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Theory of (Vector-Valued) Sato Hyperfunctions on a Real-Analytic Manifold

By

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Abstract

In this article, we realize Sato hyperfunctions and Fréchet-space-valued Sato hyperfunctions on a real-analytic manifold by the algebro-analytic method. Then we prove the equivalence of the above and the correspondent realized independently by the duality method, Ito[7]. In several points, we improve the methods of proof of important theorems. Thereby the method of constructing the theory becomes clear and evident.

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Introduction

In this article, we mention an improvement of the proof of the theory of (vector-valued) Sato hyperfunctions on a real-analytic manifold. Although, in Sato's original papers[23], [24], [25], he constructed the theory of hyperfunctions on a real-analytic manifold, there does not yet exist any clear and evident proof other than Kashiwara-Kawai-Kimura[15] and Ito[7]. Nowadays there exist all things and methods necessary for such a proof. We have only to pile up these building-blocks. We do this in this article.

The points of improvement are the following:

1. We construct the Dolbeault-Grothendieck resolution of the sheaf $\mathcal{O}$ using the sheaf of locally square integrable functions. Thereby the duality method can be used in the framework of classical functions.

2. We have succeeded in proving Malgrange's Theorem completely and in a clear and evident manner. This method of proof is far reaching and applicable for all versions of this theorem.

3. We have succeeded in proving Martineau-Harvey's Theorem for one dimensional, complex manifold, namely Silva-Köthe-Grothendieck's Theorem [3], [19], [27]. Thereby we can prove Sato's Theorem independently on the dimension of a manifold.

4. The proof of Proposition 1.5.2 becames clear and evident.

5. The method in this article is used for the first time in the construction of this theory on real-analytic manifolds.

Here we briefly take a look at the history of the theory of Sato hyperfunctions.

1958-1960, Sato established the theory of hyperfunctions [23], [24], [25]. Harvey, Komatsu and Schapira studied this theory on Euclidean Spaces [4], [18], [26]. Ion-Kawai and Ito constructed the theory of vector-valued Sato hyperfunctions on Euclidean spaces [6], [9], [11]. Ito constructed the theory of vector-valued Sato hyperfunctions on real-analytic manifolds by Schapira's method [7]. Ito treated the wider classes of Sato-Fourier hyperfunctions including (vector-valued) Sato hyperfunctions as a specialization [12]. Morimoto, Kaneko and Kashiwara-Kawai-Kimura have written very excellent text books on the theory of Sato hyperfunctions [21], [13], [15].

Nowadays we have known that Sato hyperfunctions have twofold realization as classes of elements of the dual space of the space of real-analytic functions and as "boundary values" of holomorphic functions. These two realizations are mutually independent and equivalent. In this paper we mention the realization of Sato hyperfunctions as "boundary values" of holomorphic functions. As for vector-valued Sato hyperfunctions, the situations are similar.

In section 1, we treat the scalar-valued case, and in section 2, we treat the vector-valued case.

Here we note that "isomorphisms" usually mean topological ones without explicitimention for the contrary. We also note that we use the term "A-sheaf $\mathcal{F}$" if the section module $\mathcal{F}(\Omega)$ of the sheaf $\mathcal{F}$ over every open set Ω has the property "A".

1. Case of Sato hyperfunctions

1.1. The Dolbeault-Grothendieck resolutions of $\mathcal{O}$. In this subsection we recall the soft resolutions of the sheaf $\mathcal{O}$ of holomorphic functions over an n-dimensional, complex manifold X.

If $\mathcal{F}$ is a sheaf over X, we define the sheaf $\mathcal{F}^{p,q}$ to be the sheaf of differntial forms of type (p, q) with coefficients in $\mathcal{F}$ and denote the Cauchy-Riemann operator by $\overline{\partial}$, where p and q are nonnegative integers. We denote $\mathcal{F}^p = \mathcal{F}^{p,0}$. Then we have the following.

Theorem 1.1.1 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over X*

$$0 \longrightarrow \mathcal{O}^p \longrightarrow \mathcal{E}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{p,n} \to 0$$

is exact, where $\mathcal{E}$ denotes the soft nuclear Fréchet sheaf of C^∞-functions over X and p is a nonnegative integer.

Proof. Since the assertion is local, this easily follows from the Euclidean case. As for the Euclidean case, we refer the reader to Ito [11], Theorem 1.1.1 (Q.E.D)

Corollary. *For an open set Ω in X, we have the following isomorphism :*

$$H^q(\Omega, \mathcal{O}^p) \cong \{f \in \mathcal{E}^{p,q}(\Omega); \overline{\partial} f = 0\}/\{\overline{\partial} g; g \in \mathcal{E}^{p,q-1}(\Omega)\},$$

$$(p \geq 0 \text{ and } q \geq 1).$$

Proof. It follows from Theorem 1.1.1 and Komatsu [18], Theorems II.2.9 and II.2.19 (Q.E.D.)

Next we prove one another soft resolution of the sheaf $\mathcal{O}$.

Let $L = L_{2,\mathrm{loc}}$ be the soft FS*-sheaf of locally L_2-functions over X. Then we define the sheaf $\mathcal{L}^{p,q}$ to be the sheaf $\{\mathcal{L}^{p,q}(\Omega); \Omega$ is an open set in $X\}$, where, for an open set Ω in X, the section module $\mathcal{L}^{p,q}(\Omega)$ is the space of all $f \in L^{p,q}(\Omega)$ such that $\overline{\partial} f \in L^{p,q+1}(\Omega)$. We put $\mathcal{L} = \mathcal{L}^{0,0}$. Then $\mathcal{L}^{p,q}$ is a soft FS*-sheaf with respect to the graph topology of the operator $\overline{\partial}$. Then we have the following.

Theorem 1.1.2 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over X*

$$0 \longrightarrow \mathcal{O}^p \longrightarrow \mathcal{L}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{L}^{0,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{L}^{p,n} \longrightarrow 0$$

is exact.

Proof. Since the assertion is local, this easily follows from the Euclidean case. As for the Euclidean case, we refer the reader to Ito [11], Theorem 1.1.2. (Q.E.D.)

Corollary. *For an open set Ω in X, we have the following isomorphism :*

$$H^q(\Omega, \mathcal{O}^p) \cong \{f \in \mathcal{L}^{p,q}(\Omega); \overline{\partial} f = 0\}/\{\overline{\partial} g; g \in \mathcal{L}^{p,q-1}(\Omega)\},$$

$$(p \geq 0 \text{ and } q \geq 1)$$

1.2. The Oka-Cartan Theorem B. In this subsection we prove the Oka-Cartan Theorem B for the sheaf $\mathcal{O}$ by using the soft resolution of Theorem 1.1.2. Thus we have the following.

Theorem 1.2.1(The Oka-Cartan Theorem B). *For every Stein open set V in X, we have $H^s(V, \mathcal{O}^p) = 0$, ($p \geq 0$ and $s \geq 1$).*

Proof. This is an immediate consequence of Theorem 1.1.2 and Hörmander [5], Theorem 5.2.4. (Q.E.D.)

Corollary. *For every Stein open set Ω in X, the equation $\overline{\partial}u = f$ has a solution $u \in \mathcal{E}^{p,q}(\Omega)$ for every $f \in \mathcal{E}^{p,q+1}(\Omega)$ such that $\overline{\partial}f = 0$. Here p and q are nonnegative integers.*

Proof. This is an immediate consequence of Corollary to Theorem 1.1.1 and Corollary to Theorem 1.2.1. (Q.E.D.)

Let M be an n-dimensional, real-analytic manifold countable at infinity and X its complexification. We define the sheaf $\mathcal{O}$ as above and the sheaf $\mathcal{A}$ of real-analytic functions over M is defined by $\mathcal{A} = \mathcal{O}|_M$. Then we have the following.

Theorem 1.2.2(Grauert). *Let Ω be an open set in M. Then Ω has a fundamental system of Stein open neighborhoods in X.*

Proof. See Grauert [2], Proposition 7. (Q.E.D.)

Then we have the following.

Theorem 1.2.3(Malgrange). *For every subset S of M, we have $H^s(S, \mathcal{A}^p) = 0$, ($p \geq 0$ and $s \geq 1$).*

Proof. We know, by Theorem 1.2.2, that S has a fundamental system of Stein open neighborhoods. Then it follows, from Theorem 1.2.1 and Schapira [26], Theorem B42, that, for $s > 0$, we have

$$H^s(S, \mathcal{A}^p) = \lim\operatorname{ind}_{S \subset \Omega} H^s(\Omega, \mathcal{O}^p) = 0. \qquad \text{(Q.E.D.)}$$

1.3. Malgrange's Theorem. In this subsection we prove the following.

Theorem 1.3.1 (Malgrange). *Let Ω be an open set in X. Then we have $H^n(\Omega, \mathcal{O}) = 0$.*

Proof. By virtue of Corollary to Theorem 1.1.2, we have only to prove the exactness of the sequence

$$\mathcal{L}^{0,n-1}(\Omega) \xrightarrow{\overline{\partial}} \mathcal{L}^{0,n}(\Omega) \longrightarrow 0$$

in the notation of Theorem 1.1.2. By virtue of the Serre-Komatsu duality theorem for FS*-spaces, it suffices to show the injectiveness and the closedness of the range of $-\overline{\partial}^{\Omega} = (\overline{\partial})'$ in the dual sequence

$$\mathcal{L}_c^{0,1}(\Omega) \xleftarrow{-\overline{\partial}^{\Omega}} \mathcal{L}_c^{0,0}(\Omega) \longleftarrow 0.$$

Here $\mathcal{L}_c^{p,q}(\Omega)$ denotes the space of sections with compact support of $\mathcal{L}^{p,q}$ on Ω. Since $-\overline{\partial}^{\Omega}$ is elliptic, its injectivity is an immediate consequence of the unique continuation property. Now we prove the closedness of its range. This is surely true if Ω is replaced by a Stein open set V containing Ω because then $H^p(V, \mathcal{O}) = 0$ for $p \geq 1$. Here we consider the commutative diagram:

$$\begin{array}{ccccc}
\mathcal{L}_c^{0,1}(\Omega) & \xleftarrow{-\overline{\partial}^{\Omega}} & \mathcal{L}_c^{0,0}(\Omega) & \longleftarrow & 0 \\
i\downarrow & & \downarrow & & \\
\mathcal{L}_c^{0,1}(V) & \xleftarrow{-\overline{\partial}^{V}} & \mathcal{L}_c^{0,0}(V) & \longleftarrow & 0,
\end{array}$$

where the map i is the natural injection. By the remark above, $-\overline{\partial}^V$ is of closed range. Then we have

$$\mathrm{Im}(-\bar{\partial}^{\Omega}) = \{i^{-1}(\mathrm{Im}(-\overline{\partial}^V))\} \cap \{[\mathrm{Im}(-\overline{\partial}^{\Omega})]^{c1}\},$$

where $[\ ]^{c1}$ is the closure of the set $[\ ]$. In fact, the inclusin $\mathrm{Im}(-\overline{\partial}^{\Omega}) \subset \{i^{-1}(\mathrm{Im}(-\overline{\partial}^V))\} \cap \{[\mathrm{Im}(-\overline{\partial}^{\Omega})]^{c1}\}$ is evident. Now we show the inverse inclusion. Assume $f = -\overline{\partial}u \in [\mathrm{Im}(-\overline{\partial}^{\Omega})]^{c1}$ with $u \in \mathcal{L}_c^{0,0}(V)$. Then u is holomorphic on $V\backslash\mathrm{supp}(f)$ and supp $(f) \subset\subset \Omega$. Hence $u = 0$ on the components of $V\backslash\mathrm{supp}(f)$ which are disjoint from supp (f). Hence supp $(u) \subset \Omega$. Namely we have $u \in \mathcal{L}_c^{0,0}(\Omega)$. Thus we have $f \in \mathrm{Im}(-\overline{\partial}^{\Omega})$. Therefore $\mathrm{Im}(-\overline{\partial}^{\Omega})$ is closed. Namely $-\overline{\partial}^{\Omega}$ is of closed range. This completes the proof. (Q.E.D.)

Corollary. Flabby dim $\mathcal{O} \leq n$.

1.4. Serre's Duality Theorem. In this subsection we prove Serre's Duality Theorem.

Theorem 1.4.1. *Let Ω be an open set in X such that* dim $H^p(\Omega, \mathcal{O}) < \infty$ *holds* $(p \geq 1)$. *Then we have the isomorphism* $[H^p(\Omega, \mathcal{O})]' \cong H_c^{n-p}(\Omega, \mathcal{O}), (0 \leq p \leq n)$.

Proof. By virtue of Corollary to Theorem 1.1.2, cohomology groups $H^p(\Omega, \mathcal{O})$ and $H_c^{n-p}(\Omega, \mathcal{O})$ are cohomology groups respectively of the complexes.

$$\begin{array}{ccccccccccc}
& 0 & \longrightarrow & \mathcal{L}^{0,0}(\Omega) & \xrightarrow{\overline{\partial}} & \mathcal{L}^{0,1}(\Omega) & \xrightarrow{\overline{\partial}} & \cdots & \xrightarrow{\overline{\partial}} & \mathcal{L}^{0,n}(\Omega) & \longrightarrow 0. \\
(*) & & & \updownarrow & & \updownarrow & & & & \updownarrow & \\
& 0 & \longleftarrow & \mathcal{L}_c^{0,n}(\Omega) & \xleftarrow{-\overline{\partial}} & \mathcal{L}_c^{0,n-1}(\Omega) & \xleftarrow{-\overline{\partial}} & \cdots & \xleftarrow{-\overline{\partial}} & \mathcal{L}_c^{0,0}(\Omega) & \longleftarrow 0.
\end{array}$$

Here the upper complex is composed of FS*-spaces and the lower complex is composed of DFS*-spaces. Since the ranges of operators $\overline{\partial}$ in the upper complex are all closed by virtue of Schwartz's Lemma (cf.Komatsu [17]), the ranges of operators $-\overline{\partial} = (\overline{\partial})'$ in the lower complex are also all closed. Hence we have the isomorphism

$$[H^p(\Omega, \mathcal{O})]' \cong H_c^{n-p}(\Omega, \mathcal{O})$$

by virture of Serre's Lemma (cf. Komatsu [17]). (Q.E.D.)

Remark. The conclusion of the theorem works for every open set Ω for which every $\overline{\partial}$ operator in the diagram $(*)$ is of closed range.

1.5. Martineau-Harvey's Theorem. In this subsection we prove Martineau-Harvey's Theorem.

Theorem 1.5.1 (Martineau-Harvey). *Let K be a compact set in X. Further assume the following* (i) *and* (ii) :

(i) $H^p(K, \mathcal{O}) = 0, (p \geq 1)$.

(ii) Ω *be a Stein open set with* $K \subset \Omega$.

Then we have the following :

(1) $H^p_K(\Omega, \mathcal{O}) = 0, (p \neq n)$.

(2) *If* $n \geq 2$, *we have algebraic isomorphisms*

$$H^n_K(\Omega, \mathcal{O}) \cong H^{n-1}(\Omega \backslash K, \mathcal{O}) \cong \mathcal{O}(K)'.$$

(3) *If* $n = 1$, *we have topological isomorphisms*

$$H^1_K(V, \mathcal{O}) \cong \mathcal{O}(V \backslash K)/\mathcal{O}(V) \cong \mathcal{O}(K)'$$

Remark. If a compact set K in X has a fundamental system of Stein open neighborhoods, it satisfies the assumptions in Theorem 1.5.1.

Proof. It goes in a similar way to Ito [11]. From a general theory of relative cohomology groups (cf. Komatsu [18], Theorem II.3.2), we have an exact sequence

$$\begin{aligned} 0 &\to H^0_K(\Omega, \mathcal{O}) \to H^0(\Omega, \mathcal{O}) \to H^0(\Omega \backslash K, \mathcal{O}) \\ &\to H^1_K(\Omega, \mathcal{O}) \to H^1(\Omega, \mathcal{O}) \to H^1(\Omega \backslash K, \mathcal{O}) \to \cdots \\ &\to H^n_K(\Omega, \mathcal{O}) \to H^n(\Omega, \mathcal{O}) \to H^n(\Omega \backslash K, \mathcal{O}) \to \cdots . \end{aligned}$$

Then, we have $H^p(\Omega, \mathcal{O}) = 0, (p \geq 1)$, and $H^0_K(\Omega, \mathcal{O}) = 0$ by the unique continuation theorem. Hence we have an exact sequence and algebraic isomorphisms

$$0 \to \mathcal{O}(\Omega) \to \mathcal{O}(\Omega \backslash K) \to H^1_K(\Omega, \mathcal{O}) \to 0,$$

$$H^p_K(\Omega, \mathcal{O}) \cong H^{p-1}(\Omega \backslash K, \mathcal{O}), (p \geq 2).$$

We also have the long exact sequence of cohomology groups with compact support (cf. Komatsu [18]), Theorem II.3.15):

$$\begin{aligned} 0 &\to H^0_c(\Omega \backslash K, \mathcal{O}) \to H^0_c(\Omega, \mathcal{O}) \to H^0(K, \mathcal{O}) \\ &\to H^1_c(\Omega \backslash K, \mathcal{O}) \to H^1_c(\Omega, \mathcal{O}) \to H^1(K, \mathcal{O}) \to \cdots \\ &\to H^p_c(\Omega \backslash K, \mathcal{O}) \to H^p_c(\Omega, \mathcal{O}) \to H^p(K, \mathcal{O}) \to \cdots . \end{aligned}$$

Here $H^p(K,\mathcal{O})=0,(p\geq 1)$ by the assumption on K. From Theorem 1.4.1 and the fact $H^p(\Omega,\mathcal{O})=0,(p\geq 1)$, we also have $H_c^p(\Omega,\mathcal{O})=0,(p\neq n)$. Therefore we obtain an exact sequence and topological isomorphisms:

When $n=1$,

$$0\to\mathcal{O}(K)\to H_c^1(\Omega\backslash K,\mathcal{O})\to H_c^1(\Omega,\mathcal{O})\to 0,$$

when $n\geq 2$,

$$H_c^1(\Omega\backslash K,\mathcal{O})\cong\mathcal{O}(K),$$

$$H_c^p(\Omega\backslash K,\mathcal{O})\cong H_c^p(\Omega,\mathcal{O})=0,(p\neq 1,n),$$

$$H_c^n(\Omega\backslash K,\mathcal{O})\cong H_c^n(\Omega,\mathcal{O}).$$

Now we consider the following dual complexes :

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{L}^{0,0}(\Omega\backslash K) & \xrightarrow{\overline{\partial}_0} & \mathcal{L}^{0,1}(\Omega\backslash K) & \xrightarrow{\overline{\partial}_1} \cdots \xrightarrow{\overline{\partial}_{n-2}} & \mathcal{L}^{0,n-1}(\Omega\backslash K) & \longrightarrow (*) \\
& \updownarrow & & \updownarrow & & \updownarrow & \\
0 \longleftarrow & \mathcal{L}_c^{0,n}(\Omega\backslash K) & \xleftarrow{-\overline{\partial}_{n-1}} & \mathcal{L}_c^{0,n-1}(\Omega\backslash K) & \xleftarrow{-\overline{\partial}_{n-2}} \cdots \xleftarrow{-\overline{\partial}_1} & \mathcal{L}_c^{0,1}(\Omega\backslash K) & \longleftarrow (**)
\end{array}$$

$$\begin{array}{cc}
(*) \xrightarrow{\overline{\partial}_{n-1}} & \mathcal{L}^{0,n}(\Omega\backslash K) \longrightarrow 0 \\
& \updownarrow \\
(**) \xleftarrow{-\overline{\partial}_0} & \mathcal{L}_c^{0,0}(\Omega\backslash K) \longleftarrow 0.
\end{array}$$

Then, since $H_c^p(\Omega\backslash K,\mathcal{O})=0,(p\neq 1,n)$, the range of $-\overline{\partial}_j=(\overline{\partial}_{n-j-1})'$ is closed for $j\neq 0,n-1$. However $\overline{\partial}_{n-1}$ is of closed range by the Malgrange Theorem. Hence, by the Serre-Komatsu duality theorem, $-\overline{\partial}_0$ is of closed range.

In order to prove the closedness of the range of $-\overline{\partial}_{n-1}$, we consider the following diagram:

$$\begin{array}{cccc}
0 \longleftarrow & \mathcal{L}_c^{0,n}(\Omega\backslash K) & \xleftarrow{-\overline{\partial}_{n-1}^{\Omega\backslash K}} & \mathcal{L}_c^{0,n-1}(\Omega\backslash K) \\
& i\downarrow & & \downarrow \\
0 \longleftarrow & \mathcal{L}_c^{0,n}(\Omega) & \xleftarrow{-\overline{\partial}_{n-1}^{\Omega}} & \mathcal{L}_c^{0,n-1}(\Omega),
\end{array}$$

where the map i is the natural injection.

We conclude that $\overline{\partial}_0^{\Omega}$ is of closed range because $H^1(\Omega, \mathcal{O}) = 0$. Thus $-\overline{\partial}_{n-1}^{\Omega}$ is of closed range by the Serre-Komatsu duality theorem. Therefore Im $(-\overline{\partial}_{n-1}^{\Omega\setminus K}) = i^{-1}(\mathrm{Im}(-\overline{\partial}_{n-1}^{\Omega}))$ is closed by the continuity of the map i. Therefore all $-\overline{\partial}_j^{\Omega\setminus K}$ are of closed range. Hence, by the Serre-Komatsu duality theorem, we have the isomorphisms

$$[H^p(\Omega\setminus K, \mathcal{O})]' \cong H_c^{n-p}(\Omega\setminus K, \mathcal{O}), \quad (0 \leq p \leq n).$$

If $n = 1$, by the Serre duality theorem, we have the dual complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(\Omega) & \longrightarrow & \mathcal{O}(\Omega\setminus K) & \longrightarrow & H_K^1(\Omega, \mathcal{O}) & \longrightarrow & 0 \\ & & \updownarrow & & \updownarrow & & & & \\ 0 & \longleftarrow & H_c^1(\Omega, \mathcal{O}) & \longleftarrow & H_c^1(\Omega\setminus K, \mathcal{O}) & \longleftarrow & \mathcal{O}(K) & \longleftarrow & 0. \end{array}$$

Therefore we have topological isomorphisms

$$[H_K^1(\Omega, \mathcal{O})]' \cong [\mathrm{Coker}(\mathcal{O}(\Omega) \longrightarrow \mathcal{O}(\Omega\setminus K))]'$$
$$\equiv \mathrm{Ker}(H_c^1(\Omega\setminus K, \mathcal{O}) \longrightarrow H_c^1(\Omega, \mathcal{O})) \cong \mathcal{O}(K).$$

Thus we have topological isomorphisms

$$H_K^1(\Omega, \mathcal{O}) \cong \mathcal{O}(\Omega\setminus K)\setminus\mathcal{O}(\Omega) \cong \mathcal{O}(K)'.$$

This proves (3).

If $n \geq 2$, since FS*- or DFS*-spaces are reflexive, we have

$$\mathcal{O}(\Omega\setminus K)' \cong H_c^n(\Omega\setminus K, \mathcal{O}) \cong H_c^n(\Omega, \mathcal{O}) \cong \mathcal{O}(\Omega)'.$$

Hence we have the isomorphism

$$\mathcal{O}(\Omega) \cong \mathcal{O}(\Omega\setminus K).$$

Thus we have

$$H_K^1(\Omega, \mathcal{O}) \cong \mathcal{O}(\Omega\setminus K)/\mathcal{O}(\Omega) = 0.$$

Further, for $p \geq 2, p \neq n$, we have

$$0 = [H_c^{n-p+1}(\Omega, \mathcal{O})]' \cong [H_c^{n-p+1}(\Omega\setminus K, \mathcal{O})]'$$
$$\cong H^{p-1}(\Omega\setminus K, \mathcal{O}) \cong H_K^p(\Omega, \mathcal{O}).$$

Thus we have

$$H_K^p(\Omega, \mathcal{O}) = 0, \quad (p \neq n).$$

This proves (1).

In the case $p = n$, we have algebraic isomorphisms

$$H^n_K(\Omega, \mathcal{O}) \cong H^{n-1}(\Omega \backslash K, \mathcal{O}) \cong [H^1_c(\Omega \backslash K, \mathcal{O})]' \cong \mathcal{O}(K)'.$$

This proves (2) (Q.E.D.)

Here we mention the important facts used in the proof of Theorem 1.5.1 in the following.

Proposition 1.5.2. *Assume $n \geq 2$. Let K and Ω be as in Theorem 1.5.1. Then we have (topological) isomorphisms :*

(1) $H^1_c(\Omega \backslash K, \mathcal{O}) \cong H^0(K, \mathcal{O})$,

(2) $H^p_c(\Omega \backslash K, \mathcal{O}) \cong H^p_c(\Omega, \mathcal{O}), (p \geq 2)$.

1.6. Sato's Theorem. In this subsection we prove the pure-codimensionality of M with respect to $\mathcal{O}$. Then we realize Sato hyperfunctions as "boundary values" of holomorphic functions or as relative cohomology classes of holomorphic functions.

Theorem 1.6.1 (Sato's Theorem).

(1) *M is purely n-codimensional with respect to the sheaf $\mathcal{O}$. Namely, we have $\mathcal{H}^p_M(\mathcal{O}) = 0, (p \neq n)$.*

(2) *The presheaf $\{H^n_\Omega(V, \mathcal{O}); \Omega$ is an open set in $M\}$ over M is a flabby sheaf. Here the section module $H^n_\Omega(V, \mathcal{O})$ is the relative cohomology group with coefficients in the sheaf $\mathcal{O}$ and V is an open set in X which contains Ω as its closed subset. We denote this sheaf by $\mathcal{H}^n_M(\mathcal{O})=\mathrm{Dist}^n(M, \mathcal{O})$.*

(3) *The sheaf $\mathcal{H}^n_M(\mathcal{O})$ is isomorphic to the sheaf $\mathcal{B}$ of Sato hyperfunctions realized by the duality method in Ito* [7], *Definition* 4.2 *applied to the case $E = \mathbf{C}$.*

Remark. The symbol $\mathrm{Dist}^n(M, \mathcal{O})$ is due to Sato [25].

Proof. (1) We have to prove the vanishing of the derived sheaf $\mathcal{H}^p_M(\mathcal{O})$ for $p \neq n$. This is local in nature. Thus, it is sufficient to prove $H^p_\Omega(V, \mathcal{O}) = 0, (p \neq n)$ for every relatively compact open set Ω in M. Thus, let Ω be a relatively compact open set in M. Then, by the excision theorem, we may assume that V is an open set in X which contains Ω^{c1}. Consider the following exact sequence of relative cohomology groups

$$\cdots \longrightarrow H^p_{\partial\Omega}(V, \mathcal{O}) \longrightarrow H^p_{\Omega^{c1}}(V, \mathcal{O}) \to H^p_\Omega(V, \mathcal{O})$$
$$\longrightarrow H^{p+1}_{\partial\Omega}(V, \mathcal{O}) \longrightarrow \cdots.$$

By Theorem 1.5.1, we may conclude that $H^p_{\partial\Omega}(V, \mathcal{O}) = H^p_{\Omega^{c1}}(V, \mathcal{O}) = 0, (p \neq n)$. So that, we have $H^p_\Omega(V, \mathcal{O}) = 0, (p \neq n-1, n)$. Therefore, we have the exact sequence

$$0 \longrightarrow H^{n-1}_\Omega(V, \mathcal{O}) \longrightarrow H^n_{\partial\Omega}(V, \mathcal{O})$$
$$\longrightarrow H^n_{\Omega^{c1}}(V, \mathcal{O}) \longrightarrow H^n_\Omega(V, \mathcal{O}) \longrightarrow 0.$$

By Theorem 1.5.1, we have the isomorphisms

$$H^n_{\partial\Omega}(V,\mathcal{O}) \cong \mathcal{A}(\partial\Omega)', \quad H^n_{\Omega^{c1}}(V,\mathcal{O}) \cong \mathcal{A}(\Omega^{c1})'.$$

Since the natural mapping

$$\mathcal{A}(\partial\Omega)' \longrightarrow \mathcal{A}(\Omega^{c1})'$$

is injective, we have $H^{n-1}_\Omega(V,\mathcal{O}) = 0$. Therefore, we have $H^p_\Omega(V,\mathcal{O}) = 0, (p \neq n)$.

(2) By (1) and Komatsu [18], Theorem II. 3. 24, we have the conclusion.

(3) It follows in a similar way to Ito [11], Theorem 1.6.1. (Q.E.D.)

Corollary. *Let Ω be an arbitrary open set in M and V a Stein open neighborhood of Ω. Then we have the following:*

(1) *If $n \geq 2, H^n_\Omega(V,\mathcal{O}) \cong H^{n-1}(V\backslash\Omega,\mathcal{O})$.*

(2) *If $n = 1, H^1_\Omega(V,\mathcal{O}) \cong \mathcal{O}(V\backslash\Omega)/\mathcal{O}(V)$.*

Thus we have realized Sato hyperfunctions as "boundary values" of holomorphic functions or as relative cohomology classes of holomorphic functions. They are equivalent to those which are realized by the duality method in Ito [7], [12].

2. Case of vector-valued Sato hyperfunctions

2.1. The Dolbeault-Grothendieck resolution of ${}^E\mathcal{O}$. In this section we mention the theory of vector-valued Sato hyperfunctions on a real-analytic manifold countable at infinity.

Let M be an n-dimensional real-anlytic manifold countable at infinity and X its complexification. Let E be a Fréchet space.

We define the sheaf ${}^E\mathcal{O}$ of E-valued holomorphic functions over X to be the sheaf $\{\mathcal{O}(\Omega;E);\Omega$ is an open set in $X\}$, where, for an open set Ω in X, the section module $\mathcal{O}(\Omega;E)$ is the space of all E-valued holomorphic functions on Ω.

We also define the sheaf ${}^E\mathcal{E}$ of E-valued C^∞-functions over X to be the sheaf $\{\mathcal{E}(\Omega;E);\Omega$ is an open set in $X\}$, where, for an open set Ω in X, the section module $\mathcal{E}(\Omega;E)$ is the space of all E-valued C^∞-functions on Ω. Then we have the following.

Proposition 2.1.1. *The sheaf ${}^E\mathcal{E}$ is soft.*

Proof. Since ${}^E\mathcal{E}$ is obviously an $\mathcal{E}$-module and $\mathcal{E} = {}^{\mathbf{C}}\mathcal{E}$ is a soft sheaf, we have the conclusion by virtue of Bredon [1], Chapter II, Theorem 9.12, p.50. (Q.E.D.)

Then we have the following.

Theorem 2.1.2(The Dolbeault-Grothendieck resolution of ${}^E\mathcal{O}^p$). *The sequence of sheaves over X*

$$0 \longrightarrow {}^E\mathcal{O}^p \longrightarrow {}^E\mathcal{E}^{p,0} \xrightarrow{\bar\partial} {}^E\mathcal{E}^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {}^E\mathcal{E}^{p,n} \longrightarrow 0$$

is exact, $(p \geq 0)$.

Proof. Since E is a Fréchet space and the sheaf $\mathcal{E}$ is a unclear Fréchet sheaf, this follows from Theorem 1.1.1 and Ion-Kawai [6], Theorem 1.10, p.9. (Q.E.D.)

Corollary. *For an open set* Ω *in* X, *we have the following isomorphism:*

$$H^q(\Omega, {}^E\mathcal{O}^p) \cong \{f \in \mathcal{E}^{p,q}(\Omega; E); \overline{\partial} f = 0\} \backslash \{\overline{\partial} g; g \in \mathcal{E}^{p,q-1}(\Omega; E)\},$$

$$(p \geq 0 \text{ and } q \geq 1).$$

Proof. It follows from Theorem 2.1.2 and Komatsu [18], Theorems II.2.9 and II.2.19. (Q.E.D.)

2.2. The Oka-Cartan Theorem B. In this subsection we prove the Oka-Cartan Theorem B for the sheaf ${}^E\mathcal{O}$.

Theorem 2.2.1 (The Oka-Cartan Theorem B). *For every Stein open set* Ω *in* X, *we have* $H^q(\Omega, {}^E\mathcal{O}^p) = 0, (p \geq 0$ *and* $q \geq 1)$.

Proof. By virtue of the Oka-Cartan Theorem B for the sheaf $\mathcal{O}$, we have

$$H^q(\Omega, \mathcal{O}^p) = 0, (p \geq 0 \text{ and } q \geq 1).$$

Thus the complex obtained from Theorem 1.1.1

$$\mathcal{E}^{p,0}(\Omega) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1}(\Omega) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{p,n}(\Omega) \longrightarrow 0$$

is exact. Since $\mathcal{E}^{p,q}(\Omega)$' s are unclear Fréchet spaces and E is a Fréchet space, the complex

$$\mathcal{E}^{p,0}(\Omega; E) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1}(\Omega; E) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{E}^{p,n}(\Omega; E) \longrightarrow 0$$

is also exact by virtue of the isomorphism

$$\mathcal{E}^{p,q}(\Omega; E) \cong \mathcal{E}^{p,q}(\Omega) \hat{\otimes} E$$

and Ion-Kawai [6], Theorem 1.10, p.9. Hence we obtain

$$H^q(\Omega, {}^E\mathcal{O}^p) = 0, (p \geq 0 \text{ and } q \geq 1).$$

This completes the proof. (Q.E.D.)

Corollary. *We use the notation in Theorem 2.2.1. Then the equation* $\overline{\partial} u = f$ *has a solution* $u \in \mathcal{E}^{p,q}(\Omega; E)$ *for every* $f \in \mathcal{E}^{p,q+1}(\Omega; E)$ *such that* $\overline{\partial} f = 0$. *Here* p *and* q *are nonnegative integers.*

Proof. It follows from Theorem 2.2.1 and Corollary to Theorem 2.1.2. (Q.E.D.)

2.3. The Malgrange Theorem. In this subsection we prove the Malgrange Theorem.

Theorem 2.3.1. *Let* Ω *be an open set in* X. *Then we have* $H^n(\Omega, {}^E\mathcal{O}) = 0$.

Proof. By virtue of Theorem 1.1.1 and 1.3.1, we have an exact sequence

$$\mathcal{E}^{0,n-1}(\Omega) \xrightarrow{\overline{\partial}} \mathcal{E}^{0,n}(\Omega) \longrightarrow 0.$$

Thus, by Trèves [28], Proposition 43.9, we have the exact sequence

$$\mathcal{E}^{0,n-1}(\Omega)\hat{\otimes}E \xrightarrow{\overline{\partial}} \mathcal{E}^{0,n}(\Omega)\hat{\otimes}E \to 0$$

or

$$\mathcal{E}^{0,n-1}(\Omega; E) \xrightarrow{\overline{\partial}} \mathcal{E}^{0,n}(\Omega; E) \to 0.$$

This completes the proof. (Q.E.D.)

Corollary. Flabby dim ${}^{E}\mathcal{O} \leq n$.

2.4. The Serre Duality Theorem. In this subsection we prove the Serre Duality Theorem.

Theorem 2.4.1. *Let Ω be an open set in X such that $[H^p(\Omega, \mathcal{O})]' \cong H_c^{n-p}(\Omega, \mathcal{O}), (0 \leq p \leq n)$ holds. Then we have the isomorphism $H^p(\Omega, {}^{E}\mathcal{O}) \cong L(H_c^{n-p}(\Omega, \mathcal{O}); E), (0 \leq p \leq n)$.*

Proof. Since we can easily obtain the isomorphism $H^p(\Omega, {}^{E}\mathcal{O}) \cong H^p(\Omega, \mathcal{O})\hat{\otimes}E$, we have the following isomorphisms by the assumption :

$$\begin{aligned} H^p(\Omega, {}^{E}\mathcal{O}) &\cong H^p(\Omega, \mathcal{O})\hat{\otimes}E \\ &\cong [H_c^{n-p}(\Omega, \mathcal{O})]'\hat{\otimes}E \cong L(H_c^{n-p}(\Omega, \mathcal{O}); E). \qquad \text{(Q.E.D.)} \end{aligned}$$

2.5. The Martineau-Harvey Theorem. In this subsection we prove the Martineau-Harvey Theorem.

Theorem 2.5.1. *Let K and Ω be as in Theorem* 1.5.1. *Then we have the following* :

(1) $H_K^p(\Omega, {}^{E}\mathcal{O}) = 0, (p \neq n)$.

(2) *If $n \geq 2$, we have algebraic isomorphisms*

$$H_K^n(\Omega, {}^{E}\mathcal{O}) \cong H^{n-1}(\Omega \backslash K, {}^{E}\mathcal{O}) \cong L(\mathcal{O}(K); E).$$

(3) *If $n = 1$, we have topological isomorphisms*

$$H_K^1(V, {}^{E}\mathcal{O}) \cong \mathcal{O}(\Omega \backslash K; E)/\mathcal{O}(\Omega; E) \cong L(\mathcal{O}(K); E).$$

Proof. It goes in a similar way to Theorem 1.5.1.

From a general theory of relative cohomology groups (cf. Komatsu [18], Theorem II.3.2), we have an exact sequence

$$0 \longrightarrow H_K^0(\Omega, {}^{E}\mathcal{O}) \longrightarrow H^0(\Omega, {}^{E}\mathcal{O}) \longrightarrow H^0(\Omega \backslash K, {}^{E}\mathcal{O})$$

$$\longrightarrow H^1_K(\Omega,{}^E\mathcal{O}) \longrightarrow H^1(\Omega,{}^E\mathcal{O}) \longrightarrow H^1(\Omega\backslash K,{}^E\mathcal{O}) \longrightarrow \cdots$$
$$\longrightarrow H^n_k(\Omega,{}^E\mathcal{O}) \longrightarrow H^n(\Omega,{}^E\mathcal{O}) \longrightarrow H^n(\Omega\backslash K,{}^E\mathcal{O}) \longrightarrow \cdots.$$

Then, we have $H^p(\Omega,{}^E\mathcal{O}) = 0, (p \geq 1)$, and $H^0_K(\Omega,{}^E\mathcal{O}) = 0$ by the unique continuation theorem. Hence we have an exact sequence and algebraic isomorphisms

$$0 \longrightarrow \mathcal{O}(\Omega; E) \longrightarrow \mathcal{O}(\Omega\backslash K; E) \longrightarrow H^1_K(\Omega,{}^E\mathcal{O}) \longrightarrow 0,$$
$$H^p_K(\Omega,{}^E\mathcal{O}) \cong H^{p-1}(\Omega\backslash K,{}^E\mathcal{O}), (p \geq 2).$$

If $n = 1$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}(\Omega) \longrightarrow \mathcal{O}(\Omega\backslash K) \longrightarrow H^1_K(\Omega, \mathcal{O}) \longrightarrow 0$$

by the proof of Theorem 1.5.1. Since we have $\mathcal{O}(\Omega; E) \cong \mathcal{O}(\Omega)\hat{\otimes}E$ and $\mathcal{O}(\Omega\backslash K; E) \cong \mathcal{O}(\Omega\backslash K)\hat{\otimes}E$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}(\Omega)\hat{\otimes}E \longrightarrow \mathcal{O}(\Omega\backslash K)\hat{\otimes}E \longrightarrow H^1_K(\Omega, \mathcal{O})\hat{\otimes}E \longrightarrow 0.$$

Therefore we have an exact sequence

$$0 \longrightarrow \mathcal{O}(\Omega; E) \longrightarrow \mathcal{O}(\Omega\backslash K; E) \longrightarrow H^1_K(\Omega, \mathcal{O})\hat{\otimes}E \longrightarrow 0.$$

Thus we have isomorphisms

$$H^1_K(\Omega,{}^E\mathcal{O}) \cong \mathcal{O}(\Omega\backslash K; E)/\mathcal{O}(\Omega; E)$$
$$\cong H^1_K(\Omega, \mathcal{O})\hat{\otimes}E \cong \mathcal{O}(K)'\hat{\otimes}E \cong L(\mathcal{O}(K); E).$$

This proves (3).

Now we prove (1). Assume $n \geq 2$. Then we have $H^p(V,{}^E\mathcal{O}) \cong H^p(V, \mathcal{O})\hat{\otimes}_\pi E, (0 \leq p \leq n)$ for an open set V in X. Thus we have isomorphisms

$$\mathcal{O}(\Omega\backslash K; E) \cong \mathcal{O}(\Omega\backslash K)\hat{\otimes}E \cong \mathcal{O}(\Omega)\hat{\otimes}E \cong \mathcal{O}(\Omega; E).$$

Therefore we have $H^1_K(\Omega,{}^E\mathcal{O}) = 0$.

For $p \geq 2, p \neq n$, we have

$$H^p_K(\Omega,{}^E\mathcal{O}) \cong H^{p-1}(\Omega\backslash K,{}^E\mathcal{O}) \cong H^{p-1}(\Omega\backslash K, \mathcal{O})\hat{\otimes}E = 0$$

by Theorem 1.5.1.

Now assume $n = 1$. Then we have the conclusion by virtue of the long exact sequence of relative cohomology groups, the Oka-Cartan Theorem B and the Malgrange Theorem. This proves (1).

Now we prove (2). Assume $n \geq 2$. Then, by virtue of Proposition 1.5.2 and Theorem 2.4.1, we have algebraic isomorphisms

$$H^n_K(\Omega,{}^E\mathcal{O}) \cong H^{n-1}(\Omega\backslash K,{}^E\mathcal{O})$$

$$\cong L(H_c^1(\Omega \backslash K, \mathcal{O}); E) \cong L(\mathcal{O}(K); E).$$

This completes the proof. (Q.E.D,)

2.6 The Sato Theorem. In this subsection we prove the pure-codimensionality of M with respect to ${}^E\mathcal{O}$. Then we realize E-valued Sato hyperfunctions as "boundary values" of E-valued holomorphic functions or as relative cohomology classes of E-valued holomorphic functions.

Theorem 2.6.1 (the Sato Theorem).

(1) *M is purely n-codimensional with respect to ${}^E\mathcal{O}$. Namely we have $\mathcal{H}_M^p({}^E\mathcal{O}) = 0, (p \neq n)$.*

(2) *The presheaf $\{H_\Omega^n(V, {}^E\mathcal{O}); \Omega$ is an open set in $M\}$ over M is a flabby sheaf. Here the section module $H_\Omega^n(V, {}^E\mathcal{O})$ is the relative cohomology group with coefficients in the sheaf ${}^E\mathcal{O}$ and V is an open set in X which contains Ω as its closed subset. We denote this sheaf by $\mathcal{H}_M^n({}^E\mathcal{O})=\mathrm{Dist}^n(M, {}^E\mathcal{O})$.*

(3) *The sheaf $\mathcal{H}_M^n({}^E\mathcal{O})$ is isomorphic to the sheaf ${}^E\mathcal{B}$ of E-valued Sato hyperfunctions defined in Ito* [7], *Definition* 4.2.

Remark. The symbol $\mathrm{Dist}^n(M, {}^E\mathcal{O})$ is due to Sato [25].

Proof. (1) We have to prove the vanishing of the derived sheaf $\mathcal{H}_M^p({}^E\mathcal{O})$ for $p \neq n$. This is local in nature. Thus, it is sufficient to prove $H_\Omega^p(V, {}^E\mathcal{O}) = 0, (p \neq n)$ for every relatively compact open set Ω in M. Thus, let Ω be a relatively compact open set in M. Then, by the excision theorem, we may assume that V is an open set in X which contains Ω^{c1}. Consider the following exact sequence of relative cohomology groups

$$\cdots \longrightarrow H_{\partial\Omega}^{p+1}(V, {}^E\mathcal{O}) \longrightarrow H_{\Omega^{c1}}^p(V, {}^E\mathcal{O}) \longrightarrow H_\Omega^p(V, {}^E\mathcal{O})$$
$$\longrightarrow H_{\partial\Omega}^{p+1}(V, {}^E\mathcal{O}) \longrightarrow \cdots.$$

By Theorem 2.5.1, we may conclude that $H_{\partial\Omega}^p(V, {}^E\mathcal{O}) = H_{\Omega^{c1}}^p(V, {}^E\mathcal{O}) = 0, (p \neq n)$. So that, we have $H_\Omega^p(V, {}^E\mathcal{O}) = 0, (p \neq n-1, n)$. Therefore, we have the exact sequence

$$0 \longrightarrow H_\Omega^{n-1}(V, {}^E\mathcal{O}) \longrightarrow H_{\partial\Omega}^n(V, {}^E\mathcal{O})$$
$$\longrightarrow H_{\Omega^{c1}}^n(V, {}^E\mathcal{O}) \longrightarrow H_\Omega^n(V, {}^E\mathcal{O}) \longrightarrow 0.$$

By Theorem 2.5.1, we have the isomorphisms

$$H_{\partial\Omega}^n(V, {}^E\mathcal{O}) \cong L(\mathcal{A}(\partial\Omega); E), \quad H_{\Omega^{c1}}^n(V, {}^E\mathcal{O}) \cong L(\mathcal{A}(\Omega^{c1}); E).$$

Since the natural mapping

$$L(\mathcal{A}(\partial\Omega); E) \longrightarrow L(\mathcal{A}(\Omega^{c1}); E)$$

is injective, we have $H_\Omega^{n-1}(V, {}^E\mathcal{O}) = 0$. Therefore, we have $H_\Omega^p(V, {}^E\mathcal{O}) = 0, (p \neq n)$.

(2) By (1) and Komatsu [18], Theorem II.3.24, we have the conclusion.

(3) It follows in a similar way to Ito [11], Theorem 1.6.1. (Q.E.D.)

Corollary. *Let Ω be an arbitrary open set in M and V a Stein open neighborhood of Ω. Then we have the following* :

(1) *If* $n \geq 2, H^n_\Omega(V, {}^E\mathcal{O}) \cong H^{n-1}(V \backslash \Omega, {}^E\mathcal{O})$.

(2) *If* $n = 1, H^1_\Omega(V, {}^E\mathcal{O}) \cong \mathcal{O}(V \backslash \Omega; E)/\mathcal{O}(V; E)$.

Thus we have realized E-valued Sato hyperfunctions as "boundary values" of E-valued holomorphic functions or as relative cohomology classes of E-valued holomorphic functions. They are equivalent to those which are realized by the duality method in Ito [7], [12].

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