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Signed Graphs Associated with the Lattice A_n

By

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Abstract

For any base of the root lattice A_n, we can construct a signed graph naturally. A connected graph is called a Fushimi tree if its all blocks are complete subgraphs. In the present note, a Fushimi tree is said to be simple when by deleting any cut vertex, we have always two its connected components. We prove that a signed graph corresponding to a base of A_n is a simple Fushimi tree. Cameron, Seidel and Cameron [3] defined local switchings of singned graphs. We also show that a simple Fushimi tree is transformed to a line by a sequence of local switchngs.

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Introduction

A signed graph is a graph whose edges are signed by +1 or -1. In [3], for a given signed graph, Cameron, Seidel and Cameron construct the corresponding root lattice. In the present note, we treate with signed graphs corresponding to the root lattic A_n. A connected graph is called a *Fushimi tree* if its all blocks are complete subgraphs. A Fushimi tree is said to be *simple* when by deleting any cut vertex, we have always two its connected components. *Switching* and *local switching* of signed graphs are also introduced by Cameron, Seidel and Cameron in [3]. A signed Fushimi tree is said to be a Fushimi tree with standard sign if it can be transformed to a signed Fushimi tree whose all edges are signed by +1, by a switching. We prove that any signed graph corresponding to A_n is a simple Fusimi tree with standard sign. Switching defines a equivalent relation in the set of all signed graphs. A equivalent class is called a switching class. Local switching partitions the all signed graphs on n vertexs into clusters of switching classes. Our main result is that a simple Fusimi tree with standard sign is contained in the cluster given by the line.

1. Signed graphs

Following 3], we state basic facts about signed grapphs. A graph $G = (V, E)$ consists of an n-set V(the vertices) and a set E of unordered pairs from V(the edges). A *signed graph* (G, f) is a graph G with a signing $f : E \to \{1, -1\}$ of the edges. For any subset $U \subseteq V$ of vertices, let f_U denote the signing obtained from f by reversing the sign of each edge which has one vertex in U. This defines on the set of signings an equivalence relation, called *switching.* The equivalence classes $\{f_U : U \subseteq V\}$ are the *signed swithing classes* of the graph $G = (V, E)$. The *adjacency matrix* $A = (A_{ij})$ is defined by $A_{ij} = f(\{i,j\})$ for $\{i,j\} \in E$; else $A_{ij} = 0$ otherwise. The matrix $2I + A$ is called the *intersection matrix* , and interpreted as the Gram matrix of the inner product of n vectors $a_1, \cdots, a_n$ in a (possibly indefinite) inner product space $R^{p,q}$. These vectors are roots(which have length $\sqrt{2}$) at angles $\pi/2, \pi/3$, or $2\pi/3$. Their integral linear combinations form a root lattice (an even integral lattice spanne d by vectors of norm 2), which we denote by $L(G, f)$. Let $i \in V$ be a vertex of G, and $V(i)$ be the neighbours of i. The *local graph* of (G, f) at i has $V(i)$ as its vertex set, and as edges all edges $\{j, k\}$ of G for which $f(i,j)f(j,k)f(k,i) = -1$. A *rim* of (G, f) at i is any union of connected components of local graph at i. Let J be any rim at i, and let $K = V(i)\backslash J$. *Local switching* of (G, f) with respect to (i, J) is the following operation:(i) delete all edges of G between J and K; (ii) for any $j \in J$, $k \in K$ not previously joined, introduce an edge $\{j, k\}$ with sign chosen so that $f(i,j)f(j,k)f(k,i) = -1$; (iii) change the signs of all edges from i to J; (iv) leave all other edges and signs unaltered. Let Ω_n be the set of switching classes of signed graphs of order n. Local switching , applied to any vertex and any rim at the vertex, gives a relation on Ω which is symmetric but not transitive. The equivalence classes of its tansitive closure are called the *clusters* of order n.

2. The lattice A_n and signed Fushimi trees

A connected graph $G = (V, E)$ is called *Fushimi tree* if each block of G is a perfect graph. In the present paper, a Fushimi tree G is said to be a *simple Fushimi tree* if G is divided exactly two connected components when each cut vertex in G is deleted.

A signed simple Fushimi tree is called a special Fushimi tree *with standard sign* if we can switch all signs of edges into $+1$.

The latiice A_n is spanned by vectors $e_i - e_j$, $1 \leq i \neq j \leq n+1$, where $\{e_1, \cdots, e_{n+1}\}$ is the orthonormal base of the euclidean $(n+1)$-space R^{n+1}.

There is the one-to -one correspondence between ordered root bases of A_n and connected signed graphs associated with A_n.

Theorem 1 *Any connected signed graph is a signed graph associated with A_n if and only if it is a simple Fushimi tree with standard sign.*

Proof. Let G be a signed graph corresponding to a ordered base $\{a_1, a_2, \cdots, a_n\}$. If we replace a_i by $-a_i$, then the sign of G is switched with respect to $\{a_i\}$. Hence there is no problem whether we take a_i or $-a_i$. There is no cycle in G whose length is more than 3. In fact, if $a_{i_1}, a_{i_2}, \cdots, a_{i_m}, m > 3$ make a cycle, then we can assume that $a_{i_1} = e_{j_1} - e_{j_2}, a_{i_2} = e_{j_2} - e_{j_3}, a_{i_3} = e_{j_3} - e_{j_4}, \cdots, a_{i_m} = e_{j_m} - e_{j_1}$. But this implies that $a_{i_1}, a_{i_2}, \cdots, a_{i_m}$ are not linearly independent. If $a_{i_1}, a_{i_2}, a_{i_3}$ make a cycle, then we can assume that $a_{i_1} = e_j - e_{j_1}, a_{i_2} = e_j - e_{j_2}, a_{i_3} = e_j - e_{j_3}$. We have cycles of this type only in G. Now take a block B of G consisting of vertices $a_{i_1}, a_{i_2}, \cdots, a_{i_m}$. Two vetics a_{i_1} and a_{i_2} must be on a cycle in B. We may assume that $a_{i_1}, a_{i_2}, a_{i_3}$ make a cycle, Then we can put $a_{i_1} = e_j - e_{j_1}, a_{i_2} = e_j - e_{j_2}, a_{i_3} = e_j - e_{j_3}$. Two vetics a_{i_1} and a_{i_4} are also on a cycle in B, which may be $a_{i_1}, a_{i_4}, a_{i_5}$. Then we can put $a_{i_4} = e_j - e_{j_4}, a_{i_5} = e_j - e_{j_5}$ or $a_{i_4} = e_{j_1} - e_{j_4}, a_{i_5} = e_{j_1} - e_{j_5}$, where $j_4 \neq j$. Assume that $a_{i_4} = e_{j_1} - e_{j_4}, a_{i_5} = e_{j_1} - e_{j_5}$. Two vetics a_2 and a_4 are also on a cycle in B. Then we have $j_4 = j_2$, a contradiction. Hence, we get $a_{i_4} = e_j - e_{j_4}, a_{i_5} = e_j - e_{j_5}$. By this way, we get $a_{i_k} = e_j - e_{j_k}$, $1 \le k \le m$. Hence any block of G is a perfect graph whose edges have sign $+$ 1. Suppose that $a_{i_1} = e_j - e_{j_1}$ of a block B is a cut vertex,If two vertex a_j, a_k which are not in B are adjacent with a_{i_1}, then we cab put $a_j = e_{j_1} - e_{j_1}, a_k = e_{j_1} - e_{k_1}$. Hence a_{i_1}, a_j, a_k are contained in another block of G. Hence we show that $G - a_{i_1}$ has two connected components. Thus G is a simple Fushimi tree with standard sign.

Conversely, Let G be a special Fushimi tree with standard sign. Assume that G has m blocks. If $m = 1$, it is evident that G is a connected signed graph associated with A_n. Now suppose that the result is true for simple signed Fushimi trees with m blocks. Let G be a simple signed Fushimi tree with $m+1$ blocks. Let G' be a simple Fushimi tree with sndard sign which is made from G by deleting one block B of G. Then G' is a connected signed graph associated with A_n and corresponding to a ordered base $\{a_1, a_2, \cdots, a_n\}$. Now assume that all ℓ vertices of B are adjacement with a vertex $a_i = e_{i_1} - e_{i_2}$ and that e_{i_2} is not used in any other a_j. Then, we can consider that the block B consists of $e_{i_2} - e_{n+2}, e_{i_2} - e_{n+3}, \cdots, e_{i_2} - e_{n+\ell+1}$ and a_i. Hence We regard G as a connected signed graph associated with $A_{n+\ell}$.

3. Local switching of simple Fushimi trees

Let G be a simple Fushimi tree with standard sign. Take a block B of G. In the present paper, G is said to be $(n+k, k)$- *type with respect to* B if the order of B is $n+k$ and the number of cut vertices in B is k. Let a be a cut vertex in a block B. $G/\{a\}$ has the two connected components. One is the component conntaining $B/\{a\}$. We call the other the *branch with respect to* (B, a), All such components are called *branches with respect to B*. A block of a simple Fushimi tree is said to be *pendant* if it has only one cut vertex. We call a simple Fushimi tree *line-like* if it has only one block or it has exactly two pendant blocks. A branch B_a with respect to (B, a) is called *line-like* if $B_a \cup \{a\}$ is line-like. A simple Fushimi tree is said to be a *line Fushimi tree* if the order of its every block is 2.

Lemma 2. *Let G be a simple Fushimai tree of $(n+k, k)$-type with respect to a block B. We can transform G into a simple Fushimai tree of (k, k)-type, by a sequence of local switchings.*

Proof. Let the block B consist of vertices $a_1, a_2, \cdots, a_n, a_{n+1}, \cdots, a_{n+k}$, where $a_{n+1}, \cdots, a_{n+k}$ are cut vertices. Set $J = \{a_{n+1}\}$ and $K = \{a_1, a_2, \cdots, a_{n-1}, a_{n+2}, \cdots, a_{n+k}\}$. By local switching with respect to (a_n, J), we obtain a simple Fushimai tree of $(n+k-1, k)$-type with respect to block $\{a_1, a_2, \cdots, a_n, a_{n+2}, \cdots, a_{n+k}\}$, where $a_n, a_{n+2}, \cdots, a_{n+k}$ are cut vertecis. Denote by G_1 this simple Fushimai tree of $(n+k-1, k)$-type. Applying the same procedure to G_1, we get a simple Fushimai tree of $(n+k-2, k)$-type. Repeating the same method, we get a simple Fushimai tree of (k, k)-type at last.

Lemma 3. *Let G be a simple Fushimai tree of (k, k)-type with respect to a block B. Assume that all branches with respect to B are line-like. Then, we can transform G into a simple Fushimai tree of $(k+n, k-1)$-type with respect to some block B whose all branches are line-like, by a sequence of local switchings, where n is some positive integer.*

Proof. Let the block B consist of vertices $a_1, a_2, \cdots, a_k$. Take a branch, for example, the branch B_1 with respect to (B, a_1). Assume the order of B_1 is n. Suppose $B_1 \cup \{a_1\}$ has m blocks. Firstly, assume $m = 1$. Take $i = a_1, J = B_1, K = \{a_2, \cdots, a_k\}$. Then by local swithing with resect to (a_1, B_1), G is transformed to a simple Fushimai tree of $(k+n, k-1)$-type with respect to the block $\{a_2, a_3, \cdots, a_k\}$, whose branches with respect to this block are all line-like. Suppose that the result is true for m. Assume that $B_1 \cup \{a_1\}$ has $m+1$ blocks. Let C be the pendant block of $B_1 \cup \{a_1\}$ and c be its cut vertex. Put $G_1 = \{G \backslash C\} \cup \{c\}$. Then G_1 is a simple Fushimai tree of (k, k)-type with respect to a block B. By the inductive hyposesis, G_1 is transformed into a simple Fushimai tree G_2 of $(k+n_1, k-1)$-type with respect to the block $\{a_2, a_3, \cdots, a_k\}$, whose branches with respect to this block are all line-like, by a sequence of local switchings, where $n_1 = n+1-n_2$ and n_2 is the order of the block C. By the same way, we can transform G into $G_2 \cup C$, which is a special Fushimai tree of $(k+n_1, k)$-type with respect to the block $\{c, a_2, a_3, \cdots, a_k\}$, whose branches with respect to this block are all line-like and can be transform into a simple Fushimai tree of $(k+n, k-1)$-type with respect to the block $\{a_2, a_3, \cdots, a_k\}$, whose branches with respect to this block are all line-like, by local switching.

Lemma 4. *Let B be a a imple Fushimai tree with one block. Then it can be transformed into a line Fushimai tree, by a sequence of local switchings.*

Proof. Let B consist of vertices $a_1, a_2, \cdots, a_k$. Set $J = \{a_1\}$ and $K = \{a_3, a_4, \cdots, a_k\}$. By local switching with respect to (a_2, J), we obtain a simple Fushimai tree of $(k-1, 1)$-type with respect to block $\{a_2, \cdots, a_k\}$. Next, set $J = \{a_2\}$ and $K = \{a_4, \cdots, a_k\}$. By local switching with respect to (a_3, J), we obtain a simple Fushimai tree of $(k-2, 1)$-type with respect to block $\{a_3, \cdots, a_k\}$. By this way, we can get a line Fushimi tree , by a sequence of local switchings.

Lemma 5. *Let G be a simple Fushimai tree of $(n+k, k)$-type with respect to a block B. Assume that all branches with respect to B are line-like. Then it is transformed to a line Fushimai tree, by a sequence of local switchings. Especialy, a line-like special Fushimai tree is transformed to a line Fushimai tree, by a sequence of local switchings.*

Proof. By the same way in the proof of lemma 2, G can be transformed into a simple Fushimai tree G_1 of (k, k)-type with respect to some block B_1, by a sequence of local switchings, whose branches with respect to B_1 are all line-like. By lemma 3, we can get a simple Fushimai tree of $(k+n, k-1)$-type with some block B_2 whose branches are all line-like, by a sequence of local switchings, where n is some positive integer. By a sequence of this process, we obtain a simple Fushimai tree of $(k+N, 0)$, where N is some positive integer, that is, a simple Fushimai tree with a one block, whitch is also transformed int a line Fushima tree by lemma 4.

Lemma 6. *Let G be a simple Fushimai tree. Then it has at least two pendant blocks.*

Proof. If every block has more than one cut vertex, then, as we have no cycle, the order of the graph is infinite. Hence, t has at least two pendant blocks.

Theorem 7. *Let G be a simple Fushimai tree. We can transform G into a line Fushimai tree, by a sequence of local switchings.*

Proof. Asuume G has m blocks. If $m = 1$, we get the result by Lemma 4. Suppose the result is true for $m = k$. Let $m = k + 1$. Take a pendannt block B_1 of G with cut vertex b. Let B_2 be the other block with cut vertex b. Put $i = b, J = B_1 \backslash b, K = B_2 B_1 \backslash b$. By local swiching with respect to (b, J), we obtain a simple Fushimai tree with k blocks, which can be can transformed into a line Fushimai tree, by a sequence of local switchings.

References

[1] P. J. Cameron, J. M. Goethals, J. J. Seidel and E. E. Shult Line graphs, root systems, and Elliptic geometry, J. Algebra. **43**, 305-327,1976.

[2] D. M. Cvetkovic, M. Doob , I. Gutman and A. Torgasev, Recent results

in the theory of graph spectra, Annals of Discrete Mathematics **36**, North-Holand, Amsterdam, 1991.

[3] P. J. Cameron, J. J. Seidel and S. V. Tsaranov, Signed Graphs, Root lattices, and Coxeter groups, J. Algebra. **164**, 173-209,1994.

[4] J. E. Humphreys, Reflectioon group and Coxeter groups, Cambridge Staudies in Advanced Mathematics **28**, Cambridge, 1989.