[\psi]$ for a random variable $\psi \in L_{2,P}$.

For a Borel set C in the complex plane $\mathbf{C}$, we define

$$\xi_\phi(C) = \xi(\phi^{-1}(C)), \quad \xi_\phi(C|A) = \xi_A(\phi^{-1}(C)).$$

Then $\xi_\phi(\cdot)$ is a quantum probability law of a random variable $\phi \in L_{2,P}$ and $\xi_\phi(C|A)$ is a conditional quantum probability law of a random variable $\phi \in L_{2,P}$ assuming the event $A \in \mathcal{B}$ with $P(A) > 0$.

Then we have the following.

Proposition 4.4. $\mathcal{E}_A[\phi] = \int_{\mathbf{C}} \check{\phi}\xi_\phi(d\check{\phi}|A)$ *holds.*

We can also define a conditional quantum probability law $\xi_\psi(C|\phi = \check{\phi})$ of another random variable ψ and a conditional quantum expectation $\mathcal{E}[\psi|\phi = \check{\phi}] = \mathcal{E}_{\phi=\check{\phi}}[\psi]$. Here C is a Borel set in $\mathbf{C}$ and ϕ, $\psi \in L_{2,P}$ and $\| \xi(\{\phi = \check{\phi}\}) \| > 0$.

Then, if $\phi = \Sigma_{n=1}^r a_n \chi_{A_n}$ is a $\mathcal{B}$-simple function such that $P(A_n) > 0\,(n = 1, 2, \cdots, r)$, and $\{a_n\}_{n=1}^r$ is included in a Borel set C in $\mathbf{C}$, we have

$$\xi(\phi^{-1}(C) \cap B) = \Sigma_{n=1}^r \xi_{\phi=a_n}(B) \| \xi(\phi^{-1}(a_n)) \| .$$

Thus we have

$$\begin{aligned} P(\phi^{-1}(C) \cap B) &= \| \xi(\phi^{-1}(C) \cap B) \|^2 \\ &= \| \Sigma_{n=1}^r \xi(\phi^{-1}(a_n) \cap B) \|^2 = \Sigma_{n=1}^r \| \xi(\phi^{-1}(a_n) \cap B \|^2 \\ &= \Sigma_{n=1}^r \| \xi_{\phi=a_n}(B) \|^2 \| \xi(\phi^{-1}(a_n)) \|^2 \\ &= \Sigma_{n=1}^r \| \xi_{\phi=a_n}(B) \|^2 P_\phi(\{a_n\}) \\ &= \int_C \| \xi_{\phi=\check{\phi}}(B) \|^2 P_\phi(d\check{\phi}). \end{aligned}$$

Thus we have the following.

Proposition 4.5. *If a $\mathcal{B}$-simple random variable $\phi \in L_{2,P}$ has its values with positive probability, then, for a Borel set C in $\mathbf{C}$ and any $B \in \mathcal{B}$, we have*

$$P(\phi^{-1}(C) \cap B) = \int_C \| \xi_{\phi=\check{\phi}}(B) \|^2 P_\phi(d\check{\phi}).$$

Thus $\| \xi_{\phi=\check{\phi}}(B) \|^2$ is the Radon-Nikodym derivative of $Q(C) = P(\phi^{-1}(C) \cap B)$ with respect to $P_\phi(C)$.

In general, we can show that, for any random variable $\phi \in L_{2,P}$ and any $B \in \mathcal{B}$, there exists an $\mathcal{H}$-valued measurable function $\xi_{\phi=\check{\phi}}(B)$ of $\check{\phi}$ on $\mathbf{C}$ such that, for a Borel set C in $\mathbf{C}$,

$$P(\phi^{-1}(C) \cap B) = \int_C \| \xi_{\phi=\check{\phi}}(B) \|^2 \, P_\phi(d\check{\phi})$$

holds.

In fact, putting $a_{k,l} = \frac{k-1}{2^n} + i\frac{l-1}{2^n}$ and $[a_{k,l}] = [\frac{k-1}{2^n}, \frac{k}{2^n}] \times [\frac{l-1}{2^n}, \frac{l}{2^n}]$, we have, by virtue of Proposition 1.7,

$$\begin{aligned} \xi(\phi^{-1}(C) \cap B) &= \lim_{n\to\infty} \Sigma_{k,l=-\infty}^{\infty} \xi(\phi_n^{-1}([a_{k,l}] \cap C) \cap B) \\ &= \lim_{n\to\infty} \Sigma_{k,l=-\infty}^{\infty} \xi_{\phi_n=a_{k,l}}(B) \| \xi(\phi_n^{-1}([a_{k,l}] \cap C)) \| . \end{aligned}$$

Here ϕ_n is a random variable such that $\phi_n(\omega) = a_{k,l}$ on $\phi^{-1}([a_{k,l}])$ for any k, l so that $-\infty < k, l < \infty$. Then we have

$$\begin{aligned} P(\phi^{-1}(C) \cap B) &= \| \xi(\phi^{-1}(C) \cap B) \|^2 \\ &= \lim_{n\to\infty} \| \Sigma_{k,l=-\infty}^{\infty} \xi(\phi_n^{-1}([a_{k,l}] \cap C) \cap B) \|^2 \\ &= \lim_{n\to\infty} \Sigma_{k,l=-\infty}^{\infty} \| \xi(\phi_n^{-1}([a_{k,l}] \cap C) \cap B) \|^2 \\ &= \lim_{n\to\infty} \Sigma_{k,l=-\infty}^{\infty} \| \xi_{\phi_n=a_{k,l}}(B) \|^2 \| \xi(\phi_n^{-1}([a_{k,l}] \cap C)) \|^2 \\ &= \int_C \| \xi_{\phi=\check{\phi}}(B) \|^2 \, P_\phi(d\check{\phi}). \end{aligned}$$

Thus we have the following.

Theorem 4. 6. *For any random variable $\phi \in L_{2,P}$ and any $B \in \mathcal{B}$, there exists an $\mathcal{H}$-valued measurable function $\xi_{\phi=\check{\phi}}(B)$ on $\mathbf{C}$ as a function of $\check{\phi}$ such that, for a Borel set C in $\mathbf{C}$,*

$$P(\phi^{-1}(C) \cap B) = \int_C \| \xi_{\phi=\check{\phi}}(B) \|^2 \, P_\phi(d\check{\phi})$$

holds. Thus $\| \xi_{\phi=\check{\phi}}(B) \|^2$ is the Radon-Nikodym derivative of $Q(C) = P(\phi^{-1}(C) \cap B)$ with respect to $P_\phi(C)$.

5. Independence

Let ξ be a quantum probability over a given probability space $(\Omega, \mathcal{B}, P)$.

We say that events A and B are independent if $P(A \cap B) = P(A)P(B)$ holds. In general, we say that a finite number of events $A_1, A_2, \cdots A_n$ are independent if, for every subset $\{i_1, i_2, \cdots, i_k\}$ of $\{1, 2, \cdots, n\}$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

holds. Then we have the following.

Proposition 5.1. *Two events A and B are independent if*

$$\xi(A \cap B) = \| \xi(A) \| \xi(B) = \| \xi(B) \| \xi(A)$$

holds.

Corollary 1. *In the case $\| \xi(A) \| > 0$, two events A and B are independent if $\xi_A(B) = \xi(B)$.*

Corollary 2. *Two event A and B are independent if and only if*

$$\| \xi(A \cap B) \| = \| \xi(A) \| \| \xi(B) \| .$$

Proposition 5.2. *A finite number of events $A_1, A_2, \cdots, A_n$ are independent if, for every subset $\{i_1, i_2, \cdots, i_k\}$ of $\{1, 2, \cdots, n\}$,*

$$\xi(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \| \xi(A_{i_1}) \| \| \xi(A_{i_2}) \| \cdots \| \xi(A_{i_{k-1}}) \| \xi(A_{i_k})$$

holds.

Corollary 1. *A finite number of events $A_1, A_2, \cdots, A_n$ are independent if, for every subset $\{i_1, i_2, \cdots, i_k, j\}$ of $\{1, 2, \cdots, n\}$,*

$$\| \xi(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) \| > 0$$

and

$$\xi_{A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}}(A_j) = \xi(A_j)$$

holds.

Corollary 2. *A finite number of events $A_1, A_2, \cdots, A_n$ are independent if and only if, for every subset $\{i_1, i_2, \cdots, i_k\}$ of $\{1, 2, \cdots, n\}$,*

$$\| \xi(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) \| = \| \xi(A_{i_1}) \| \| \xi(A_{i_2}) \| \cdots \| \xi(A_{i_k}) \|$$

holds.

We say that an infinite number of events are independent if every finite subsystem of events is independent.

Proposition 5.3. *Every subsystem of a system of independent events is independent.*

6. Quantum probabilities and quantum expectation over real intervals: Functions with orthogonal increments

The general concept of an $\mathcal{H}$-valued function with orthogonal increments is as follows.

Definition 6.1(Masani). Let Λ be any subinterval of $(-\infty, \infty)$. A function x on Λ to a Hilbert space $\mathcal{H}$ is said to have orthogonal increments if

$$a, b, c, d \in \Lambda \text{ and } a < b \leq c < d \;\; x(b) - x(a) \perp x(d) - x(c).$$

In the following, we assume that

(A) For an $\mathcal{H}$-valued function x with orthogonal increments on a subinterval Λ of $(-\infty, \infty)$,

$$\sup\{\| x(t) - x(s) \|^2; s, t \in \Lambda, s \leq t\} = 1$$

holds.

Then we can show that an $\mathcal{H}$-valued function x on Λ with orthogonal increments generates an $\mathcal{H}$-valued quantum probability ξ on a certain class of Borel sets of Λ. This hinges on the following known lemma:

Lemma 6.2. *Let x be an $\mathcal{H}$-valued function with orthogonal increments on a subinterval Λ of $(-\infty, \infty)$ which satisfies the condition* (A). *Then*

(1) *There exists only one function f on Λ such that*

$$s, t \in \Lambda \text{ and } s \leq t \Longrightarrow 1 \geq f(t) - f(s) = \| x(t) - x(s) \|^2 \geq 0.$$

(2) *The function f is a probability distribution function.*

(3) *For any $t \in \mathrm{Int}(\Lambda)$, $x(t-0) = \lim_{s \to t-0} x(s)$ and $x(t+0) = \lim_{s \to t+0} x(s)$ exist. With obvious amendments this holds also for the end point t of Λ.*

(4) *$s, t \in \Lambda$ and $s \leq t \Longrightarrow f(t^*) - f(s^\#) = \| x(t^*) - x(s^\#) \|^2$, where "*" can be anyone of the symbols "$+0$", "-0" or a blank, and likewise for "#".*

(5) *The functions $x(\cdot - 0)$, $x(\cdot + 0)$ are left-continuous and right-continuous, respectively, on Λ.*

(6) *For any t outside a certain countable subset of Λ, $x(t - 0) = x(t) = x(t + 0)$.*

Proof. cf. Doob[2], p.425. Q.E.D.

Now let x be an $\mathcal{H}$-valued function with orthogonal increments on a subinterval Λ of $(-\infty, \infty)$ which satisfies the condition (A). Introduce the pre-ring

$$\mathcal{P} = \{J; J = (a, b] \subset \Lambda\} \tag{6.1}$$

of bounded, open-closed subintervals of Λ, and define, for any $J = (a, b] \in \mathcal{P}$,

$$\xi(J) = x(b+0) - x(a+0), \;\; P(J) = \| x(b+0) - x(a+0) \|^2 . \tag{6.2}$$

Here, as to the definition of a pre-ring, see Masani[11], p.63.

We assert the following lemma.

Lemma 6.3. *If $x, \mathcal{P}, \xi, P$ are as in Lemma 6.2,* (6.1) *and* (6.2), *then ξ is an $\mathcal{H}$-valued quantum probability over $\mathcal{P}$ with $P_\xi = P$.*

Proof. By Lemma 6.2(4), $P(J) = f(b+0) - f(a+0)$, where f is nondecreasing and $0 \leq f(t) \leq 1$. Hence, as is well-known,

(6.3) P is a countably additive probability measure on $\mathcal{P}$.

Obviously

(6.4) ξ is a function on $\mathcal{P}$ to $\mathcal{H}$.

Hence, in view of Theorem 1.8 in Masani[11] and Theorem 1.5, in order to complete the proof, we need only show that

(6.5) For any $I, J \in \mathcal{P}$, $(\xi(I), \xi(J)) = P(I \cap J)$.

To prove (6.5), let $I = (a, b]$, $J = (c, d]$ with $a \leq c$ for definiteness. If $I \cap J = \emptyset$, we have

$$\xi(I) = x(b+0) - x(a+0) \perp x(d+0) - x(c+0) = \xi(J)$$

and obviously (6.5) holds. When $I \cap J \neq \emptyset$, we must have $c < b$, and therefore

$$\begin{aligned}\xi(I) &= x(b+0) - x(a+0)\\ &= x(b+0) - x(c+0) + x(c+0) - x(a+0)\\ &= \xi(I \cap J) + \xi(I \backslash J)\end{aligned}$$

and similarly

$$\xi(J) = \xi(I \cap J) + \xi(J \backslash I).$$

Appealing to the fact that (6.5) holds for disjoint sets, we easily infer that

$$\begin{aligned}(\xi(I), \xi(J)) &= (\xi(I \cap J), \xi(I \cap J)) = \| \xi(I \cap J) \|^2\\ &= \| x(b+0) - x(c+0) \|^2 = P(I \cap J),\end{aligned}$$

assuming $b \leq d$ for definiteness. Thus we have (6.5). Q.E.D.

Now the Hahn Extension Theorems 2.3 and 2.5 in Masani[11] guarantee that ξ can be uniquely extended to a quantum probability $\tilde{\xi}$ over a probability space $(\Lambda, \mathcal{B}, \tilde{P})$ where $\mathcal{B}$ is the σ-algebra of all Borel subsets of Λ, and $\tilde{P}$ is the Hahn extension of P to $\mathcal{B}$. We have thus established the following.

Theorem 6.4. *Let x be an $\mathcal{H}$-valued function with orthogonal increments on a subinterval Λ of $(-\infty, \infty)$ which satisfies the condition* (A)*and $\mathcal{B}$ be the σ-algebra of all Borel subsets of Λ. Then*

(1) *There exists a unique probability measure P on $\mathcal{B}$ such that for any $(a, b] \subset \Lambda$, $P((a, b]) = \| x(b+0) - x(a+0) \|^2$.*

(2) *There exists a unique $\mathcal{H}$-valued quantum probability ξ over $(\Lambda, \mathcal{B}, P)$ such that for any $(a, b] \subset \Lambda$, $\xi((a, b]) = x(b+0) - x(a+0)$.*

We now turn to the quantum expectation of a complex-valued function ϕ on a real interval Λ with respect to ξ. Several authors define this quantum expectation, in the Stieltjes style, with respect to the function x with orthogonal increments, rather than with respect to the quantum probability ξ generated by x. This approach, though limited, has the advantage of suggesting analogs of formulas valid for ordinary Stieltjes integrals, e.g. integration by parts. Several such analogs are actually correct. For instance when the integrand ϕ is continuous, our quantum expectation can be defined in the Riemann-Stieltjes fashion. This fact is important and we outline a proof along the line of Masani[11].

Let $\pi : a = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = b$ be a finite net over $[a, b]$. We call $\Delta_k = [\lambda_{k-1}, \lambda_k]$ the subintervals of π, and a set $\pi^* = \{t_1, \cdots, t_n\}$ such that $t_k \in \Delta_k$ a dual of π. With a slight inconsistency we write "$t_k \in \Delta_k$" and also "$\Delta_k \in \pi$". We also employ the notation $|\Delta_k| = \lambda_k - \lambda_{k-1}$ and $|\pi| = \max\{|\Delta_1|, \cdots, |\Delta_n|\}$. Given an $\mathcal{H}$-valued function x with orthogonal increments on $[a, b]$, and a complex-valued function ϕ on $[a, b]$, let

$$(6.6) \qquad E(\pi, \pi^*) = \Sigma_{\Delta_k \in \pi, t_k \in \pi^*} \phi(t_k)\{x(\lambda_k) - x(\lambda_{k-1})\}.$$

We then assert

Lemma 6.5. *If ϕ is bounded on $[a, b]$, then for any nets π, π' such that $\pi \subset \pi'$ and for any duals π^*, π'^*,*

$$\| E(\pi, \pi^*) - E(\pi', \pi'^*) \|^2 \leq \max_{\Delta \in \pi}\{\mathrm{Osc}(\phi, \Delta)\}^2 \cdot \{f(b) - f(a)\},$$

where f is associated with our function as in Lemma 6.2.

Proof. Let $\Delta_1, \cdots, \Delta_n$ be the subintervals of π, and let π' split Δ_k subintervals $\Delta_k^1, \cdots, \Delta_k^{m_k}$. Put

$$\pi^* = \{t_k; 1 \leq k \leq n\}, \text{ where } t_k \in \Delta_k,$$

$$\pi'^* = \{t_k^i; 1 \leq k \leq n \text{ and } 1 \leq i \leq m_k\}, \text{ where } t_k^i \in \Delta_k^i.$$

Then obviously

$$E(\pi, \pi^*) - E(\pi', \pi'^*) = \Sigma_{k=1}^n \Sigma_{i=1}^{m_k} \{\phi(t_k) - \phi(t_k^i)\} \Delta_k^i x$$

where Δx means $x(d) - x(c)$ when $\Delta = [c, d]$. Since the increments $\Delta_k^i x$ are mutually orthogonal, the Pythagorean identity yields

$$\| E(\pi, \pi^*) - E(\pi', \pi'^*) \|^2 = \Sigma_{k=1}^n \Sigma_{i=1}^{m_k} |\phi(t_k) - \phi(t_k^i)|^2 \| \Delta_k^i x \|^2 .$$

From this, the desired inequality is immediate, since $\| \Delta_k^i x \|^2 = \Delta_k^i f$ and $|\phi(t_k) - \phi(t_k^i)| \leq \mathrm{Osc}(\phi, \Delta_k)$. Q.E.D.

We can now state our theorem.

Theorem 6.6. *Let x be an $\mathcal{H}$-valued function with orthogonal increments on $(-\infty, \infty)$, satisfying the condition* (A)*and P and ξ be the probability measure and the $\mathcal{H}$-valued quantum probability generated by x (cf. Theorem 6.4) and ϕ be a complex-valued, continuous function on $[a, b]$. Then the Riemann-Stieltjes integral*

$$\int_a^b \phi(\lambda) dx(\lambda) = \lim_{|\pi| \to 0} E(\pi, \pi^*)$$

exists, and equals

$$\phi(a)\{x(a+0) - x(a)\} + \int_{(a,b)} \phi(\lambda)\xi(d\lambda) + \phi(b)\{x(b) - x(b-0)\}.$$

Note. The asserted equality is simplified in cases of common occurence. For instance, for a right-continuous x, it reduces to

$$\int_a^b \phi(\lambda)dx(\lambda) = \int_{(a,b]} \phi(\lambda)\xi(d\lambda).$$

In case $\xi\{a\} = x(a+0) - x(a)$, $\xi\{b\} = x(b) - x(b-0)$, it becomes

$$\int_a^b \phi(\lambda)dx(\lambda) = \int_{[a,b]} \phi(\lambda)\xi(d\lambda).$$

Proof. Let f be associated with x as in Lemma 6.2, and $\epsilon > 0$. Since ϕ is uniformly continuous on $[a,b]$, there exists $\delta > 0$ such that

(6.7) Δ is a subinterval and $|\Delta| < \delta \Longrightarrow \mathrm{Osc}(\phi, \Delta) \leq \epsilon/[2\{f(b) - f(a)\}]^{1/2}$.

Now let π_1, π_2 be nets such that $|\pi_i| < \delta$ and let π_i^* and $(\pi_1 \cup \pi_2)^*$ be duals for π_i and $\pi_1 \cup \pi_2 (i = 1, 2)$. Then by the last lemma and (6.7),

$$\| E(\pi_i, \pi_i^*) - E(\pi_1 \cup \pi_2, (\pi_1 \cup \pi_2)^*) \|^2$$

$$\leq \max_{\Delta \in \pi_i} \{\mathrm{Osc}(\phi, \Delta)\}^2 \cdot \{f(b) - f(a)\} < \frac{1}{2}\epsilon^2;$$

whence

$$\| E(\pi_1, \pi_1^*) - E(\pi_2, \pi_2^*) \| < \epsilon.$$

Thus $\{E(\pi, \pi^*)\}$ is an $\mathcal{H}$-valued Cauchy directed system as $|\pi| \to 0$, and therefore has a limit, which by definition is the Riemann-Stieltjes integral, $\int_a^b \phi(\lambda)dx(\lambda)$.

To relate this integral to $\int_{(a,b)} \phi(\lambda)\xi(d\lambda)$, we first show that

$$\lim_{|\pi| \to 0} \int_{(a,b)} \phi_{\pi,\pi^*}(\lambda)\xi(d\lambda) = \int_{(a,b)} \phi(\lambda)\xi(d\lambda), \tag{6.8}$$

$$\text{where } \phi_{\pi,\pi^*} = \Sigma_{\lambda_k \in \pi, t_k \in \pi^*} \phi(t_k)\chi_{(\lambda_{k-1}, \lambda_k]}.$$

Since

$$\phi(\lambda) - \phi_{\pi,\pi^*}(\lambda) = \Sigma_{k=1}^n \{\phi(\lambda) - \phi(t_k)\}\chi_{(\lambda_{k-1}, \lambda_k]}(\lambda),$$

therefore

$$|\phi(\lambda) - \phi_{\pi,\pi^*}(\lambda)| \leq \max_{\Delta \in \pi} \mathrm{Osc}(\phi, \Delta).$$

Hence

$$\| \int_{(a,b)} \phi(\lambda)\xi(d\lambda) - \int_{(a,b)} \phi_{\pi,\pi^*}(\lambda)\xi(d\lambda) \|^2$$

$$= \int_{(a,b)} |\phi(\lambda) - \phi_{\pi,\pi^*}(\lambda)|^2 P(d\lambda)$$
$$\leq \{\max_{\Delta\in\pi} \mathrm{Osc}(\phi,\Delta)\}^2 P(a,b).$$

But the right hand side $\longrightarrow 0$ as $|\pi| \longrightarrow 0$ (cf.(6.7)). Thus we have (6.8).

We now consider only nets $\pi : a = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = b$, for which $\lambda_1, \cdots, \lambda_{n-1}$ are points of right continuity of x, i.e., $x(\lambda_k) = x(\lambda_k + 0)$. Then, by a routine calculation using (6.6) and (6.2), we have

$$E(\pi,\pi^*) = \phi(t_1)\{x(a+0) - x(a)\} + \Sigma_{k=1}^{n-1}\phi(t_k)\xi(\lambda_{k-1},\lambda_k]$$
$$+\phi(t_n)\xi(\lambda_{n-1},\lambda_n) + \phi(t_n)\{x(b) - x(b-0)\}$$
$$= \phi(t_1)\{x(a+0) - x(a)\} + \int_{(a,b)} \phi_{\pi,\pi^*}(\lambda)\xi(d\lambda)$$
$$+\phi(t_n)\{x(b) - x(b-0)\},$$

where the last equality follows from Step 1 of Definition of $\mathcal{E}[\phi]$ (cf. section 3). Now let $|\pi| \longrightarrow 0$. Then by (6.8) and the continuity of ϕ at a and b, the right hand side tends to

$$\phi(a)\{x(a+0) - x(a)\} + \int_{(a,b)} \phi(\lambda)\xi(d\lambda) + \phi(b)\{x(b) - x(b-0)\}.$$

Q.E.D.

We now prove a generalized version of the law of integration by parts, which ties up our $\mathcal{H}$-valued Riemann-Stieltjes integral with an $\mathcal{H}$-valued Bochner integral. The formulation and proof are suggested by Masani[11], p.104 and Doob's treatment[2], p.432.

Theorem 6.7(Integration by parts). *Let x be an $\mathcal{H}$-valued function with orthogonal increments on $[a,b]$ satisfying the condition* (A),*and ϕ be a complex-valued absolutely continuous function on $[a,b]$. Then*

$$\int_a^b \phi(\lambda)dx(\lambda) = \phi(b)x(b) - \phi(a)x(a) - \int_{[a,b]} \phi'(\lambda)x(\lambda)d\lambda,$$

the last being an $\mathcal{H}$-valued Bochner integral with respect to the Lebesgue measure.

Proof. Let $\Lambda = [a,b]$, $\mathcal{B}$ be the σ-algebra of all Borel sets of Λ, and ξ be the $\mathcal{H}$-valued quantum probability and P the probability measure associated with ξ, cf. Theorem 6.4. Then we have

$$P(\Lambda) = \| x(b) - x(a) \|^2 = 1.$$

Since x is defined only on $[a,b]$, therefore

$$\xi\{a\} = x(a+0) - x(a), \ \xi\{b\} = x(b) - x(b-0),$$

and hence, by the note to Theorem 6.6, our Riemann-Stieltjes integral equals $\mathcal{E}[\phi] = \int_\Lambda \phi(\lambda)\xi(d\lambda)$. Hence we must show that

$$(6.9)\ \mathcal{E}[\phi] = \int_\Lambda \phi(\lambda)\xi(d\lambda) = \phi(b)x(b) - \phi(a)x(a) - \int_{[a,b]} \phi'(\lambda)x(\lambda)d\lambda.$$

For this we introduce the function ψ on $\Lambda \times \Lambda$:

$$(6.10) \qquad \psi(\omega, \lambda) = \begin{cases} \phi'(\omega)\chi_{[a,\lambda]}(\omega), & \omega \in \Lambda \backslash N, \lambda \in N, \\ 0, & \omega \in N, \lambda \in N, \end{cases}$$

where N is the set (of Lebesgue measure zero) on which ϕ' does not exist. It is a straightforward matter to verify that ψ satisfies the hypotheses (iii) $\sim$ (v) of Theorem 5.20 of Masani[11] on iterated integration. Hence

$$(6.11) \qquad \int_\Lambda \{\int_\Lambda \psi(\omega, \lambda)d\omega\}\xi(d\lambda) = \int_\Lambda \{\int_\Lambda \psi(\omega, \lambda)\xi(d\lambda)\}d\omega.$$

An easy caculation shows that

the left hand side of (6.11) $= \int_\Lambda \phi(\lambda)\xi(d\lambda) - \phi(a)\{x(b) - x(a)\}$,

the right hand side of (6.11) $= \{\phi(b) - \phi(a)\}x(b) - \int_\Lambda \phi'(\omega)x(\omega - 0)d\omega$.

In the last integral $x(\omega - 0)$ is replaceable by $x(\omega)$, since the two are equal except on a countable set, cf. Lemma 6.2(6). The equation (6.11) thus reduces to (6.9). Q.E.D.

7. L_2-valued quantum probabilities

The Hilbert space of complex-valued random variables over a probability space $(\Omega, \mathcal{B}, P)$ with absolute moments of degree 2 has a natural quantum probability basis, viz, the one given by the indicator functions, cf. Example 1.3(b) and Definition 7.2. This gives the theory of L_2-valued quantum probabilities an indivisual flavor, especially in regard to the isometric and unitary transformations between L_2-spaces. In this section we study this matter.

Fundamental to the entire development is the well known result:Theorem 3.1.

Notation. Let $(\Lambda, \mathcal{F}, Q)$ be a probability space and $L_{2,Q} = L_2(\Lambda, \mathcal{F}, Q)$ the Hilbert space of all complex-valued, random variables f on Λ such that $\int_\Lambda |f(\lambda)|^2 Q(\, d\lambda) < \infty$ with the inner product

$$(f, g)_Q = \int_\Lambda f(\lambda)\overline{g(\lambda)}Q(d\lambda).$$

Let ξ be an $L_{2,Q}$-valued quantum probability over a probability space $(\Omega, \mathcal{B}, P)$. Then since the value of ξ at B, where $B \in \mathcal{B}$, is a function on Λ, it is more convenient to denote this value by $\xi_{(B)}$ than by $\xi(B)$. We can then write equations such as:

$$E_Q(\xi_{(A)}\overline{\xi_{(B)}}) = \int_\Lambda \xi_{(A)}(\lambda)\overline{\xi_{(B)}(\lambda)}Q(d\lambda)$$
$$= (\xi_{(A)}, \xi_{(B)})_Q = P(A \cap B), \text{ for any } A, B \in \mathcal{B}.$$

Our quantum expectation is written $\mathcal{E}_P(\phi) = \int_\Omega \phi(\omega)\xi_{(d\omega)}$ and not $\int_\Omega \phi(\omega)\xi(d\omega)$ as before.

The fact alleged to at the start of this section can be stated as follows:

Proposition 7.1. *Let $(\Lambda, \mathcal{F}, Q)$ be a probability space and, for any $A \in \mathcal{F}$, $\chi_{(A)}$ be the indicator function of A. Then χ is an $L_2(\Lambda, \mathcal{F}, Q)$-basic quantum probability over $(\Lambda, \mathcal{F}, Q)$. Moreover, we have*

$$f = \int_\Lambda f(\lambda)\chi_{(d\lambda)}, \text{ for any } f \in L_{2,Q}.$$

Proof. Obviously, for any $A, B \in \mathcal{F}$, $\chi_A, \chi_B \in L_{2,Q}$ and

$$(\chi_{(A)}, \chi_{(B)})_Q = E_Q(\chi_{(A)}\overline{\chi_{(B)}}) = Q(A \cap B).$$

Therefore, by Theorem 1.5, χ is an $L_{2,Q}$-valued quantum probability over $(\Lambda, \mathcal{F}, Q)$.

Next, let $f \in L_{2,Q}$ and, for any $A \in \mathcal{B}$,

$$m_f(A) = (f, \chi_{(A)})_Q = \int_A f(\lambda)Q(d\lambda).$$

Then $m_f(\cdot)$ is a finite countably additive measure on A. Letting M_χ be the projection on $L_{2,Q}$ onto $\mathcal{S}_\chi$, it follows from Theorem 3.6 that

$$M_\chi(f) = \int_\Lambda \frac{dm_f}{dQ}(\lambda)\chi_{(d\lambda)} = \int_\Lambda f(\lambda)\chi_{(d\lambda)}. \tag{7.1}$$

Thus, by Lemma 3.2(1), we have

$$\| M_\chi(f) \|^2 = \int_\Lambda |f(\lambda)|^2 Q(d\lambda) = \| f \|^2 .$$

Since $f - M_\chi(f) \perp \chi_A$, $A \in \mathcal{F}$ and the set of all $\mathcal{F}$-simple functions are dense in $L_{2,Q}$, this shows that $f = M_\chi(f) \in \mathcal{S}_\chi$; whence $\mathcal{S}_\chi = L_{2,Q}$, and (7.1) reduces to the desired equality. Q.E.D.

Definition 7.2. We call the $L_{2,Q}$-valued quantum probability χ given in Proposition 7.1 the indicator quantum probability basis for $L_2(\Lambda, \mathcal{F}, Q)$.

By Theorem 3.7, if V is an isometry on an L_2-space, then $V(\chi)$ is an L_2-valued quantum probability in the range of V. This yields the following.

Theorem 7.3(Isometry between L_2-spaces). *Let $(\Lambda_i, \mathcal{B}_i, P_i)$ be two probability spaces and $\mathcal{H}_i = L_2(\Lambda_i, \mathcal{B}_i, P_i)$ for $i = 1, 2$. Then*

(1) *To every isometry V on $\mathcal{H}_1$ into $\mathcal{H}_2$ corresponds an $\mathcal{H}_2$-valued quantum probability η over $(\Lambda_1, \mathcal{B}_1, P_1)$ such that $\mathcal{S}_\eta = V(\mathcal{H}_1)$ and*

$$V(f) = \int_{\Lambda_1} f(\lambda_1)\eta_{(d\lambda_1)}, \text{ for any } f \in \mathcal{H}_1. \tag{7.2}$$

(2) *To every $\mathcal{H}_2$-valued quantum probability η over $(\Lambda_1, \mathcal{B}_1, P_1)$ corresponds an isometry V on $\mathcal{H}_1$ onto $\mathcal{S}_\eta(\subset \mathcal{H}_2)$ satisfying* (7.2).

Proof. (1) This follows at once from Theorem 3.7 on setting

$$\xi_{(B)} = \chi_{(B)},\ \eta_{(B)} = V(\chi_{(B)}), \text{ for any } B \in \mathcal{B}_1, \tag{7.3}$$

and noting, cf. Proposition 7.1, that, for any $f \in \mathcal{H}_1$,

$$V(f) = V(\int_{\Lambda_1} f(\lambda_1)\chi_{(d\lambda_1)}) = \int_{\Lambda_1} f(\lambda_1)\eta_{(d\lambda_1)}.$$

(2) χ, η are $\mathcal{H}_1$-valued, $\mathcal{H}_2$-valued quantum probabilities, respectively, over the same $(\Lambda_1, \mathcal{B}_1, P_1)$. Hence, by Theorem 3.8, there exists an isometry V on $\mathcal{S}_\chi$, i.e., $\mathcal{H}_1$, onto $\mathcal{S}_\eta$ such that (7.3) holds. But as just shown, (7.3) entails (7.2). Q.E.D.

When the isometry V is "onto", we get the following more symmetric result:

Corollary 1(Isomorphism between L_2-spaces). *Let $(\Lambda_i, \mathcal{B}_i, P_i)$ and $\mathcal{H}_i$ be as in Theorem 7.3 for $i = 1, 2$. Then given a unitary operator V on $\mathcal{H}_1$ onto $\mathcal{H}_2$, there exist η, ξ such that*

(1) *η is an $\mathcal{H}_2$-basic quantum probability over $(\Lambda_1, \mathcal{B}_1, P_1)$, ξ is an $\mathcal{H}_1$-basic quantum probability over $(\Lambda_2, \mathcal{B}_2, P_2)$,*

(2) *for any $f \in \mathcal{H}_1$ and any $g \in \mathcal{H}_2$,*

$$V(f) = \int_{\Lambda_1} f(\lambda_1)\eta_{(d\lambda_1)},\ V^*(g) = \int_{\Lambda_2} g(\lambda_2)\xi_{(d\lambda_2)},$$

(3) *for any $B_1 \in \mathcal{B}_1$ and any $B_2 \in \mathcal{B}_2$,*

$$(\eta_{(B_1)}, \chi_{(B_2)})_{\mathcal{H}_2} = (\chi_{(B_1)}, \xi_{(B_2)})_{\mathcal{H}_1}.$$

Proof. Let V be a unitary operator on $\mathcal{H}_1$ onto $\mathcal{H}_2$, and η be the $\mathcal{H}_2$-valued quantum probability over $(\Lambda_1, \mathcal{B}_1, P_1)$ given in Theorem 7.3(1). Then $\mathcal{S}_\eta = V(\mathcal{H}_1) = \mathcal{H}_2$, and the first equation in (2) holds. In other words, we have an $\mathcal{H}_2$-basic quantum probability η over $(\Lambda_1, \mathcal{B}_1, P_1)$ for which the first equation in (2) holds.

Next, since V is unitary, $V^* = V^{-1}$ is a unitary operator on $\mathcal{H}_2$ onto $\mathcal{H}_1$. Hence we can apply to V^* the result just proved for V, and conclude that there exists an $\mathcal{H}_1$-basic quantum probability ξ over $(\Lambda_2, \mathcal{B}_2, P_2)$ for which the second equation in (2) holds.

Finally, to prove (3) we note that, by (2), for any $B_1 \in \mathcal{B}_1$ and any $B_2 \in \mathcal{B}_2$,

$$V(\chi_{(B_1)}) = \eta_{(B_1)},\ V^*(\chi_{(B_2)}) = \xi_{(B_2)},$$

and so

$$(\eta_{(B_1)}, \chi_{(B_2)})_{\mathcal{H}_2} = (V(\chi_{(B_1)}), \chi_{(B_2)})_{\mathcal{H}_2}$$
$$= (\chi_{(B_1)}, V^*(\chi_{(B_2)}))_{\mathcal{H}_1} = (\chi_{(B_1)}, \xi_{(B_2)})_{\mathcal{H}_1}.$$

Q.E.D.

Corollary 1 to Theorem 7.3 shows that the condition (3) is necessary in order that our isometry V be "onto". The next corollary shows that the condition (3) is also sufficient for this. This is nice, because the condition is easy to check in many practical cases.

Corollary 2. *Let $(\Lambda_i, \mathcal{B}_i, P_i)$ and $\mathcal{H}_i$ be as in Theorem 7.3, for $i = 1, 2$. Let η be an $\mathcal{H}_2$-valued quantum probability over $(\Lambda_1, \mathcal{B}_1, P_1)$ and, ξ be an $\mathcal{H}_1$-valued quantum probability over $(\Lambda_2, \mathcal{B}_2, P_2)$ and, for any $B_1 \in \mathcal{B}_1$ and any $B_2 \in \mathcal{B}_2$, $(\eta_{(B_1)}, \chi_{(B_2)})_{\mathcal{H}_2} = (\chi_{(B_1)}, \xi_{(B_2)})_{\mathcal{H}_1}$ holds. Then*

(1) $\mathcal{S}_\eta = \mathcal{H}_2$, $\mathcal{S}_\xi = \mathcal{H}_1$, i.e., η, ξ are $\mathcal{H}_2$-basic, $\mathcal{H}_1$-basic quantum probabilities, respectively.

(2) The isometry V given in Theorem 7.3(2) is a unitary operator on $\mathcal{H}_1$ onto $\mathcal{H}_2$.

Proof. (1) In view of Proposition 7.1, to prove that $\mathcal{S}_\eta = \mathcal{H}_2$ it suffices to show that, for any $B_2 \in \mathcal{B}_2$, $\chi_{(B_2)} \in \mathcal{S}_\eta$. Now for a given $B_2 \in \mathcal{B}_2$, let

$$Q(B_1) = (\chi_{(B_2)}, \eta_{(B_1)})_{\mathcal{H}_2},\ B_1 \in \mathcal{B}_1.$$

Then by the assumption we have

$$Q(B_1) = (\xi_{(B_2)}, \chi_{(B_1)})_{\mathcal{H}_1} = \int_{B_1} \xi_{(B_2)}(\lambda_1) P_1(d\lambda_1).$$

Hence we have $dQ/dP_1 = \xi_{(B_2)}$, a.s.(P_1). Hence, by Theorem 3.6, we have

$$M_\eta(\chi_{(B_2)}) = \int_{\Lambda_1} \frac{dQ}{dP_1}(\lambda_1)\eta_{(d\lambda_1)} = \int_{\Lambda_1} \xi_{(B_2)}(\lambda_1)\eta_{(d\lambda_1)},$$

and therefore, by the assumptions and Lemma 3.2(1), we have

$$\| M_\eta(\chi_{(B_2)}) \|^2 = \int_{\Lambda_1} |\xi_{(B_2)}(\lambda_1)|^2 P_1(d\lambda_1)$$
$$= \| \xi_{(B_2)} \|^2 = P_2(B_2) = \| \chi_{(B_2)} \|^2 .$$

This shows that $\chi_{(B_2)} = M_\eta(\chi_{(B_2)}) \in \mathcal{S}_\eta$, as desired. Thus $\mathcal{S}_\eta = \mathcal{H}_2$. In exactly the same way we can show that $\mathcal{S}_\xi = \mathcal{H}_1$.

(2) follows at once from (1), since, by Theorem 7.3(2), V is an isometry on $\mathcal{H}_1$ onto $\mathcal{S}_\eta$, and now $\mathcal{S}_\eta = \mathcal{H}_2$. Q.E.D.

8. The resolution of the identity

We now mention the resolution of the identity as an example of quantum probability. We mention this following Chapter 11 of Fujita-Ito-Kuroda[3].

Definition 8.1. Let $(\Omega, \mathcal{B})$ be a measurable space and $\mathcal{H}$ a complex Hilbert space. We call a system $\{E(\Lambda); \Lambda \in \mathcal{B}\}$ of projection operators $E(\Lambda)$ on $\mathcal{H}$ the resolution of the identity (over the measurable space $(\Omega, \mathcal{B})$), if, for every $\Lambda \in \mathcal{B}$, a projection operator $E(\Lambda)$ on $\mathcal{H}$ is assigned and the following conditions (E.1), (E.2) and (E.3) are satisfied:

(E.1) If $\Lambda_1 \cap \Lambda_2 = \emptyset$, $E(\Lambda_1) \perp E(\Lambda_2)$ holds, namely, for every $x, y \in \mathcal{H}$, $(E(\Lambda_1)x, E(\Lambda_2)y) = 0$ holds.

(E.2) If $\Lambda = \Sigma_{n=1}^{\infty}\Lambda_n$ (disjoint sum), $E(\Lambda) = \Sigma_{n=1}^{\infty}E(\Lambda_n)$ holds in the strong convergence topology, namely, for every $x \in \mathcal{H}$, $E(\Lambda)x = \Sigma_{n=1}^{\infty}E(\Lambda_n)x$ holds in the norm topology of $\mathcal{H}$.

(E.3) $E(\Omega) = I$ (=the identity operator).

Theorem 8.2. *Let $(\Omega, \mathcal{B})$ be a measurable space and $\{E(\Lambda); \Lambda \in \mathcal{B}\}$ a system of a projection operators on a Hilbert space $\mathcal{H}$. For every $x \in \mathcal{H}$ such as $\| x \|= 1$, put $\xi_x(\Lambda) = E(\Lambda)x$ for $\Lambda \in \mathcal{B}$. Then $\xi_x(\cdot)$ is an $\mathcal{H}$-valued quantum probability over the measurable space $(\Omega, \mathcal{B})$.*

For $\phi \in L_{2,P}$ where $P = P_{\xi_x}$, we can define

$$\mathcal{E}[\phi] = \int_{\Omega} \phi(\omega)\xi_x(d\omega),$$

which has the properties in Lemma 3.2. Then we have

$$\mathcal{E}[\phi] = \int_{\Omega} \phi(\omega)dE(\omega)x$$

in the notation of Fujita- Ito-Kuroda[3]. Thus the theory of spectral resolutions of selfadjoint operators can be treated within the framework of our theory. But the restriction that x is normalized is allowed in the point that wave functions in old quantum theory are meanigful only when they are normalized.

References

[1] P. A. M. Dirac, The Principles of Quantum Mechanics, 4th ed., Oxford UP, Oxford, 1958(Japanese translation by S. Tomonaga et al., Iwanami, Tokyo, 1968).

[2] J. L. Doob, Stochastic Processes, John Wiley & Sons, Inc., New York, 1953.

[3] H. Fujita, S. Ito, and S. Kuroda, Functional Analysis I $\sim$ III, Iwanami, Tokyo, 1978(in Japanese).

[4] P. R. Halmos, Measure Theory, Springer, Berlin, 1950.

[5] K. Ito, Theory of Probability, Iwanami, Tokyo, 1953(in Japanese).

[6] Y. Ito, Theory of Hypoprobability Measures, RIMS Kokyuroku **558**(1985), 96-113(in Japanese).

[7] Y. Ito, New Axiom of Quantum Mechanics — Hilbert's 6th Problem —, Simposium on Real Analysis 1998 Takamatsu, pp. 96-103.

[8] Y. Ito, New Axiom of Quantum Mechanics — Hilbert's 6th Problem —, J. Math. Tokushima Univ., **32**(1998), 43-51.

[9] Y. Ito, The Mathematical Principles of Quantum Mechanics. New Theory, Sciencehouse, Tokyo, 2000(in Japanese).

[10] A. N. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933(Japanese translation by S. Nemoto and Y. Ichijo, Tokyo-Tosho, Tokyo, 1969).

[11] P. Masani, Orthogonally Scattered Measures, Adv. in Math., **2**(1968), 61-117.

[12] J. v. Neumann, Mathematische Grundlagen der Quantenmechanik, Springer, Berlin, 1981(Japanese translation by T. Inoue et al., Misuzu-Shobou, Tokyo, 1957).