J. Math. Univ. Tokushima
Vol.39 (2005), 23-46

A Geometrical Formulation for Classical Mechanics in Gauge Fields*

By

RUISHI KUWABARA

Faculty of Integrated Arts and Sciences,
The University of Tokushima,
Minami-Josanjima, Tokushima 770-8502, JAPAN
e-mail: kuwabara@ias.tokushima-u.ac.jp

(Received September 30, 2005)

Abstract

We give a geometrical formulation for the classical mechanics in a non-abelian gauge field on a Riemannian manifold. The formulation is based on the reduction procedure associated to the non-abelian symmetry in the principal bundle which describes the gauge field. In the formulation we present explicitly the equation of the motion (called Wong's equation) of a charged particle by using a local coordinate system.

2000 Mathematics Subject Classification. 53D20

Introduction

Let (M, m) be an n dimensional smooth Riemannian manifold without boundary, and let $\pi : P \to M$ be a principal G-bundle, where G is a compact Lie group with $\mathfrak{g}$ its Lie algebra. Suppose P is endowed with a connection $\widetilde{\nabla}$. Take an open covering $\{U_\alpha\}$ of M with $\{\varphi_{\alpha\beta}\}$ being the transition functions of P. Then the curvature of $\widetilde{\nabla}$ is regarded as a family of $\mathfrak{g}$-valued two forms $\bar{\Theta}_\alpha$ defined on U_α which satisfies

$$\bar{\Theta}_\beta = \mathrm{Ad}(\varphi_{\alpha\beta}^{-1})\bar{\Theta}_\alpha \tag{0.1}$$

on $U_\alpha \cap U_\beta (\neq \phi)$, where $\mathrm{Ad}(\cdot)$ denotes the adjoint action of G on $\mathfrak{g}$. Such a family of $\mathfrak{g}$-valued two forms $\{\bar{\Theta}_\alpha\}$ on M satisfying (0.1) is called a *gauge field.* When G is the abelian group $U(1)$, $\bar{\Theta}_\alpha = \bar{\Theta}_\beta$ holds, and accordingly we have a two-form $\bar{\Theta}$ globally defined on M, which is called a *magnetic field.*

In the previous papers [10], [11], [12] we have considered the case where $G = U(1)$, namely the classical and the quantum mechanics in magnetic fields, and clarified some relations between the classical orbits and the energy levels of the Schrödinger operator. In those papers the geometrical formulation for the magnetic dynamical system based on the reduction procedure associated

*This research is partially supported by Grant-in-Aid for Scientific Research (C) (No.14540210) of JSPS.

to the $U(1)$-symmetry of the system plays key role in the investigations. In the present article we generalize the formulation for the magnetic systems to the case of a non-abelian compact Lie group G, which describe the motion of a (classical) particle in a non-abelian gauge field.

Various mathematical formulations of the equation of classical motion of a particle in the (non-abelian) gauge field (or the Yang-Mills field) have been presented by Kerner [5], Guillemin-Sternberg [3], [4], Kummer [9] and so on. (See also Montgomery [14].) We give in this article a slightly different formulation based on the reduction procedure for the symplectic G-action on the cotangent bundle T^*P, and present the equation of the motion (called Wong's equation) explicitly by using a local coordinate system.

The organization of the article is as follows. In §1 we introduce the so-called Kaluza-Klein metric on P associated to the connection $\widetilde{\nabla}$, and the metrics on M and G. Thus we get the Hamiltonian system $(T^*P, \Omega_P, \widetilde{H})$ of geodesic flow on T^*P. Then in §2 because of the G-invariance the system $(T^*P, \Omega_P, \widetilde{H})$ is reduced to $(P_\mu, \Omega_\mu, H_\mu)$ according to the Marsden-Weinstein reduction procedure (see [13], [1]). We clarify in §3 the reduced phase space P_μ is the fiber bundle over T^*M with the fiber being a co-adjoint orbit through μ in $\mathfrak{g}^*$. In §4 we show that the reduced system $(P_\mu, \Omega_\mu, H_\mu)$ is realized as the subsystem of $(T^*M_\mu, \widetilde{\Omega}_\mu, \widetilde{H}_\mu)$, where the manifold M_μ is a union of the spaces of (external) configurations and of internal degrees of freedom, and the symplectic structure $\widetilde{\Omega}_\mu$ is derived explicitly from the connection form (or the gauge potential) of $\widetilde{\nabla}$. Section 5 gives a explicit expression of the flow or the equation of motion (called Wong's equation) in the system $(T^*M_\mu, \widetilde{\Omega}_\mu, \widetilde{H}_\mu)$ by using local coordinates. Finally in §6 we consider the Hopf bundle over the quaternionic projective space, which is a typical example in non-abelian gauge theory.

1 Kaluza-Klein metric on the principal bundle

Let $\pi : P \to M$ be a principal G-bundle over an n-dimensional Riemannian manifold (M, m) without boundary, where G is an r-dimensional compact Lie group. Suppose P is endowed with a connection $\widetilde{\nabla}$, i.e., the direct decomposition of each tangent space T_uP $(u \in P)$ as

$$T_uP = H_u \oplus V_u, \tag{1.1}$$

where V_u is tangent to the fiber, and H_u is linearly isomorphic with $T_{\pi(u)}M$ through $\pi_*|_{H_u}$ and satisfies

$$H_{u\cdot g} = R_{g*}(H_u) \tag{1.2}$$

for the right action R_g of $g \in G$ on P (cf. [8]). Note that the tangent space V_u to the fiber is linearly isomorphic with $\mathfrak{g}$ by the correspondence $\mathfrak{g} \ni A \mapsto$

$A_u^P := \frac{d}{dt}(u \cdot \exp tA)|_{t=0} \in V_u$. Let us take an inner product $(\,,\,)_{\mathfrak{g}}$ on $\mathfrak{g} \cong T_eG$ (e : the identity of G) which is invariant under the adjoint action of G (such inner product induces a right- and left-invariant metric on G). Then, $(\,,\,)_{\mathfrak{g}}$ induces the inner product $(\,,\,)_{V,u}$ on V_u ($u \in P$) as $(A^P, B^P)_{V,u} = (A, B)_{\mathfrak{g}}$ ($A, B \in \mathfrak{g}$). On the other hand, we have the inner product $(\,,\,)_{H,u}$ on H_u from the metric m on M such that $\pi_*|_{H_u}$ is an isometry. Finally, we define an inner product $\widetilde{m}$ in each T_uP ($u \in P$) by defining H_u and V_u to be orthogonal each other. The metric $\widetilde{m}$ on P (which is induced from the metric m on M, the Ad-invariant metric on $\mathfrak{g}$, and the connection (1.1)) is called the *Kaluza-Klein metric* (cf. [5]). Note that $\widetilde{m}$ is invariant under the G-action on P because of the Ad-invariance of the metric of $\mathfrak{g}$ and the property (1.2).

Let $\Omega_P = d\omega_P$ be the standard symplectic structure on the cotangent bundleT^*P of P, where ω_P is called the canonical one form on T^*P. We have the natural Hamiltonian function $\widetilde{H}$ on T^*P defined by the Kaluza-Klein metric $\widetilde{m}$. Thus, we have the Hamiltonian system $(T^*P, \Omega_P, \widetilde{H})$, which is just the system of geodesic flow on T^*P generated by the Hamiltonian vector field $X_{\widetilde{H}}$ induced from $\widetilde{H}$, i.e., $i(X_{\widetilde{H}})\Omega_P = -d\widetilde{H}$, where $i(X_{\widetilde{H}})\Omega_P$ stands for the interior product of $X_{\widetilde{H}}$ and Ω_P.

2 The momentum map and the reduction of the system

The action $p \mapsto p \cdot g = R_g(p)$ ($p \in P$, $g \in G$) of G on P is naturally lifted to the action $R^*_{g^{-1}} := (R_{g^{-1}})^*$ on T^*P (so that $R^*_{g^{-1}} : T^*_pP \to T^*_{p\cdot g}P$ for each $p \in P$), and the action $R^*_{g^{-1}}$ preserves ω_P (and accordingly Ω_P), i.e., $R^*_{g^{-1}}\omega_P = \omega_P$ holds for every $g \in G$. (We call such action a *symplectic action.*) Moreover, we notice that the Hamiltonian $\widetilde{H}$ is also invariant under the action $R^*_{g^{-1}}$.

A momentum map for the symplectic G-action is a map $J : T^*P \to \mathfrak{g}^*$ satisfying

$$\langle J(p), A\rangle = \langle p_u, A_u^P\rangle = i(A_p^{T^*P})\omega_P \quad (p \in T^*P,\ p_u \in T_u^*P\ (u \in P)), \tag{2.1}$$

for all $A \in \mathfrak{g}$, where $A_p^{T^*P} := \frac{d}{dt}(R^*_{g(t)^{-1}}(p))\big|_{t=0}$ with $g(t) = \exp tA$.

Lemma 2.1 *(1) The momentum map J is surjective onto $\mathfrak{g}^*$, and every $\mu \in \mathfrak{g}^*$ is a regular value of J.*

(2) The momentum map J is Ad^*-equivariant, i.e.,

$$J \circ R^*_{g^{-1}} = \mathrm{Ad}^*(g^{-1}) \circ J \tag{2.2}$$

holds for $g \in G$. Here we define $\mathrm{Ad}^*(g) := (\mathrm{Ad}(g^{-1}))^*$ (the adjoint of $\mathrm{Ad}(g^{-1}) : \mathfrak{g} \to \mathfrak{g}$).

(3) The momentum map J is invariant under the geodesic flow on T^*P, i.e., $X_{\widetilde{H}}J = 0$ holds.

Proof. (1) Note the definition (2.1) of J, we can easily derive the assertion from the fact that the map $\mathfrak{g} \ni A \mapsto A_u^P \in T_u P$ is surjective.

(2) For $p \in T_u^* P$, $A \in \mathfrak{g}$ we have

$$\langle J(R_{g^{-1}}^*(p)), A\rangle = \langle R_{g^{-1}}^*(p_u), A_{u\cdot g}^P\rangle = \langle p_u, R_{g^{-1}*}(A_{u\cdot g}^P)\rangle.$$

Since

$$\begin{aligned} R_{g^{-1}*}(A_{u\cdot g}^P) &= \frac{d}{dt}(u \cdot g \exp(tA) g^{-1})|_{t=0} \\ &= \frac{d}{dt}(u \cdot \exp(t\mathrm{Ad}(g)A))|_{t=0} = [\mathrm{Ad}(g)A]_u^P, \end{aligned}$$

we have

$$\langle J(R_{g^{-1}}^*(p)), A\rangle = \langle p_u, [\mathrm{Ad}(g)A]_u^P\rangle = \langle J(p), \mathrm{Ad}(g)A\rangle = \langle \mathrm{Ad}^*(g^{-1})J(p), A\rangle.$$

(3) For $A \in \mathfrak{g}$ define the function J_A on T^*P as $J_A(p) = \langle J(p), A\rangle = i(A^{T^*P})\omega_P$. We show $X_{\widetilde{H}} J_A = 0$. First note that the Lie derivative $\mathcal{L}_{A^{T^*P}}\omega_P = d(i(A^{T^*P})\omega_P) + i(A^{T^*P})d\omega_P = 0$ because G-action on T^*P preserves ω_P. We have

$$\begin{aligned} X_{\widetilde{H}} J_A &= X_{\widetilde{H}}(i(A^{T^*P})\omega_P) = \langle d(i(A^{T^*P})\omega_P), X_{\widetilde{H}}\rangle \\ &= -\langle i(A^{T^*P})d\omega_P, X_{\widetilde{H}}\rangle = \langle i(X_{\widetilde{H}})d\omega_P, A^{T^*P}\rangle \\ &= -\langle d\widetilde{H}, A^{T^*P}\rangle = 0. \end{aligned}$$

Here note that G-action also preserves $\widetilde{H}$. □

Now, we apply the reduction procedure associated to the momentum map J by Marsden and Weinstein. For $\mu \in \mathfrak{g}^*$, it follows from Lemma 2.1 that $J^{-1}(\mu)$ is a $(2n+r)$-dimensional submanifold of T^*P, which is invariant under the geodesic flow. Let $G_\mu := \{g \in G | \mathrm{Ad}^*(g)\mu = \mu\} \subset G$, which is a closed subgroup of G. Then, by virtue of Lemma 2.1,(2) G_μ preserves the submanifold $J^{-1}(\mu)$, and the action of G_μ on $J^{-1}(\mu)$ is free. Hence the quotient set $P_\mu := J^{-1}(\mu)/G_\mu$ is a smooth manifold, and the natural projection

$$\pi_\mu : J^{-1}(\mu) \to P_\mu$$

is a submersion. In this situation we have the following.

Proposition 2.2 *The quotient manifold P_μ has a uniquely defined symplectic form Ω_μ with*

$$\pi_\mu^*\Omega_\mu = i_\mu^*\Omega_P,$$

where $i_\mu : J^{-1}(\mu) \hookrightarrow T^*P$ is the inclusion.

Sketch of the proof. Note that the following facts: For $p \in T^*P$,

(i) $T_p(J^{-1}(\mu)) = \mathrm{Ker}(J_*) \cong \{X \in T_p(T^*P) |\, \Omega_P(X, A_p^{T^*P}) = 0 \text{ for } \forall A \in \mathfrak{g}\}$, and

(ii) $T_p(J^{-1}(\mu))/T_p(G_\mu \cdot p) \cong T_{\pi_\mu(p)}(P_\mu)$.

For $X \in T_p(J^{-1}(\mu))$, let $[X] = \pi_{\mu *}(X)$ be the associated equivalence class in $T_p(J^{-1}(\mu))/T_p(G_\mu \cdot p)$. Let us define Ω_μ on P_μ as

$$\Omega_\mu([X],[Y]) := \Omega_P(X,Y) \qquad (\, X, Y \in T_p(J^{-1}(\mu))\,).$$

Here we can easily check that Ω_μ is well-defined by the above fact (i) and that Ω_P is invariant under the G_μ-action. Moreover, we can check that Ω_μ is closed and non-degenerate. These properties are guaranteed by those of Ω_P. □

The Hamiltonian $\widetilde{H}$ is G_μ-invariant, and accordingly induces the (Hamiltonian) function H_μ on P_μ. Thus, we have a reduced Hamiltonian system $(P_\mu, \Omega_\mu, H_\mu)$, where $\dim P_\mu = 2n + r - r_\mu$ ($r_\mu := \dim G_\mu$).

Let $\mathcal{O}_\mu$ be the coadjoint orbit of μ in $\mathfrak{g}^*$, i.e.,

$$\mathcal{O}_\mu := \{\mathrm{Ad}^*(g)\mu \mid g \in G\},$$

which is diffeomorphic with G/G_μ. Then, we have the following.

Proposition 2.3 *Suppose ν is an element of $\mathcal{O}_\mu$. Then, the reduced Hamiltonian system $(P_\nu, \Omega_\nu, H_\nu)$ associated to $\nu \in \mathfrak{g}^*$ is isomorphic with $(P_\mu, \Omega_\mu, H_\mu)$.*

Proof. Suppose $\nu = \mathrm{Ad}^*(g)\mu$ for $g \in G$. Note that $G_\mu \cong G_\nu$ by the isomorphism $G_\mu \ni h \mapsto ghg^{-1} \in G_\nu$. By virtue of Lemma 2.1,(2), we have the map $R_g^* : J^{-1}(\mu) \to J^{-1}(\nu)$. Then, we can easily see that R_g^* induces the isomorphism of $(P_\mu, \Omega_\mu, H_\mu)$ onto $(P_\nu, \Omega_\nu, H_\nu)$. □

3 Geometrical structure of the reduced space

We can define the surjective map $\Phi : T^*P \to T^*M$ associated to the connection $\widetilde{\nabla}$ on P as follows. Let p be a point in T^*P with $\pi_P(p) = u \in P$, $\pi(u) = x \in M$, where $\pi_P : T^*P \to P$ is a natural projection. For a tangent vector $X \in T_xM$, let $X_u^\#$ be the horizontal lift of X relative to the connection $\widetilde{\nabla}$, i.e., $X^\#$ belongs to H_u in (1.1) and $\pi_*(X_u^\#) = X$. We define $\Phi(p) \in T_x^*M$ as

$$\langle \Phi(p), X\rangle := \langle p, X_u^\#\rangle \qquad (X \in T_xM).$$

Concerning the horizontal lifts we have $X_{u\cdot g}^\# = R_{g*}(X_u^\#)$, and accordingly see that Φ is G-invariant, i.e., $\Phi(R_{g^{-1}}^*(p)) = \Phi(p)$ as follows:

$$\langle \Phi(R_{g^{-1}}^*(p)), X\rangle = \langle R_{g^{-1}}^*(p), X_{u\cdot g}^\#\rangle = \langle p, X_u^\#\rangle = \langle \Phi(p), X\rangle.$$

By virtue of the G-invariance (hence, the G_μ-invariance) of Φ we have the surjective map $\Phi_\mu : P_\mu \to T^*M$ induced from Φ. The purpose of this subsection is to show the following.

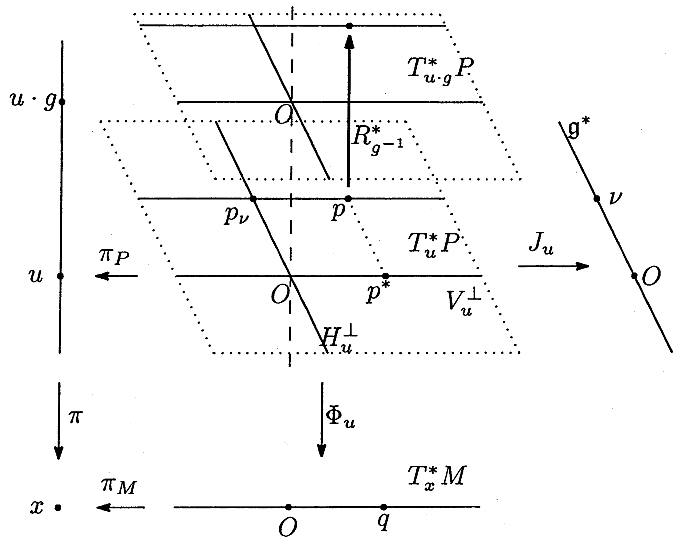

Figure 1: Reduced space

Proposition 3.1 *$\Phi_\mu : P_\mu \to T^*M$ is a fiber space with the fiber being the coadjoint orbit $\mathcal{O}_\mu$ of μ in $\mathfrak{g}^*$.*

We have the decomposition of T_u^*P associated to (1.1):

$$T_u^*P = H_u^\perp \oplus V_u^\perp,$$

where

$$\begin{aligned} H_u^\perp &:= \{p \in T_u^*P | \langle p, X\rangle = 0 \text{ for } \forall X \in H_u\}, \\ V_u^\perp &:= \{p \in T_u^*P | \langle p, A^P\rangle = 0 \text{ for } \forall A^P \in V_u\}. \end{aligned}$$

Note that $\dim H_u^\perp = r$ and $\dim V_u^\perp = n$. The following lemma is easily obtained.

Lemma 3.2 *(1) The map $J_u := J|_{H_u^\perp} : H_u^\perp \to \mathfrak{g}^*$ is a linear isomorphism.*

(2) For $\nu \in \mathfrak{g}^*$ let $p_\nu = J_u^{-1}(\nu) \in H_u^\perp$. Then,

$$J^{-1}(\nu) \cap T_u^*P = p_\nu + V_u^\perp := \{p_\nu + p^* \mid p^* \in V_u^\perp\}.$$

(3) The map $\Phi_{u,\nu} := \Phi|_{p_\nu + V_u^\perp} : J^{-1}(\nu) \cap T_u^*P \to T_x^*M$ is a bijection with $\Phi_{u,\nu}(p_\nu) = 0$.

Proof. (1)It is obvious that J_u is linear and surjective. Suppose $J(p) = J(p')$ for $p,\ p' \in H_u^{\perp}$. Then, $\langle p, A_u^P\rangle = \langle p', A_u^P\rangle$ for $\forall A \in \mathfrak{g}$. On the other hand, $\langle p, X\rangle = \langle p', X\rangle = 0$ for $\forall X \in H_u$. Hence, $\langle p, V\rangle = \langle p', V\rangle$ for $\forall V \in T_uP$, and accordingly $p = p'$

(2) is obvious.

(3) The surjectivity is obvious. Suppose $\Phi(p) = \Phi(p')$ for $p,\ p' \in J^{-1}(\nu) \cap T_u^*P$. Then, $\langle p, X_u^{\#}\rangle = \langle p', X_u^{\#}\rangle$ for $\forall X \in T_xM$. On the other hand, $\langle p, A_u^P\rangle = \langle p', A_u^P\rangle = \langle \nu, A\rangle$ for $\forall A \in \mathfrak{g}$. Hence, $\langle p, V\rangle = \langle p', V\rangle$ for $\forall V \in T_uP$, and accordingly $p = p'$ □

Let u be a point in P with $\pi(u) = x \in M$. We define the map ψ_u of $T_x^*M \times \mathcal{O}_\mu$ to $J^{-1}(\mu)$ as

$$T_x^*M \times \mathcal{O}_\mu \ni (q, \nu) \mapsto R_{g^{-1}}^*(\Phi_{u,0}^{-1}(q) + J_u^{-1}(\nu)) \in T_{u\cdot g}^*P \cap J^{-1}(\mu),$$

where g is an element of G satisfying $\nu = \mathrm{Ad}^*(g)\mu$. Here we can check by noticing (2.2) and $\Phi_{u,0}^{-1}(q) \in V_u^{\perp}$ that the image of Ψ_u belongs to $J^{-1}(\mu)$ as follows:

$$J(R_{g^{-1}}^*(\Phi_{u,0}^{-1}(q) + J_u^{-1}(\nu))) = J(R_{g^{-1}}^*(J_u^{-1}(\nu))) = \mathrm{Ad}^*(g^{-1})\nu = \mu.$$

If $\nu = \mathrm{Ad}^*(g)\mu = \mathrm{Ad}^*(g')\mu$ for $g \neq g'$, then $g' = gh$ for some $h \in G_\mu$. Therefore, we obtain the bijective map

$$\Psi_u : T_x^*M \times \mathcal{O}_\mu \to \Big[\big(\bigcup_{g\in G} T_{u\cdot g}^*P\big) \cap J^{-1}(\mu)\Big] \Big/ G_\mu$$

from ψ_u. It is easily see that Ψ_u is bijective and satisfies $\Phi_\mu \circ \Psi_u(q,\nu) = q$. Thus it is shown that $\Phi_\mu : P_\mu \to T^*M$ is a fiber space with the fiber being the coadjoint orbit $\mathcal{O}_\mu$. Moreover, by taking a local section $u = u(x)$ $(x \in U \subset M)$ of P we have a local triviality of P_μ:

$$\Psi_u : T^*U \times \mathcal{O}_\mu \xrightarrow{\cong} \Phi_\mu^{-1}(T^*U).$$

In particular, we have a local section $s(q)$ $(q \in T^*U)$ of the fiber space $\Phi_\mu : P_\mu \to T^*M$ by

$$s_u(q) := \Psi_u(q, \mu) = [\Phi_{u,0}^{-1}(q) + J_u^{-1}(\mu)]. \tag{3.1}$$

associated to a local section $u(x)$ $(x \in U)$ of P. Thus we complete the proof of Proposition 3.1. □

4 Dynamical structure of the reduced system

Let θ be the connection form on P of the connection $\widetilde{\nabla}$. The connection form θ is a $\mathfrak{g}$-valued one form satisfying (i) $\theta(A^P) = A$ for $\forall A \in \mathfrak{g}$, and (ii)

$R_g^*\theta = \mathrm{Ad}(g^{-1})\theta$ for $\forall g \in G$. For $\mu \in \mathfrak{g}^*$ we define the $\mathbb{R}$-valued one form θ_μ on P by

$$\theta_\mu(X) := \langle \mu, \theta(X) \rangle \qquad (X \in T_uP).$$

It is easy to see that θ_μ is G_μ-invariant, i.e., $R_g^*\theta_\mu = \theta_\mu$ for $\forall g \in G_\mu$. Let $\mathfrak{g}_\mu$ be the Lie algebra of G_μ. Then we have

Lemma 4.1 *$d\theta_\mu(A^P, X) = 0$ holds for any $A \in \mathfrak{g}_\mu$ and $X \in T_uP$.*

Proof. We have

$$d\theta_\mu(A^P, X) = (i(A^P)d\theta_\mu)(X) = (\mathcal{L}_{A^P}\theta_\mu)(X) - d(i(A^P)\theta_\mu)(X) = 0$$

because θ_μ is G_μ-invariant and $i(A^P)\theta_\mu = \theta_\mu(A^P) = \langle \mu, A \rangle = \text{constant}$. □

Let $M_\mu := P/G_\mu$ be the quotient manifold by the G_μ-action on P. By noticing the above lemma the two form $d\theta_\mu$ can be regarded as that on M_μ as

$$d\theta_\mu([X],[Y]) := d\theta_\mu(X, Y) \qquad (X, Y \in T_uP).$$

Now, we consider the cotangent bundle T^*M_μ with the twisted symplectic form

$$\widetilde{\Omega}_\mu := \Omega_{M_\mu} + \pi_{M_\mu}^*(d\theta_\mu), \tag{4.1}$$

where Ω_{M_μ} is the canonical symplectic two form on T^*M_μ and $\pi_{M_\mu} : T^*M_\mu \to M_\mu$ is the projection.

Proposition 4.2 *([1, Theorem 4.3.3], [9, Theorem 3]) There exists a symplectic embedding*

$$\chi_\mu : (P_\mu, \Omega_\mu) \hookrightarrow (T^*M_\mu, \widetilde{\Omega}_\mu),$$

that is, χ_μ is an embedding satisfying $\chi_\mu^*\widetilde{\Omega}_\mu = \Omega_\mu$.

Proof. For each $u \in P$ let

$$(V_\mu)_u^\perp := \{p \in T_u^*P \mid \langle p, A_u^P \rangle = 0 \text{ for } \forall A \in \mathfrak{g}_\mu\} \ (\subset T_u^*P),$$

which can be identified with $T_q^*M_\mu$ ($q = \pi'(u) \in M_\mu$ for the projection $\pi' : P \to M_\mu$) because we have the linear isomorphism

$$R_{g^{-1}}^* : (V_\mu)_u^\perp \to (V_\mu)_{u\cdot g}^\perp$$

for each $g \in G_\mu$. Thus we have $T^*M_\mu \cong V_\mu^\perp / G_\mu$.

Take $p \in T^*P$ such that $\pi_P(p) = u \in P$, i.e., $p \in T_u^*P$. We define $\bar{\chi}_\mu(p) = p_u - (\theta_\mu)_u \in T_u^*P$. Then we can easily see that (i) $\bar{\chi}_\mu(p)$ belongs to $V_u^\perp$ (and accordingly to $(V_\mu)_u^\perp$) if $p_\mu \in J^{-1}(\mu)$, and (ii) $\bar{\chi}_\mu(R_{g^{-1}}^*(p)) = R_{g^{-1}}^*(\bar{\chi}_\mu(p))$ for $g \in G_\mu$, i.e., $\bar{\chi}_\mu$ is G_μ-equivariant. In fact, (i) is shown as $\langle p_u, A_u^P \rangle - \langle (\theta_\mu)_u, A_u^P \rangle = \langle J(p), A \rangle - \langle \mu, A \rangle = 0$. As (ii) we have only to see the equality

$(\theta_\mu)_{u\cdot g} = R^*_{g^{-1}}((\theta_\mu)_u)$, that is shown as $(\theta_\mu)_{u\cdot g}(X) = R^*_{g^{-1}}((\theta_\mu)_u)(X) = 0$ for $\forall X \in H_{u\cdot g}$ and $(\theta_\mu)_{u\cdot g}(A^P_{u\cdot g}) = R^*_{g^{-1}}((\theta_\mu)_u)(A^P_{u\cdot g}) = \langle \mu, A\rangle$ for $\forall A \in \mathfrak{g}$. As a result of (i) and (ii) $\bar{\chi}_\mu$ induces the map $\chi_\mu : P_\mu(= J^{-1}(\mu)/G_\mu) \to T^*M_\mu(= V^\perp_\mu/G_\mu)$. It is obvious that χ_μ is an injection.

Now, we will show that $\chi^*_\mu \widetilde{\Omega}_\mu = \Omega_\mu$. Let X be a vector in $T_p(T^*P)$ ($p \in T^*P, \pi_P(p) = u$). Then, X is expressed as

$$X = \bar{X} + X^* \quad \text{with } \bar{X} \in T_uP,\ X^* \in T^*_uP(= T_p(T^*_uP)).$$

Here X^* belongs to $V^\perp_u$ if $X \in T_pJ^{-1}(\mu)$ (Lemma 3.2,(2)). For two vector fields $X = X(p)$, $Y = Y(p)$ on a neighborhood of p_0 in $J^{-1}(\mu)$ we have

$$\begin{aligned}\Omega_P(X,Y) &= \frac{1}{2}\{X\langle\omega_P, Y\rangle - Y\langle\omega_P, X\rangle - \langle\omega_P, [X,Y]\rangle\}\\ &= \frac{1}{2}\{X\langle p, \bar{Y}\rangle - Y\langle p, \bar{X}\rangle - \langle p, \overline{[X,Y]}\rangle\}.\end{aligned}$$

Put $p'(= \bar{\chi}_\mu(p)) = p - \theta_\mu$, and we have

$$\begin{aligned}\Omega_P(X,Y) &= \frac{1}{2}\{X\langle p', \bar{Y}\rangle - Y\langle p', \bar{X}\rangle - \langle p', \overline{[X,Y]}\rangle\}\\ &+ \frac{1}{2}\{\bar{X}\langle\theta_\mu, \bar{Y}\rangle - \bar{Y}\langle\theta_\mu, \bar{X}\rangle - \langle\theta_\mu, \overline{[X,Y]}\rangle\}.\end{aligned}$$

Noticing $V^\perp_u \subset (V_\mu)^\perp_u$ we can regard $X = \bar{X} + X^*$ as a vector in $T_{p'}(T^*M_\mu)$, and see that the first term is nothing but $\Omega_{M_\mu}(\chi_{\mu*}([X]), \chi_{\mu*}([Y]))$. By noticing $\overline{[X,Y]} = [\bar{X}, \bar{Y}]$ we see that the second term is just
$d\theta_\mu((\pi_{M_\mu}\circ\chi_\mu)_*([X]), (\pi_{M_\mu}\circ\chi_\mu)_*([Y]))$. □

Next, we define the Riemannian metric m_μ on M_μ as follows. Put $(V_\mu)_u := T_u(G_\mu \cdot u)$ for $u \in P$, which is a subspace of V_u. Then we have the orthogonal decomposition

$$T_uP = H_u \oplus V_u = H_u \oplus (H_\mu)_u \oplus (V_\mu)_u \tag{4.2}$$

from (1.1) and the G-invariant metric on G, where $(H_\mu)_u$ is the orthogonal compliment of $(V_\mu)_u$ in V_u. By identifying T_uM_μ with $H_u \oplus (H_\mu)_u$ we obtain the metric m_μ on T_uM_μ.

Let $\widetilde{H}_\mu$ be the Hamiltonian function on T^*M_μ naturally induced from the metric m_μ. Then,

Lemma 4.3 $H_\mu = \chi^*_\mu \widetilde{H}_\mu + \|\mu\|^2_{\mathfrak{g}^*}$, *where* $\|\cdot\|_{\mathfrak{g}^*}$ *is the naturally induced norm from the inner product in* $\mathfrak{g}$.

Proof. For $p \in T^*_uP \cap J^{-1}(\mu)$ we have $p = \bar{\chi}_\mu(p) + (\theta_\mu)_u$ with $\bar{\chi}_\mu(p) \in V^\perp_u$ and $(\theta_\mu)_u \in H^\perp_u$. Since $V^\perp_u$ and $H^\perp_u$ are orthogonal each other, we have

$$H_\mu([p]) = \|\bar{\chi}_\mu(p)\|^2 + \|(\theta_\mu)_u\|^2 = \widetilde{H}_\mu(\chi_\mu([p])) + \|(\theta_\mu)_u\|^2.$$

$$
\begin{tikzcd}
 & & & T^*P \arrow[r, "J"] \arrow[llld, "\pi_P"'] & \mathfrak{g}^* \\
P \arrow[dr] \arrow[dd, "\pi"'] & & & J^{-1}(\mu) \arrow[u, "\text{incl.}"'] \arrow[d, "\pi_\mu"] & \\
 & M_\mu \arrow[dl] & T^*M_\mu \arrow[l, "\pi_{M_\mu}"'] & P_\mu(=J^{-1}(\mu)/G_\mu) \arrow[l, hook', "\chi_\mu"'] \arrow[d, "\Phi_\mu"] & \\
M & & & T^*M \arrow[lll, "\pi_M"'] &
\end{tikzcd}
$$

Figure 2: Reduction procedure

Note that $(\theta_\mu)_u(A_u^P) = \langle \mu, A\rangle$ for $\forall A \in \mathfrak{g}$, and we have $\|(\theta_\mu)_u\| = \|\mu\|_{\mathfrak{g}^*}$. $\square$

As a consequence, we have the following (cf. Figure 2).

Proposition 4.4 *The reduced Hamiltonian system $(P_\mu, \Omega_\mu, H_\mu)$ is regarded as a Hamiltonian subsystem of $(T^*M_\mu, \widetilde{\Omega}_\mu, \widetilde{H}_\mu)$.*

Let X_μ be the Hamiltonian vector field on P_μ associated to H_μ, that is, $i(X_\mu)\Omega_\mu = -dH_\mu$. The flow of X_μ is regarded as embedded Hamiltonian flow in $(T^*M_\mu, \widetilde{\Omega}_\mu, \widetilde{H}_\mu)$, and represents the motion of a classical particle of "charge" μ in the gauge field given by the connection $\widetilde{\nabla}$. The (external) configuration space of the system is the manifold M and the fiber of $M_\mu \to M$ is the space of internal degrees of freedom.

5 Expressions in local coordinate systems

5.1 Basic formulas

Let $\{Y_\alpha\} = \{Y'_\alpha, Y''_\beta\} = \{Y'_1, \ldots, Y'_{r_1}, Y''_{r_1+1}, \ldots, Y''_r\}$ be the orthonormal basis of $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_\mu$ with $\{Y'_\alpha\}$, $\{Y''_\beta\}$ being the basis of $\mathfrak{m}$ and $\mathfrak{g}_\mu$, respectively, where $\mathfrak{g}_\mu$ is the Lie algebra of G_μ. Note that $\dim G_\mu = r - r_1 (= r_\mu)$. We have coordinates $y = (y', y'') = (y'^1, \ldots, y'^{r_1}, y''^{r_1+1}, \ldots, y''^r)$ in a neighborhood V of the identity $(y = 0)$ of G by

$$
\begin{aligned}
y &= (y'^1, \ldots, y'^{r_1}, y''^{r_1+1}, \ldots, y''^r) \\
&\longleftrightarrow \ g = \exp\Big(\sum_{\alpha=1}^{r_1} y'^\alpha Y'_\alpha\Big) \exp\Big(\sum_{\beta=r_1+1}^{r} y''^\beta Y''_\beta\Big).
\end{aligned}
$$

By virtue of the local triviality of the bundle P we take a local coordinate $(x^i, y^\alpha) \in U \times V$ $(U \subset M,\ V \subset G)$ of P. For the basis $\{Y_\alpha\}$ of $\mathfrak{g}$ we have the associated basis $\{Y_\alpha^P\}$ of V_u, which is expressed as

$$Y_\alpha^P(x, y) = \sum_{\beta=1}^{r} \Gamma_\alpha^\beta(y) \frac{\partial}{\partial y^\beta} \tag{5.1}$$

with $\Gamma_\alpha^\beta(0) = \delta_\alpha^\beta$. Let $(x^i, y^\alpha; \xi_i, \eta_\alpha)$ the local canonical coordinates of T^*P. Then, the momentum map J is represented as

$$J(x, y; \xi, \eta) = \sum_{\alpha=1}^{r} \Big(\sum_{\beta=1}^{r} \Gamma_\alpha^\beta(y) \eta_\beta \Big) Y^\alpha \in \mathfrak{g}^*, \tag{5.2}$$

where $\{Y^1, \ldots, Y^r\}$ is the dual basis of $\mathfrak{g}^*$ associated to $\{Y_\alpha\}$.

We remark that $\Gamma(y) := (\Gamma_\alpha^\beta(y))$ in (5.1) is non-singular near $y = 0$ because $\Gamma_\alpha^\beta(0) = \delta_\alpha^\beta$. Let $\Lambda(y) = (\Lambda_\alpha^\gamma(y))$ be the inverse matrix of $\Gamma(y)$. Then, the following is easy to see.

Lemma 5.1 *(1) If $1 \le \alpha \le r_1$, $r_1 + 1 \le \kappa \le r$, then we have*

$$\Gamma_\kappa^\alpha(y) = \Lambda_\kappa^\alpha(y) = 0, \tag{5.3}$$

i.e.,

$$\Gamma(y) = \left[\begin{array}{c|c} \Gamma_1(y) & O \\ \hline \Gamma_{21}(y) & \Gamma_2(y) \end{array} \right], \quad \Lambda(y) = \left[\begin{array}{c|c} \Lambda_1(y) & O \\ \hline \Lambda_{21}(y) & \Lambda_2(y) \end{array} \right],$$

(2) If $r_1 + 1 \le \kappa \le r$, then we have

$$\Gamma_\kappa^\nu(y', 0) = \Lambda_\kappa^\nu(y', 0) = \delta_\kappa^\nu, \quad i.e., \Gamma_2(y', 0) = \Lambda_2(y', 0) = E. \tag{5.4}$$

and

$$\frac{\partial \Gamma_\kappa^\nu}{\partial y'^\beta}(y', 0) = \frac{\partial \Lambda_\kappa^\nu}{\partial y'^\beta}(y', 0) = 0. \tag{5.5}$$

Let $C_{\alpha\beta}^\kappa$ be the structure constants of $\mathfrak{g}$ with respect to the basis $\{Y_\alpha\} = \{Y'_\beta, Y''_\gamma\}$, i.e., $[Y_\alpha, Y_\beta] = \sum_\kappa C_{\alpha\beta}^\kappa Y_\kappa$. Then, the following formulas concerning the functions $\Gamma_\beta^\alpha(y)$ and $\Lambda_\beta^\alpha(y)$ are derived from the fact that $\{Y_\alpha^P(y)\}$ is a family of left-invariant vector fields on G.

Lemma 5.2

$$\sum_\gamma \Big(\Gamma_\alpha^\gamma \frac{\partial \Gamma_\beta^\kappa}{\partial y^\gamma} - \Gamma_\beta^\gamma \frac{\partial \Gamma_\alpha^\kappa}{\partial y^\gamma} \Big) = \sum_\gamma C_{\alpha\beta}^\gamma \Gamma_\gamma^\kappa, \tag{5.6}$$

$$\frac{\partial \Lambda_\beta^\kappa}{\partial y^\alpha} - \frac{\partial \Lambda_\alpha^\kappa}{\partial y^\beta} = -\sum_{\gamma,\nu} \Lambda_\alpha^\gamma C_{\gamma\nu}^\kappa \Lambda_\beta^\nu. \tag{5.7}$$

Proof. The first formula is obtained from the relation $[Y_\alpha^P, Y_\beta^P] = \sum_\gamma C_{\alpha\beta}^\gamma Y_\gamma^P$. The second is derived from the first by noticing $\Lambda\Gamma = E$. □

5.2 Kaluza-Klein metric and the system of geodesic flow on T^*P

Now let us consider the connection form θ of the connection $\widetilde{\nabla}$. Noticing $\theta(A^P) = A$ for $\forall A \in \mathfrak{g}$, we put

$$\theta = \sum_{\alpha=1}^{r} \Big(\sum_{i=1}^{n} \theta_i^\alpha(x,y) dx^i + \sum_{\beta=1}^{r} \Lambda_\beta^\alpha(y) dy^\beta \Big) \otimes Y_\alpha. \tag{5.8}$$

Note that the property $R_g^* \theta = \mathrm{Ad}(g^{-1})\theta$ ($g \in G$), and we put for $G \ni g \leftrightarrow (y_1, \dots, y_r)$

$$\mathrm{Ad}(g^{-1}) Y_\alpha = \sum_{\beta=1}^{r} A_\alpha^\beta(y) Y_\beta \quad (\alpha = 1, \dots, r). \tag{5.9}$$

Then, we have the following.

Lemma 5.3 *Put $\bar{\theta}_i^\alpha(x) := \theta_i^\alpha(x, 0)$, and we have*

$$\theta_i^\alpha(x,y) = \sum_{\beta=1}^{r} A_\beta^\alpha(y) \bar{\theta}_i^\beta(x) \quad (i = 1, \dots, n;\ \alpha = 1, \dots, r), \tag{5.10}$$

where $A_\beta^\alpha(y)$ satisfies

$$\frac{\partial A_\beta^\alpha}{\partial y^\gamma} = - \sum_{\kappa,\nu} \Lambda_\gamma^\nu C_{\nu\kappa}^\alpha A_\beta^\kappa \quad \textit{with} \quad A_\beta^\alpha(0) = \delta_\beta^\alpha, \tag{5.11}$$

and accordingly

$$\frac{\partial \theta_i^\alpha}{\partial y^\gamma} = - \sum_{\beta,\kappa} \Lambda_\gamma^\beta C_{\beta\kappa}^\alpha \theta_i^\kappa. \tag{5.12}$$

Proof. By virtue of the property:

$$\theta_{p\cdot g}\Big(\frac{\partial}{\partial x^i}\Big) = \mathrm{Ad}(g^{-1}) \theta_p \Big(\frac{\partial}{\partial x^i}\Big)$$

we get (5.10). The equations (5.11) is obtained by differentiate the formula

$$\mathrm{Ad}(g(t)^{-1}) Y_\alpha = \sum_{\beta=1}^{r} A_\alpha^\beta(y(t)) Y_\beta$$

with respect to t for $g(t) = g \cdot \exp(tY_\gamma)$. □

From (5.8) we see that the horizontal space H_u in $T_u P$ ($u = (x,y)$) is generated by the vectors

$$X_i^\#(x,y) := \frac{\partial}{\partial x^i} - \sum_{\alpha,\beta} \Gamma_\alpha^\beta(y) \theta_i^\alpha(x,y) \frac{\partial}{\partial y^\beta} \quad (i = 1, \dots, n).$$

The Kaluza-Klein metric $\widetilde{m}$ on P is defined by

$$(X_i^\#, X_j^\#) = m_{ij}, \quad (X_i^\#, Y_\alpha^P) = 0, \quad (Y_\alpha^P, Y_\beta^P) = \delta_{\alpha\beta},$$

and is represented by

$$\left.\begin{aligned} \widetilde{m}_{ij} &= \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = m_{ij} + \sum_\alpha \theta_i^\alpha(x,y)\theta_j^\alpha(x,y), \\ \widetilde{m}_{i\alpha} &= \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha}\right) = \sum_\gamma \theta_i^\gamma(x,y)\Lambda_\alpha^\gamma(y), \\ \widetilde{m}_{\alpha\beta} &= \left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right) = \sum_\gamma \Lambda_\alpha^\gamma(y)\Lambda_\beta^\gamma(y). \end{aligned}\right\} \tag{5.13}$$

As a consequence, we get the Hamiltonian system $(T^*P, \Omega_P, \widetilde{H})$ with

$$\Omega_P = \sum_i d\xi_i \wedge dx^i + \sum_\alpha d\eta_\alpha \wedge dy^\alpha, \tag{5.14}$$

$$\begin{aligned} \widetilde{H}(x,y;\xi,\eta) = & \sum m^{ij}(x)\xi_i\xi_j - 2\sum m^{ij}(x)\theta_j^\beta(x,y)\Gamma_\beta^\alpha(y)\xi_i\eta_\alpha \\ & + \sum \Gamma_\gamma^\alpha(y)\theta_j^\gamma(x,y)m^{ji}(x)\theta_i^\kappa(x,y)\Gamma_\kappa^\beta(y)\eta_\alpha\eta_\beta \\ & + \sum \Gamma_\gamma^\alpha(y)\Gamma_\gamma^\beta(y)\eta_\alpha\eta_\beta. \end{aligned} \tag{5.15}$$

The Hamiltonian flow (the geodesic flow) on T^*P is governed by the canonical equation

$$\left.\begin{aligned} \dot{x}^i &= 2\sum m^{ij}\xi_j - 2\sum m^{ij}\theta_j^\beta\Gamma_\beta^\alpha\eta_\alpha, \\ \dot{y}^\alpha &= -2\sum m^{ij}\theta_j^\beta\Gamma_\beta^\alpha\xi_i + 2\sum \Gamma_\gamma^\alpha\theta_j^\gamma m^{ji}\theta_i^\kappa\Gamma_\kappa^\beta\eta_\beta + 2\sum \Gamma_\gamma^\alpha\Gamma_\gamma^\beta\eta_\beta, \\ \dot{\xi}_i &= -\sum \frac{\partial m^{kj}}{\partial x^i}\xi_k\xi_j + 2\sum \frac{\partial}{\partial x^i}(m^{kj}\theta_j^\beta)\Gamma_\beta^\alpha\xi_k\eta_\alpha \\ &\quad - \sum \Gamma_\gamma^\alpha \frac{\partial}{\partial x^i}(\theta_j^\gamma m^{jk}\theta_k^\kappa)\Gamma_\kappa^\beta\eta_\alpha\eta_\beta, \\ \dot{\eta}_\alpha &= 2\sum m^{ij}\frac{\partial}{\partial y^\alpha}(\theta_j^\gamma\Gamma_\gamma^\beta)\xi_i\eta_\beta - 2\sum m^{ji}\frac{\partial}{\partial y^\alpha}(\Gamma_\kappa^\beta\theta_j^\kappa)\theta_i^\nu\Gamma_\nu^\gamma\eta_\beta\eta_\gamma \\ &\quad - 2\sum \frac{\partial\Gamma_\kappa^\beta}{\partial y^\alpha}\Gamma_\kappa^\gamma\eta_\beta\eta_\gamma. \end{aligned}\right\} \tag{5.16}$$

We can directly see that the momentum map $J(x,y;\xi,\eta)$ is invariant under the flow governed by this equation, i.e.,

$$\frac{d}{dt}J(x(t),y(t);\xi(t),\eta(t)) = \frac{d}{dt}\left(\sum \Gamma_\alpha^\beta(y(t))\eta_\beta(t)\otimes Y^\alpha\right) = 0$$

by virtue of Lemmas 5.2 and 5.3.

5.3 Reduced system $(P_\mu, \Omega_\mu, H_\mu)$

Now we consider the reduced system $(P_\mu, \Omega_\mu, H_\mu)$. Note that $\mathrm{Ad}^*(g)\mu = \mu$ ($\mu = \sum \mu_\alpha Y^\alpha \in \mathfrak{g}^*$) for $\forall g \in G_\mu$, and we have the following.

Lemma 5.4 *If* $1 \leq \alpha \leq r$, $r_1 + 1 \leq \beta \leq r$, *then*

$$\sum_{\gamma=1}^{r} \mu_\gamma C^\gamma_{\alpha\beta} = 0. \tag{5.17}$$

Proof. Put $g(t) = \exp(tY''_\beta)$ ($Y''_\beta \in \mathfrak{g}_\mu$). Then for $\forall X \in \mathfrak{g}$ we have $\langle \mu, \mathrm{Ad}(g(t))_* X\rangle = \langle \mu, X\rangle$. Differentiate this equation with respect to t, and we get $\langle \mu, [Y''_\beta, X]\rangle = 0$, which derives (5.17). □

If $(x, y; \xi, \eta) = (x, y', y''; \xi, \eta', \eta'')$ belongs to $J^{-1}(\mu)$, then it follows from (5.2) that

$$\sum_{\gamma=1}^{r} \Gamma^\gamma_\alpha(y)\eta_\gamma = \mu_\alpha \quad (\alpha = 1, \ldots, r), \tag{5.18}$$

and accordingly

$$\eta_\alpha = \sum_{\gamma=1}^{r} \Lambda^\gamma_\alpha(y)\mu_\gamma \quad (\alpha = 1, \ldots, r). \tag{5.19}$$

Thus, we can take $(x, y; \xi) = (x, y', y''; \xi)$ as local coordinates of $J^{-1}(\mu)$ in the neighborhood W of $p_0 = (x_0, 0, 0; \xi_0, \mu', \mu'')$. From (5.19) we have

$$d\eta_\alpha = \sum_{\beta,\gamma=1}^{r} \mu_\gamma \frac{\partial \Lambda^\gamma_\alpha}{\partial y^\beta} dy^\beta. \tag{5.20}$$

Therefore we have

$$\begin{aligned} i^*_\mu \Omega_P &= \sum_i d\xi_i \wedge dx^i + \frac{1}{2}\sum_\gamma \mu_\gamma \Big[\sum_{\alpha,\beta}\Big(\frac{\partial \Lambda^\gamma_\alpha}{\partial y^\beta} - \frac{\partial \Lambda^\gamma_\beta}{\partial y^\alpha}\Big) dy^\beta \wedge dy^\alpha\Big] \\ &= \sum_i d\xi_i \wedge dx^i + \frac{1}{2}\sum_{\alpha,\beta}\Big(\sum_{\gamma,\kappa,\nu} \mu_\gamma \Lambda^\kappa_\beta C^\gamma_{\kappa\nu}\Lambda^\nu_\alpha\Big) dy^\alpha \wedge dy^\beta \end{aligned}$$

on $J^{-1}(\mu)$ by virtue of (5.7). Here notice Lemmas 5.1 and 5.4, and we get

$$\begin{aligned} &(i^*_\mu \Omega_P)(x, y', y'', \xi) \\ &= \sum_i d\xi_i \wedge dx^i + \frac{1}{2}\sum_{\alpha,\beta=1}^{r_1}\Big(\sum_{\kappa,\nu=1}^{r_1}\sum_{\gamma=1}^{r} \mu_\gamma \Lambda^\kappa_\beta(y) C^\gamma_{\kappa\nu}\Lambda^\nu_\alpha(y)\Big) dy'^\alpha \wedge dy'^\beta. \end{aligned}$$

Two points $(x_1, y'_1, y''_1; \xi_1)$ and $(x_2, y'_2, y''_2; \xi_2)$ are in the same G_μ-orbit if and only if $x_1 = x_2, y'_1 = y'_2, \xi_1 = \xi_2$. Hence we take $(x, y'; \xi)(= (x, y', 0; \xi))$ as local coordinates of P_μ, and have the following.

Proposition 5.5 *The symplectic form Ω_μ on P_μ is locally expressed as*

$$\Omega_\mu(x, y'; \xi) = \sum_i d\xi_i \wedge dx^i$$
$$+\frac{1}{2}\sum_{\alpha,\beta=1}^{r_1}\Big(\sum_{\kappa,\nu=1}^{r_1}\sum_{\gamma=1}^{r}\mu_\gamma\Lambda^\kappa_\beta(y',0)C^\gamma_{\kappa\nu}\Lambda^\nu_\alpha(y',0)\Big)dy'^\alpha \wedge dy'^\beta. \quad (5.21)$$

Remark. For a fixed $p \in T^*U$ we have the bijection $\kappa_q : \mathcal{O}_\mu \to \Phi_\mu^{-1}(q)(\subset P_\mu)$ given by $\kappa_q(q) := \Psi_u(q, \nu)$ $(\nu \in \mathcal{O}_\mu)$ (see §3). Then, the two form $\kappa_q^*\Omega_\mu$ is just the symplectic form on $\mathcal{O}_\mu$ called the *Kirillov-Kostant form*(cf. [6]). The coadjoint orbit $\mathcal{O}_\mu$ is locally parameterized by the coordinates $(y'^1, \ldots, y'^{r_1})$, and the second term in (5.21) is just the Kirillov-Kostant form.

By plugging (5.18) into (5.15) we have

$$H_\mu(x, y'; \xi) = \sum_{i,j=1}^{n} m^{ij}(x)\xi_i\xi_j - 2\sum_{i,j=1}^{n}\sum_{\gamma=1}^{r} m^{ij}(x)\theta^\gamma_j(x, y', 0)\xi_i\mu_\gamma$$
$$+\sum_{i,j=1}^{n}\sum_{\gamma,\kappa=1}^{r}\theta^\gamma_j(x, y', 0)m^{ji}(x)\theta^\kappa_i(x, y', 0)\mu_\gamma\mu_\kappa + \sum_{\gamma=1}^{r}(\mu_\gamma)^2. \quad (5.22)$$

As a consequence, we get the equation of the motion in the reduced Hamiltonian system $(P_\mu, \Omega_\mu, H_\mu)$.

Proposition 5.6 *The Hamiltonian flow on the reduced phase space P_μ is governed by*

$$\left.\begin{aligned}
\dot{x}^i &= 2\sum_{i,j=1}^{n}\Big[m^{ij}(x)\xi_j - \sum_{\gamma=1}^{r} m^{ij}(x)\theta^\gamma_j(x, y', 0)\mu_\gamma\Big],\\
\dot{\xi}_i &= -\sum_{j,k=1}^{n}\Big[\frac{\partial m^{kj}}{\partial x^i}\xi_k\xi_j - 2\sum_{\gamma=1}^{r}\frac{\partial}{\partial x^i}(m^{kj}\theta^\gamma_j)\xi_k\mu_\gamma\\
&\quad + 2\sum_{\gamma,\kappa=1}^{r}\frac{\partial}{\partial x^i}(m^{jk}\theta^\gamma_j)\theta^\kappa_k\mu_\gamma\mu_\kappa\Big],\\
\dot{y}'^\alpha &= -2\sum_{i,j=1}^{n}\Big[\sum_{\beta=1}^{r_1} m^{ij}\Gamma^\alpha_\beta\theta^\beta_j\xi_i - \sum_{\beta=1}^{r_1}\sum_{\gamma=1}^{r} m^{ji}\Gamma^\alpha_\beta\theta^\beta_j\theta^\gamma_i\mu_\gamma\Big] \quad (\alpha = 1, \ldots, r_1).
\end{aligned}\right\} \quad (5.23)$$

Proof. The Hamiltonian vector field $X_{H_\mu} = \sum(X^i\partial/\partial x^i + \Xi^i\partial/\partial\xi_i + Y^\alpha\partial/\partial y^\alpha)$ corresponding to H_μ is defined by the equation $i(X_{H_\mu})\Omega_\mu = -dH_\mu$, which directly derives the first and second equations of (5.23) and

$$\sum_{\alpha,\kappa,\nu=1}^{r_1}\sum_{\gamma=1}^{r}\mu_\gamma\Lambda^\kappa_\beta C^\gamma_{\kappa\nu}\Lambda^\nu_\alpha Y^\alpha = 2\sum_{i,j=1}^{n}\Big[\sum_{\gamma=1}^{r} m^{ij}\frac{\partial\theta^\gamma_j}{\partial y'^\beta}\xi_i\mu_\gamma - \sum_{\gamma,\sigma=1}^{r} m^{ij}\frac{\partial\theta^\gamma_j}{\partial y'^\beta}\theta^\sigma_i\mu_\gamma\mu_\sigma\Big].$$

By virtue of (5.12) we get

$$\begin{aligned}\sum_{\alpha,\kappa,\nu=1}^{r_1}\sum_{\gamma=1}^{r}\mu_\gamma\Lambda^\kappa_\beta C^\gamma_{\kappa\nu}\Lambda^\nu_\alpha\, Y^\alpha = -2\sum_{i,j=1}^{n}\Big[&\sum_{\kappa,\nu=1}^{r_1}\sum_{\gamma=1}^{r} m^{ij}\Lambda^\kappa_\beta\mu_\gamma C^\gamma_{\kappa\nu}\theta^\kappa_j\xi_i\\ &-\sum_{\kappa,\nu=1}^{r_1}\sum_{\gamma,\sigma=1}^{r} m^{ji}\Lambda^\kappa_\beta\mu_\gamma C^\gamma_{\kappa\nu}\theta^\nu_j\theta^\sigma_i\mu_\sigma\Big]\end{aligned}$$

Here, we notice that the $r_1\times r_1$ matrices $\Lambda_1 = (\Lambda^\kappa_\beta)$ and $\widetilde{C} = (\widetilde{C}_{\kappa\nu}) = (\sum_\gamma \mu_\gamma C^\gamma_{\kappa\nu})$ are non-singular, and we get the last equation of (5.23). □

5.4 Gauge field and Wong's equation

Finally, we treat the system $(T^*M_\mu, \widetilde{\Omega}_\mu, \widetilde{H}_\mu)$, in which the curvature Θ of the connection $\widetilde{\nabla}$ appears explicitly in the equation of the motion of the charged particle. The curvature form Θ of $\widetilde{\nabla}$ is defined by

$$\Theta(X,Y) := d\theta(X,Y) + \frac{1}{2}[\theta(X),\theta(Y)]$$

for $X, Y \in T_uP$ $(u\in P)$ (see [8] for example). From (5.8) we have the local expression

$$\begin{aligned}\Theta &= \sum_{\alpha=1}^{r}\Big[\frac{1}{2}\sum_{i,j=1}^{n}\Theta^\alpha_{ij}(x,y)dx^i\wedge dx^j\Big]\otimes Y_\alpha\\ &= \sum_{\alpha=1}^{r}\Big[\frac{1}{2}\sum_{i,j=1}^{n}\Big\{\Big(\frac{\partial\theta^\alpha_j}{\partial x^i}-\frac{\partial\theta^\alpha_i}{\partial x^j}\Big)+\sum_{\beta,\gamma=1}^{r_1}C^\alpha_{\beta\gamma}\theta^\beta_i\theta^\gamma_j\Big\}dx^i\wedge dx^j\Big]\otimes Y_\alpha\end{aligned} \tag{5.24}$$

by noticing (5.7) and (5.12). Moreover since Θ satisfies $R^*_g\Theta = \mathrm{Ad}(g^{-1})\Theta$, we have

$$\Theta^\alpha_{ij}(x,y) = \sum_{\beta=1}^{r} A^\alpha_\beta(y)\bar{\Theta}^\beta_{ij}(x),$$

for $\bar{\Theta}^\beta_{ij}(x) := \Theta^\beta_{ij}(x,0)$, where

$$\bar{\Theta} = \sum_{\alpha=1}^{r}\Big[\frac{1}{2}\sum_{i,j=1}^{n}\bar{\Theta}^\alpha_{ij}(x)dx^i\wedge dx^j\Big]\otimes Y_\alpha = s^*\Theta$$

is a $\mathfrak{g}$-valued two-form on $U \subset M$ (the so-called gauge field) pulled-back by the local section $s : U \ni x \mapsto (x, 0) \in U \times G \cong \pi^{-1}(U)$.

For $\mu \in \mathfrak{g}$ we define the $\mathbb{R}$-valued two-form Θ_μ on P by

$$\Theta_\mu(X, Y) := \langle \mu, \Theta(X, Y) \rangle \qquad (X, Y \in T_u P).$$

By virtue of the following lemma we can regard Θ_μ as a two-form (globally defined) on M_μ.

Lemma 5.7 *$\Theta_\mu(A^P, X) = 0$ holds for any $A \in \mathfrak{g}_\mu$ and $X \in T_u P$.*

Proof. Note Lemma 3.2, and we see that $\langle \mu, [\theta(A^P), \theta(X)] \rangle = 0$. In fact, we have

$$\langle \mu, [\theta(A^P), \theta(X)] \rangle = \langle \mu, [A, \theta(X)] \rangle = \langle \mu, \mathrm{ad}(A)(\theta(X)) \rangle = 0$$

because $\langle \mu, \mathrm{Ad}(\exp(tA))(\cdot) \rangle = \langle \mu, \cdot \rangle$ $(t \in \mathbb{R})$. □

The one-form θ_μ on P is given by

$$\theta_\mu(x, y) = \sum_{\alpha=1}^{r} \Big(\sum_{i=1}^{n} \mu_\alpha \theta_i^\alpha(x, y) dx^i + \sum_{\beta=1}^{r} \mu_\alpha \Lambda_\beta^\alpha(y) dy^\beta \Big),$$

and we can directly check that $\mathcal{L}_{Y''^P} \theta_\mu = 0$ for $Y'' \in \mathfrak{g}_\mu$, which means that θ_μ is G_μ-invariant. We can take $(x^1, \ldots, x^n, y'^1, \ldots, y'^{r_1})$ as coordinates of M_μ. Then, the two forms $d\theta_\mu$ and Θ_μ on M_μ are represented as

$$\begin{aligned}
d\theta_\mu(x, y') &= d\theta_\mu(x, y', 0) \\
&= \frac{1}{2} \sum_{i,j=1}^{n} \sum_{\gamma=1}^{r} \mu_\gamma \Big(\frac{\partial \theta_j^\gamma}{\partial x^i} - \frac{\partial \theta_i^\gamma}{\partial x^j} \Big) dx^i \wedge dx^j \\
&\quad + \sum_{i=1}^{n} \sum_{\alpha,\kappa,\nu=1}^{r_1} \sum_{\gamma=1}^{r} \mu_\gamma \Lambda_\alpha^\kappa C_{\kappa\nu}^\gamma \theta_i^\nu \, dx^i \wedge dy'^\alpha \\
&\quad - \frac{1}{2} \sum_{\alpha,\beta,\kappa,\nu=1}^{r_1} \sum_{\gamma=1}^{r} \mu_\gamma \Lambda_\alpha^\kappa C_{\kappa\nu}^\gamma \Lambda_\beta^\nu \, dy'^\alpha \wedge dy'^\beta, \\
\Theta_\mu(x, y') &= \frac{1}{2} \sum_{i,j=1}^{n} \sum_{\gamma=1}^{r} \mu_\gamma \Theta_{ij}^\gamma(x, y') dx^i \wedge dx^j \quad (\Theta_{ij}^\gamma(x, y') := \Theta_{ij}^\gamma(x, y', 0))
\end{aligned}$$

by means of (5.7), (5.12) and (5.24). Let $(x, y'; \bar{\xi}, \bar{\eta}) = (x^1, \ldots, x^n, y'^1, \ldots, y'^{r_1}; \bar{\xi}_1, \ldots, \bar{\xi}_{r_1}, \bar{\eta}_1, \ldots, \bar{\eta}_{r_1})$ be canonical coordinates of $T^* M_\mu$. Then, we have

$$\widetilde{\Omega}_\mu = \sum_{i=1}^{n} d\bar{\xi}_i \wedge dx^i + \sum_{\alpha=1}^{r_1} d\bar{\eta}_\alpha \wedge dy'^\alpha + d\theta_\mu(x, y').$$

The metric m_μ is defined by

$$(\bar{X}_i^\#, \bar{X}_j^\#) = m_{ij}, \quad (\bar{X}_i^\#, Y_\alpha^P) = 0, \quad (Y_\alpha^P, Y_\beta^P) = \delta_{\alpha\beta},$$

for $1 \le i, j \le n,\ 1 \le \alpha, \beta \le r_1$ with

$$\bar{X}_i^\#(x, y') := \frac{\partial}{\partial x^i} - \sum_{\alpha,\beta=1}^{r_1} \Gamma_\alpha^\beta(y', 0)\theta_i^\alpha(x, y', 0)\frac{\partial}{\partial y'^\beta}$$

and is represented by

$$\left.\begin{aligned} (m_\mu)_{ij} &= m_{ij} + \sum_{\alpha=1}^{r_1} \theta_i^\alpha(x, y', 0)\theta_j^\alpha(x, y', 0), \\ (m_\mu)_{i\alpha} &= \sum_{\gamma=1}^{r_1} \theta_i^\gamma(x, y', 0)\Lambda_\alpha^\gamma(y', 0), \\ (m_\mu)_{\alpha\beta} &= \sum_{\gamma=1}^{r_1} \Lambda_\alpha^\gamma(y', 0)\Lambda_\beta^\gamma(y', 0). \end{aligned}\right\} \tag{5.25}$$

Hence, we get

$$\begin{aligned} \widetilde{H}_\mu(x, y'; \bar{\xi}, \bar{\eta}) = &\sum m^{ij}\bar{\xi}_i\bar{\xi}_j - 2\sum m^{ij}\theta_j^\beta\Gamma_\beta^\alpha\bar{\xi}_i\bar{\eta}_\alpha \\ &+ \sum \Gamma_\gamma^\alpha\theta_j^\gamma m^{ji}\theta_i^\kappa\Gamma_\kappa^\beta\bar{\eta}_\alpha\bar{\eta}_\beta + \sum \Gamma_\gamma^\alpha\Gamma_\gamma^\beta\bar{\eta}_\alpha\bar{\eta}_\beta. \end{aligned} \tag{5.26}$$

By straightforward calculations we see that the flow $(x(t), y'(t); \bar{\xi}(t), \bar{\eta}(t))$ of $(T^*M_\mu, \widetilde{\Omega}_\mu, \widetilde{H}_\mu)$ satisfies the equation in the form

$$\frac{d}{dt}\bar{\eta}_\alpha = \sum_\beta F^{\alpha\beta}(x, y', \bar{\xi})\bar{\eta}_\beta \quad (1 \le \alpha \le r_1)$$

for some functions $F^{\alpha\beta}$. Hence, we restrict the flow on the submanifold: $\bar{\eta} \equiv 0$, (which is invariant under the flow). Then, we have the following.

Theorem 5.8 *(1) The flow of $(T^*M_\mu, \widetilde{\Omega}_\mu, \widetilde{H}_\mu)$ restricted on the submanifold: $\bar{\eta} \equiv 0$ is governed by the equation*

$$\left.\begin{aligned} \dot{x}^i &= 2\sum_{i,j=1}^n m^{ij}(x)\bar{\xi}_j, \\ \dot{\bar{\xi}}_i &= -\sum_{j,k=1}^n \frac{\partial m^{kj}}{\partial x^i}(x)\bar{\xi}_k\bar{\xi}_j - 2\sum_{j,k=1}^n\sum_{\gamma=1}^r m^{jk}(x)\mu_\gamma\Theta_{ji}^\gamma(x, y')\bar{\xi}_k, \\ \dot{y}'^\alpha &= -2\sum_{i,j=1}^n\sum_{\beta=1}^{r_1} m^{ij}(x)\theta_j^\beta(x, y')\Gamma_\beta^\alpha(y')\bar{\xi}_i \qquad (\alpha = 1, \dots, r_1). \end{aligned}\right\} \tag{5.27}$$

(2) The map $T^*P \to T^*M_\mu$ defined by

$$(x^1,\ldots,x^n,y^1,\ldots,y^r;\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_r) \longmapsto (x^1,\ldots,x^n,y'^1,\ldots,y'^{r_1};\bar{\xi}_1,\ldots,\bar{\xi}_n,\bar{\eta}_1,\ldots,\bar{\eta}_{r_1})$$

with

$$\left.\begin{aligned} y'^\alpha &= y^\alpha \qquad (\alpha = 1,\ldots,r_1), \\ \bar{\xi}_i &= \xi_i - \sum_{\gamma=1}^{r} \theta_i^\gamma(x,y)\mu_\gamma \qquad (i=1,\ldots,n), \\ \bar{\eta}_\alpha &= \eta_\alpha - \sum_{\gamma=1}^{r} \Lambda_\alpha^\gamma(y)\mu_\gamma \qquad (\alpha = 1,\ldots,r_1) \end{aligned}\right\} \tag{5.28}$$

induces the map $\chi_\mu : P_\mu \to T^*M_\mu$, under which the canonical equation (5.23) on P_μ is transformed to the equation (5.27).

Proof. We get the assertion by straightforward calculations. □

The equation (5.27) is (essentially same as) Wong's equation (see [14]), which describes the motion of a particle with charge $\mu \in \mathfrak{g}^*$ in the gauge field Θ with the potential θ. From (5.27) we get the following.

Corollary 5.9 *The motion of the particle with charge μ in the gauge field Θ is governed by the following equation in M_μ:*

$$\left.\begin{aligned} &\ddot{x}^i + \sum_{j,k=1}^{n} \Gamma_{jk}^i(x)\dot{x}^j\dot{x}^k - 2\sum_{j,k=1}^{n}\sum_{\gamma=1}^{r} m^{ij}(x)\mu_\gamma\Theta_{jk}^\gamma(x,y')\dot{x}^k = 0, \\ &\dot{y}'^\alpha = -\sum_{j=1}^{n}\sum_{\beta=1}^{r_1} \theta_j^\beta(x,y')\Gamma_\beta^\alpha(y')\dot{x}^j \qquad (\alpha=1,\ldots,r_1), \end{aligned}\right\} \tag{5.29}$$

where $\Gamma_{jk}^i(x)$ denotes the Christoffel symbol defined from the Riemannian structure m on M.

6 An example - $Sp(1)$-gauge fields associated to the Hopf bundles

Let $\mathbb{H}$ be the division algebra of quaternions, i.e.,

$$\mathbb{H} = \{q = s + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \mid s,x,y,z \in \mathbb{R},\ \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1\}.$$

Consider the product space $\mathbb{H}^{n+1} = \{\boldsymbol{q} = (q_0, q_1, \ldots, q_n)\}$ with the Hermitian inner product:

$$\langle \boldsymbol{q}, \boldsymbol{q}' \rangle = \sum_{j=0}^{n} \bar{q}_j q'_j = \sum_{j=0}^{n} (s_j - x_j\boldsymbol{i} - y_j\boldsymbol{j} - z_j\boldsymbol{k})(s'_j + x'_j\boldsymbol{i} + y'_j\boldsymbol{j} + z'_j\boldsymbol{k}),$$

and the real inner product:

$$\langle \boldsymbol{q}, \boldsymbol{q}' \rangle_{\mathbb{R}} = \mathrm{Re}\langle \boldsymbol{q}, \boldsymbol{q}' \rangle = \sum_{j=0}^{n} (s_j s_j' + x_j x_j' + y_j y_j' + z_j z_j').$$

Note that $\mathbb{H}^{n+1}$ with $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ is identified with $\mathbb{R}^{4n+4}$. Let

$$S_{[2]}^{4n+3} := \{ \boldsymbol{q} \mid |\boldsymbol{q}|^2 (= \overline{\boldsymbol{q}}\boldsymbol{q}) = |q_0|^2 + \cdots + |q_n|^2 = 4 \} \subset \mathbb{H}^{n+1} \cong \mathbb{R}^{4n+4}$$

be the $(4n+3)$-dimensional sphere with radius 2, and let $\widetilde{m}_0$ be the Riemannian metric on it induced from $\langle \cdot, \cdot \rangle_{\mathbb{R}}$.

A quaternion λ acts on $\mathbb{H}^{n+1}$ from right:

$$\boldsymbol{q} \cdot \lambda = (q_0, \ldots, q_n) \cdot \lambda = (q_0 \lambda, \ldots, q_n \lambda).$$

Then, the Hermitian product $\langle \cdot, \cdot \rangle$ is left invariant under this action by every unit quaternions, that is, every elements of

$$Sp(1) = \{ \lambda \in \mathbb{H} \mid |\lambda| = 1 \}$$

which is a three-dimensional Lie group called the symplectic group ($\cong SU(2) \cong S^3$). Thus $Sp(1)$ acts freely and isometrically on $S_{[2]}^{4n+3}$, and we get the Hopf fiber bundle:

$$Sp(1) \to S_{[2]}^{4n+3} \xrightarrow{\pi} \mathbb{H}P^n \tag{6.1}$$

over the quaternionic projective space. The tangent bundle of $S_{[2]}^{4n+3}$ is given by

$$TS_{[2]}^{4n+3} = \{ (\boldsymbol{q}, \boldsymbol{u}) \mid \boldsymbol{q} \in S_{[2]}^{4n+3}, \boldsymbol{u} \in \mathbb{H}^{n+1}, \langle \boldsymbol{q}, \boldsymbol{u} \rangle_{\mathbb{R}} = 0 \}.$$

For $\boldsymbol{q} \in S_{[2]}^{4n+3}$, let $V_{\boldsymbol{q}} = (d\pi)^{-1}(0) \subset T_{\boldsymbol{q}} S_{[2]}^{4n+3}$, and it is easy to see that

$$V_{\boldsymbol{q}} = \{ (\boldsymbol{q}, \boldsymbol{q}v) \mid v \in \mathbb{H}, \mathrm{Re}(v) = 0 \}.$$

Let $H_{\boldsymbol{q}}$ be the orthogonal compliment of $V_{\boldsymbol{q}}$ in $T_{\boldsymbol{q}} S_{[2]}^{4n+3}$ with respect to the metric $\widetilde{m}_0$, and we have

$$T_{\boldsymbol{q}} S_{[2]}^{4n+3} = H_{\boldsymbol{q}} \oplus V_{\boldsymbol{q}}. \tag{6.2}$$

Then, we have

$$H_{\boldsymbol{q}} = \{ (\boldsymbol{q}, \boldsymbol{u}) \mid \boldsymbol{u} \in \mathbb{H}^{n+1}, \langle \boldsymbol{q}, \boldsymbol{u} \rangle = 0 \}.$$

We can easily check that the horizontal space $H_{\boldsymbol{q}}$ is invariant under the $Sp(1)$ action on $S_{[2]}^{4n+3}$, and accordingly, the decomposition (6.2) defines the connection $\widetilde{\nabla}$ on the principal $Sp(1)$-bundle $\pi : S_{[2]}^{4n+3} \to \mathbb{H}P^n$. Furthermore, $\mathbb{H}P^n$ endowed with the Riemannian metric m_0 such that π is a Riemannian submersion.

Let $\mathfrak{sp}(1)$ denote the Lie algebra of $Sp(1)$. Then, $\mathfrak{sp}(1)$ consists of pure imaginary quaternions, i.e.,

$$\mathfrak{sp}(1) = \{v \in \mathbb{H} \mid \mathrm{Re}(v) = 0\} = \{v = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k} \mid v_1, v_2, v_3 \in \mathbb{R}\} \cong \mathbb{R}^3.$$

Note that we have the natural correspondence between the vertical space $V_{\boldsymbol{q}}$ and $\mathfrak{sp}(1)$.

Proposition 6.1 *The connection form θ of $\widetilde{\nabla}$ (which is a $\mathfrak{sp}(1)$-valued one-form on $S^{4n+3}_{[2]}$) is given by*

$$\theta_{\boldsymbol{q}}(\boldsymbol{u}) = \frac{1}{4}\langle \boldsymbol{q}, \boldsymbol{u}\rangle \in \mathfrak{sp}(1) \quad ((\boldsymbol{q}, \boldsymbol{u}) \in T_{\boldsymbol{q}}S^{4n+3}_{[2]}). \tag{6.3}$$

Here, note that $\langle \boldsymbol{q}, \boldsymbol{u}\rangle$ belongs to $\mathfrak{sp}(1)$ because $\langle \boldsymbol{q}, \boldsymbol{u}\rangle_{\mathbb{R}} = 0$. By using the coordinates $(q_0, \dots, q_n)$ in $\mathbb{H}^{n+1}$ we have

$$\theta = \frac{1}{8}\sum_{j=0}^{n}(\overline{q}_j dq_j - d\overline{q}_j\, q_j) = \frac{1}{8}(\overline{\boldsymbol{q}} \cdot d\boldsymbol{q} - d\overline{\boldsymbol{q}} \cdot \boldsymbol{q}). \tag{6.4}$$

Proof. Put $v = \theta_{\boldsymbol{q}}(\boldsymbol{u}) \in \mathfrak{sp}(1)$. Then, $\boldsymbol{u} - \boldsymbol{q}v$ is a horizontal vector, hence

$$0 = \langle \boldsymbol{q}, \boldsymbol{u} - \boldsymbol{q}v\rangle = \langle \boldsymbol{q}, \boldsymbol{u}\rangle - \langle \boldsymbol{q}, \boldsymbol{q}v\rangle = \langle \boldsymbol{q}, \boldsymbol{u}\rangle - \langle \boldsymbol{q}, \boldsymbol{q}\rangle\, v = \langle \boldsymbol{q}, \boldsymbol{u}\rangle - 4v.$$

Therefore, we obtain (6.3). □

Let us introduce a local coordinate of $\mathbb{H}P^n$ as follows. For a point $\boldsymbol{q} = (q_0, q_1, \dots, q_n) \in S^{4n+3}_{[2]}$, denote $[\boldsymbol{q}] = [q_0, q_1, \dots, q_n] = \pi(q_0, q_1, \dots, q_n) \in \mathbb{H}P^n$. Put $U_0 = \{[\boldsymbol{q}] = [q_0, \dots, q_n] \in \mathbb{H}P^n \mid q_0 \neq 0\}$, which is a open subset of $\mathbb{H}P^n$. Then,

$$\varphi_0 : U_0 \to \mathbb{H}^n;\ [q_0, q_1, \dots, q_n] \mapsto (p_1, p_2, \dots, p_n) = (q_1 q_0^{-1}, q_2 q_0^{-1}, \dots, q_n q_0^{-1})$$

gives a local coordinate of $\mathbb{H}P^n$. Take a local section

$$s : U_0 \to S^{4n+3}_{[2]};\ \boldsymbol{p} = (p_1, \dots, p_n) \mapsto \Big(\frac{2}{\sqrt{1+|\boldsymbol{p}|^2}}, \frac{2p_1}{\sqrt{1+|\boldsymbol{p}|^2}}, \dots, \frac{2p_n}{\sqrt{1+|\boldsymbol{p}|^2}}\Big).$$

The connection form $\theta_{U_0} = s^*\theta$ on $U_0 \subset \mathbb{H}P^n$ is given by

$$\theta_{U_0} = \frac{1}{2(1+|\boldsymbol{p}|^2)}\sum_{j=1}^{n}(\overline{p}_j dp_j - d\overline{p}_j\, p_j) = \frac{1}{2(1+|\boldsymbol{p}|^2)}(\overline{\boldsymbol{p}} \cdot d\boldsymbol{p} - d\overline{\boldsymbol{p}} \cdot \boldsymbol{p}). \tag{6.5}$$

Let Θ be the curvature form of $\widetilde{\nabla}$, which is $\mathfrak{sp}(1)$-valued two-form on $S^{4n+3}_{[2]}$, and let $\Theta_{U_0} := s^*\Theta = d\theta_{U_0} + \theta_{U_0} \wedge \theta_{U_0}$ (a gauge field on U_0). Then, we have the following.

Proposition 6.2

$$\Theta_{U_0} = \frac{1}{(1+|\boldsymbol{p}|^2)^2}\sum_{j=1}^{n} d\overline{p}_j \wedge dp_j. \tag{6.6}$$

The dual space $\mathfrak{sp}(1)^*$ of the Lie algebra $\mathfrak{sp}(1)$ is identified with $\mathfrak{sp}(1)$ by the correspondence $\mathfrak{sp}(1) \ni v \leftrightarrow v^* \in \mathfrak{sp}(1)^*$ with $v^*(w) = \langle v, w\rangle_{\mathbb{R}} = \mathrm{Re}(\overline{v}w)$ $(w \in \mathfrak{sp}(1))$. Thus we have

$$\mathfrak{sp}(1)^* = \{\nu_1 \boldsymbol{i}^* + \nu_2 \boldsymbol{j}^* + \nu_3 \boldsymbol{k}^* \mid \nu_1, \nu_2, \nu_3 \in \mathbb{R}\} \cong \mathbb{R}^3.$$

Similarly, we have

$$\begin{aligned} T^*S^{4n+3}_{[2]} &= \{(\boldsymbol{q}, \boldsymbol{u}^*) \mid (\boldsymbol{q}, \boldsymbol{u}) \in TS^{4n+3}_{[2]}\} \\ &= \{(\boldsymbol{q}, \boldsymbol{u}^*) \mid \boldsymbol{q} \in S^{4n+3}_{[2]}, \boldsymbol{u} \in \mathbb{H}^{n+1},\ \langle \boldsymbol{q}, \boldsymbol{u}\rangle_{\mathbb{R}} = 0\} \end{aligned}$$

through the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$.

Proposition 6.3 *The momentum map* $J : T^*S^{4n+3}_{[2]} \to \mathfrak{sp}(1)^*$ *is given by*

$$J(\boldsymbol{q}, \boldsymbol{u}^*) = \langle \boldsymbol{q}, \boldsymbol{u}\rangle^*. \tag{6.7}$$

Proof. Note that the vector field on $S^{4n+3}_{[2]}$ associated to $v \in \mathfrak{sp}(1)$ is given by $(\boldsymbol{q}, \boldsymbol{q}v)$. Hence, by the definition of J we have

$$\begin{aligned} \langle J(\boldsymbol{q}, \boldsymbol{u}^*), v\rangle &= \langle (\boldsymbol{q}, \boldsymbol{u}^*), \boldsymbol{q}v\rangle = \langle \boldsymbol{u}, \boldsymbol{q}v\rangle_{\mathbb{R}} \\ &= \mathrm{Re}\Big[\sum_j \overline{u}_j q_j v\Big] = \mathrm{Re}\Big[\big(\overline{\textstyle\sum_j \overline{q}_j u_j}\big) \cdot v\Big] = \big\langle \sum_j \overline{q}_j u_j, v\, \big\rangle_{\mathbb{R}}. \end{aligned}$$

Therefore, $J(\boldsymbol{q}, \boldsymbol{u}^*) = \big(\sum_j \overline{q}_j u_j\big)^* = \langle \boldsymbol{q}, \boldsymbol{u}\rangle^*$. □

Next, we consider the (co-)adjoint action of $Sp(1)$ on $\mathfrak{sp}(1)$ (or $\mathfrak{sp}(1)^*$) and its orbit. It is easy to see that

$$\mathrm{Ad}^*(\lambda)v^* = (\mathrm{Ad}(\lambda)v)^* \quad (\lambda \in Sp(1),\ v \in \mathfrak{sp}(1)).$$

Take $\lambda = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} \in Sp(1)$. Then, $[\mathrm{Ad}(\lambda^{-1})\boldsymbol{i}, \mathrm{Ad}(\lambda^{-1})\boldsymbol{j}, \mathrm{Ad}(\lambda^{-1})\boldsymbol{k}]$ $= [\,\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\,]R_\lambda$ with R_λ being a 3×3 matrix:

$$R_\lambda = \begin{bmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_0x_3 + x_1x_2) & 2(-x_0x_2 + x_1x_3) \\ 2(-x_0x_3 + x_1x_2) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_0x_1 + x_2x_3) \\ 2(x_0x_2 + x_1x_3) & 2(-x_0x_1 + x_2x_3) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{bmatrix}.$$

Here, R_λ is an element of $SO(3)$, and $\lambda \mapsto R_\lambda$ gives a homomorphism from $Sp(1)$ onto $SO(3)$. More precisely, if $\lambda = \cos(\phi/2) + \sin(\phi/2)(v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k})$, then R_λ is the rotation about the axis $v = (v_1, v_2, v_3)$ through the angle $-\phi$. Therefore, we see that the co-adjoint orbit $\mathcal{O}_\mu$ through $\mu \in \mathfrak{sp}(1)^*$ is the sphere in $\mathfrak{sp}(1)^* \cong \mathbb{R}^3$ with the center being the origin and the radius $|\mu|$. The isotropy subgroup G_μ for $\mu = \mu_1 \boldsymbol{i}^* + \mu_2 \boldsymbol{j}^* + \mu_3 \boldsymbol{k}^*$ is given by

$$G_\mu = \left\{ \cos\psi + \frac{1}{|\mu|} \sin\psi(\mu_1 \boldsymbol{i} + \mu_2 \boldsymbol{j} + \mu_3 \boldsymbol{k}) \;\middle|\; 0 \le \psi < 2\pi \right\} \cong U(1)$$

if $\mu \neq 0$, and $G_\mu = Sp(1)$ if $\mu = 0$. Suppose $\mu = c\boldsymbol{k}^*$ $(c > 0)$. Then, $G_\mu = \{\cos\psi + \sin\psi \boldsymbol{k} = e^{\psi \boldsymbol{k}} \mid 0 \le \psi < 2\pi\}$ and $\mathfrak{g}_\mu = \mathbb{R}\boldsymbol{k}$. Take $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ as a orthonormal basis of $\mathfrak{sp}(1)$, and we have local coordinates (ϕ_1, ϕ_2, ψ) of $g \in Sp(1)$ given by

$$\begin{aligned} g &= \exp(\phi_1 \boldsymbol{i} + \phi_2 \boldsymbol{j}) \exp(\psi \boldsymbol{k}) \\ &= \left\{ \cos\sqrt{\phi_1^2 + \phi_2^2} + \sin\sqrt{\phi_1^2 + \phi_2^2} \left(\frac{\phi_1}{\sqrt{\phi_1^2 + \phi_2^2}} \boldsymbol{i} + \frac{\phi_2}{\sqrt{\phi_1^2 + \phi_2^2}} \boldsymbol{j} \right) \right\} \\ &\qquad \times \Big(\cos\psi + \sin\psi \boldsymbol{k} \Big). \end{aligned}$$

Thus we have local coordinates $(p_1, \ldots, p_n, \phi_1, \phi_2)$ of $M_\mu = S^{4n+3}_{[2]}/G_\mu$, and can explicitely represent the equation (5.29) (or (5.27)) of the motion.

Finally, we give some remarks on the case $n = 1$, that is, the Hopf bundle $\pi : S^7 \to \mathbb{H}P^1$. Note that $\mathbb{H}P^1$ is diffeomorphic with the unit sphere $S^4 = \{(p, a) \in \mathbb{H} \times \mathbb{R} \mid |p|^2 + a^2 = 1\}$ in $\mathbb{R}^5 = \mathbb{H} \times \mathbb{R}$ by the stereographic projection

$$\mathbb{H}P^1 \supset U_0 (= \mathbb{H}) \ni p \longmapsto \left(\frac{2p}{|p|^2 + 1}, \frac{|p|^2 - 1}{|p|^2 + 1} \right) \in S^4 \backslash \{(0, 1)\}.$$

Furthermore, the Riemannian metric m_0 previously introduced on $\mathbb{H}P^1$ is nothing but the canonical metric on S^4. The connection given by (6.3) (or the gauge field (6.6)) is an anti-self-dual Yang-Mills connection, i.e., $*\Theta_{U_0} = -\Theta_{U_0}$ holds for Hodge's $*$ operator, and is called the *Belavin-Polyakov-Schwartz-Tyupkin anti-instanton* (cf. [2], [7]).

References

[1] R. Abraham and J. Marsden, *Foundations of Mechanics*, 2nd edition, Benjamin/Cummings (1978).

[2] M.F. Atiyah, *Geometry of Yang-Mills Fields* (Fermi Lectures), Accad. Naz. Lincei, Scuola Norm. Sup., Pisa, 1979.

[3] V. Guillemin and S. Sternberg, On the equations of motion of a particle in a Yang-Mills field and the principle of general covariance, Hadronic J., **1**(1978), 1-32.

[4] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge Univ. Press (1984).

[5] R. Kerner, Generalization of the Kaluza-Klein theory for an arbitrary non-abelian gauge group, Ann. Inst. H. Poincaré Sect. A(N.S.), **9**(1968), 143-152.

[6] A. Kirillov, *Lectures on the Orbit Method*, Graduate Studies in Math. Vol.64, AMS (2004).

[7] S. Kobayashi, *Differential Geometry of Connections and Gauge Theory*, Shokabou(1989)(in Japanese).

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol.I, John Wiley & Sons, Inc. (Interscience Devision) (1963).

[9] M. Kummer, On the construction of the reduced phase space of a Hamiltonian system with symmetry, Indiana Univ. Math. J., **30**(1981), 281-291. 439-458.

[10] R. Kuwabara, On the classical and the quantum mechanics in a magnetic field, J. Math. Tokushima Univ., **29**(1995), 9-22; Correction and addendum, J. Math. Tokushima Univ., **30**(1996), 81-87.

[11] R. Kuwabara, Difference spectrum of the Schrödinger operator in a magnetic field, Math. Z., **233**(2000), 579-599.

[12] R. Kuwabara, Eigenvalues associated with a periodic orbit of the magnetic flow, Contemporary Math., **348**(2004), 169-180.

[13] J. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys., **5**(1974), 121-130.

[14] R. Montgomery, Canonical formulations of a classical particle in a Yang-Mills field and Wong's equations, Lett. Math. Phys., **8**(1984), 59-67.