J. Math. Univ. Tokushima
Vol.40 (2006), 9-14

On the Diophantine Equation $(x^2+1)(y^2+1)=(z^2+1)^2$

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(Received September 29, 2006)

Abstract

In their paper [4], A. Schinzel and W. Sierpiński have investigated the diophantine equation $(x^2-1)(y^2-1)=(z^2-1)^2$. In this paper, we shall investigate an analogous equation $(x^2+1)(y^2+1)=(z^2+1)^2$.

2000 Mathematics Subject Classification. Primary 11D25; Secondary 11B37, 11D09

Introduction

In their paper [4], A. Schinzel and W. Sierpiński firstly investigated the following diophantine equation

$$(1) \qquad (x^2-1)(y^2-1)=(z^2-1)^2.$$

They have found all the integer solutions for which $x-y=2z$. But they could not find all the integer solutions of (1), and the problem to find all the integer solutions of this diophantine equation still remains as an open problem. In this paper, we shall show the following diophantine equation

$$(2) \qquad (x^2+1)(y^2+1)=(z^2+1)^2$$

has infinitely many integer solutions. Though we could not find all of the integer solutions of this equation, we found all the solutions with the additional condition $x-y=2z$. It is obvious that the above diophantine equation (2) has the solution $|x|=|y|=|z|$. Throughout this paper, we shall call these solutions *trivial* and other solutions *nontrivial.* Without loss of generality, we may restrict ourselves to the nontrivial and nonnegative solutions.

We shall also show the following slightly generalized diophantine equation has infinitely many integer solutions for any fixed positive integer t

$$(x^2+1)(y^2+1) = (z^2+t^2)^2. \tag{3}$$

The equation $(x^2+1)(y^2+1) = [((x-y)/2)^2+1]^2$

In this section, we shall show the diophantine equation (2) has infinitely many integer solutions with $x-y=2z$.
The left hand side of the equation (2) can be written as

$$(x^2+1)(y^2+1) = (xy+1)^2 + (x-y)^2.$$

Since $z=(x-y)/2$, the right hand side of the equation (2) is

$$z^2+1 = \left(\frac{x-y}{2}\right)^2 + 1 = \frac{(x-y)^2+4}{4}.$$

Thus we have

$$16(xy+1)^2 + 16(x-y)^2 = (x-y)^4 + 8(x-y)^2 + 16,$$

and then

$$16(xy+1)^2 = (x-y)^4 - 8(x-y)^2 + 16 = ((x-y)^2-4)^2.$$

Therefore we have

$$(x-y)^2 - 4 = \pm 4(xy+1)$$
$$\Longleftrightarrow \quad \begin{cases} (x-y)^2 - 4 = x^2 - 2xy + y^2 - 4 = -4xy - 4 \\ \text{or} \\ (x-y)^2 - 4 = x^2 - 2xy + y^2 - 4 = 4xy + 4. \end{cases}$$

We note that
$x^2 - 2xy + y^2 - 4 = -4xy - 4$
if and only if
$x^2 + 2xy + y^2 = (x+y)^2 = 0.$
Hence we have $y=-x$ and then $z=(x-y)/2=x$. Now we see these solutions are trivial. Let us consider another case $x^2-2xy+y^2-4 = 4xy+4$. Then we have $x^2-6xy+y^2=8$. Since $x^2-6xy+y^2 = (x-3y)^2-8y^2$, we can conclude $x-3y$ must be divisible by 4. Put $w=(x-3y)/4$. Then we have

$16w^2 - 8y^2 = 8$, that is, $y^2 - 2w^2 = -1$. Let us denote $\epsilon = 1+\sqrt{2}$ and $\bar{\epsilon} = 1-\sqrt{2}$. Define the binary recurrence sequences $\{t_n\}$ and $\{s_n\}$ by putting

$$\begin{cases} t_n = (\varepsilon^n + \bar{\varepsilon}^n)/2, \\ s_n = (\varepsilon^n - \bar{\varepsilon}^n)/2\sqrt{2}. \end{cases}$$

Then $\{t_n\}$ and $\{s_n\}$ satisfy

$$t_{n+1} = 2t_n + t_{n-1}, \quad s_{n+1} = 2s_n + s_{n-1},$$

and

$$t_n^2 - 2s_n^2 = (-1)^n.$$

Combining the fact that $y^2 - 2w^2 = -1$ and y is nonnegative, we see $y = t_{2n-1}$ and $|w| = s_{2n-1}$ for some positive integer n. Then we have

$$w = \frac{x-3y}{4} = \pm s_{2n-1} \Longleftrightarrow x = 3t_{2n-1} \pm 4s_{2n-1}.$$

From the fact that $\varepsilon^{2n-1} = t_{2n-1} + s_{2n-1}\sqrt{2}$ and $\varepsilon^2 = 3+2\sqrt{2}$, we have

$$\begin{aligned} \varepsilon^{2n+1} &= t_{2n+1} + s_{2n+1}\sqrt{2} = (t_{2n-1} + s_{2n-1}\sqrt{2})(3+2\sqrt{2}) \\ &= 3t_{2n-1} + 4s_{2n-1} + (3s_{2n-1} + 2t_{2n-1})\sqrt{2}, \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{2n-3} &= t_{2n-3} + s_{2n-3}\sqrt{2} = (t_{2n-1} + s_{2n-1}\sqrt{2})(3-2\sqrt{2}) \\ &= 3t_{2n-1} - 4s_{2n-1} + (3s_{2n-1} - 2t_{2n-1})\sqrt{2}). \end{aligned}$$

Hence we have verified $3t_{2n-1} + 4s_{2n-1} = t_{2n+1}$ and $3t_{2n-1} - 4s_{2n-1} = t_{2n-3}$. Thus we have $x = t_{2n+1}$ or t_{2n-3}. In the case $x = t_{2n-3}$, we have $x = t_{2n-3} < y = t_{2n-1}$, which contradicts to the assumption $2z = x - y \geq 0$. Hence any nonnegative solution of (2) can be written as $x = t_{2n+1}$ and $y = t_{2n-1}$ for some positive integer n. From the recurrence relation $t_{2n+1} = 2t_{2n} + t_{2n-1}$, we have

$$z = \frac{x-y}{2} = \frac{t_{2n+1} - t_{2n-1}}{2} = t_{2n}.$$

Thus we have shown the following theorem.

Theorem 1. *With the above notation, the diophantine equation* (2) *has infinitely many positive integer solutions. Moreover any positive integer solution* (x, y, z) *which satisfies* $2z = x - y$ *can be written as* $x = t_{2n+1}, y = t_{2n-1}, z = t_{2n}$ *for some positive integer* n.

Generalization

Let t be a positive integer. Then we shall generalize the above results to the following diophantine equation (3)

$$(x^2+1)(y^2+1) = (z^2+t^2)^2.$$

We note that t^2+1 is not square for any $t \geq 1$ and $\sqrt{t^2+1} \notin \boldsymbol{Q}$. Let us denote $\eta = t + \sqrt{t^2+1}$ and $\bar{\eta} = t - \sqrt{t^2+1}$. Now we shall define binary recurrence sequences by putting

$$\begin{cases} v_n = (\eta^n + \bar{\eta}^n)/2, \\ u_n = (\eta^n - \bar{\eta}^n)/2\sqrt{t^2+1}. \end{cases}$$

Then we have $\{u_n\}$ and $\{v_n\}$ satisfy

$$\begin{cases} u_0 = 0, \quad u_1 = 1, \quad u_2 = 2t, \dots\dots\dots, \quad u_{n+1} = 2tu_n + u_{n-1}, \\ v_0 = 1, \quad v_1 = t, \quad v_2 = 2t^2+1, \dots, \quad v_{n+1} = 2tv_n + v_{n-1}, \end{cases}$$

and

$$\begin{cases} v_{2n-1}^2 + 1 = u_{2n-1}^2(t^2+1), \\ v_{2n+1}^2 + 1 = u_{2n+1}^2(t^2+1). \end{cases}$$

Then we obtain

$$(v_{2n+1}^2+1)(v_{2n-1}^2+1) = [(t^2+1)u_{2n+1}u_{2n-1}]^2.$$

Here we see

$$(t^2+1)u_{2n+1}u_{2n-1} = \frac{\eta^{2n+1}-\bar{\eta}^{2n+1}}{2} \cdot \frac{\eta^{2n-1}-\bar{\eta}^{2n-1}}{2}$$

$$= \frac{1}{2}\left(\frac{\eta^{4n}+\bar{\eta}^{4n}+\eta^2+\bar{\eta}^2}{2}\right) = \frac{1}{2}(v_{4n}+2t^2+1).$$

On the other hand we have

$$v_{2n}^2 = \left(\frac{\eta^{2n}+\bar{\eta}^{2n}}{2}\right)^2 = \frac{\eta^{4n}+\bar{\eta}^{4n}+2}{4} = \frac{1}{2}(v_{4n}+1).$$

Thus we have shown

$$(t^2+1)u_{2n+1}u_{2n-1} = v_{2n}^2 + t^2 = (t^2+1)(u_{2n}^2+1).$$

Hence we have

$$(v_{2n+1}^2+1)(v_{2n-1}^2+1) = (v_{2n}^2+t^2)^2 = [(t^2+1)(u_{2n}^2+1)]^2.$$

Therefore we have obtained the following theorem.

Theorem 2. *With the above notation, the diophantine equation* (3) *has infinitely many positive integer solutions* $x = v_{2n+1}, y = v_{2n-1}, z = v_{2n}$ *with some positive integer* n.

From the above argument, we have the following corollary.

Corollary. *The diophantine equation*

$$(x^2+1)(y^2+1) = [(t^2+1)(z^2+1)]^2$$

has infinitely many parameterized positive integer solutions $(x, y, z, t) = (v_{2n+1}, v_{2n-1}, u_{2n}, t)$.

Concluding remarks

Finally, we shall recall the classical results on Shinzel-Sierpiński equation (1) with $x - y = 2z$. We have $(t_{2n+2}^2 - 1)(t_{2n}^2 - 1) = (2s_{2n+2}s_{2n})^2$.
Here

$$2s_{2n+2}s_{2n} = \frac{1}{4}(\varepsilon^{2n+2} - \bar{\varepsilon}^{2n+2})(\varepsilon^{2n} - \bar{\varepsilon}^{2n})$$

$$= \frac{1}{4}(\varepsilon^{4n+2} + \bar{\varepsilon}^{4n+2} - \varepsilon^2 - \bar{\varepsilon}^2) = \frac{1}{4}(\varepsilon^{4n+2} - \bar{\varepsilon}^{4n+2} - 6)$$

$$= \frac{\varepsilon^{4n+2} + \bar{\varepsilon}^{4n+2} - 2}{4} - 1 = \left(\frac{\varepsilon^{2n+1} + \bar{\varepsilon}^{2n+1}}{2}\right)^2 - 1$$

$$= t_{2n+1}^2 - 1 = \left(\frac{t_{2n+2} - t_{2n}}{2}\right)^2 - 1.$$

Thus we have recalled the elementary fact that any positive integer solution of

$$(x^2-1)(y^2-1) = (z^2-1)^2 \quad \text{with } 2z = x - y,$$

is given by $(x, y, z) = (t_{2n+2}, t_{2n}, t_{2n+1})$.

Here we shall combine this classical result and Theorem 1 as follows. Put $e = \pm 1$. Then the following diophantine equation

$$(4) \qquad (x^2+e)(y^2+e) = (z^2+e)^2.$$

with $2z = x - y$ has infinitely many positive integer solutions $(x, y, z) = (t_{n+2}, t_n, t_{n+1})$. Here, n is even for the case $e = -1$ and n is odd for the case $e = 1$.

In [2], the positive integer solutions of (1) not of the form $(t_{2n+2}, t_{2n}, t_{2n+1})$ are called *sporadic* solutions. For example, there are several sporadic solutions $(31, 4, 11), (97, 2, 13), (48049, 155, 2729)$ quoted by Szymiczek. On the contrary, it seems rare that our equation (2) has sporadic solutions. We have verified the only positive integer solutions with $0 < y < z < 300000$ are

$$\begin{aligned}(x,y,z) &= (7,1,3),\\ &\quad (41,7,17),\\ &\quad (239,41,99),\\ &\quad (1393,239,577),\\ &\quad (8119,1393,3363),\\ &\quad (47321,8119,19601),\\ &\quad (275867,47321,114243).\end{aligned}$$

Hence, there is no sporadic solution for $0 < y < z < 300000$.

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