On Decay Properties of Solutions for the Vlasov–Poisson System

By

Kosuke Ono *

Department of Mathematical Sciences
The University of Tokushima
Tokushima 770-8502, JAPAN
e-mail : *ono@ias.tokushima-u.ac.jp*

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Abstract

We study decay properties of solutions to the Cauchy problem for the collision-less Vlasov–Poisson system which appears Vlasov plasma physics and stems from Liouville's equation coupled with Poisson's equation for the determining the self-consistent electrostatics or gravitational forces.

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1 Introduction

We consider the Cauchy problem for the following kinetic system

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0 \qquad \text{in } \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty) \tag{1.1}$$

$$E(x,t) = -\nabla_x U(x,t) \qquad \text{in } \mathbb{R}^N \times (0,\infty) \tag{1.2}$$

$$f(x,v,0) = \phi(x,v) \geq 0\,, \tag{1.3}$$

where $U = U(x,t)$ is a potential which generates the force field $E = E(x,t)$. Then, the system (1.1)–(1.3) describes the evolution of a microscopic density $f = f(x,v,t) \geq 0$ of particles subject to the action of the force field E. We will be mainly interested in the Vlasov–Poisson system where the force field is self-consistent and given by

$$\begin{aligned} &-\Delta_x U(x,t) = \gamma \rho(x,t)\,, \quad U(x,t) \to 0 \text{ as } |x| \to \infty\,, \\ &\rho(x,t) = \int f(x,v,t)\,dv\,. \end{aligned} \tag{1.4}$$

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where $\nabla_x = (\partial_{x_1}, \cdots, \partial_{x_N})$, $\nabla_v = (\partial_{v_1}, \cdots, \partial_{v_N})$, Δ_x is the Laplacian in the x variable, and γ is a constant. The sign $\gamma = +1$ represents to electrostatic (repulsive) interaction between the particles of the same species, while $\gamma = -1$ represents gravitational (attractive) interaction (see Risken [11], Glassy [5] for physical interpretations).

From (1.2) and (1.4), we have

$$E(x,t) = \frac{\gamma}{S_{N-1}} \frac{x}{|x|^N} * \rho(x,t)\,, \tag{1.5}$$

where S_{N-1} is $(N-1)$-dimensional volume of the N-dimensional unit sphere, and the symbol $*$ is the convolution in the x variable.

The existence of local solutions of the system is known for every $N \in \mathbb{N}$ (e.g. [3], [4], [6], [8]). The Global existence problem has been studied by several authors under suitable restrictions (see [1], [2], [6], [7], [12], [14]).

In this paper we study decay properties of solutions to the Cauchy problem for the Vlasov–Poisson system.

Let $f = f(x,v,t) \geq 0$ be a strong solution of the Vlasov–Poisson system with non-negative initial datum $\phi(x,v) \in C_0^1(\mathbb{R}^N \times \mathbb{R}^N)$, where $C_0^1(\mathbb{R}^N \times \mathbb{R}^N)$ denotes the space of compactly supported, continuously differentiable functions (see [9], [10]).

Our main result is as follows.

Theorem 1.1 *Let $N \geq 4$ and $\gamma > 0$. Then the solution $f = f(x,v,t) \geq 0$ of the Vlasov–Poisson system satisfies that*

$$\| |x/t - v|^2 f \|_{L^1_{x,v}} \leq C_1 t^{-2}\,, \qquad t > 0\,, \tag{1.6}$$

and for $1 \leq q \leq 1 + 2/N$,

$$\|\rho(t)\|_{L^q_x} \leq C_1 t^{-N(1-1/q)}\,, \qquad t > 0\,, \tag{1.7}$$

and for $N/(N-1) < p \leq N(N+2)/(N^2-N-2)$,

$$\|E(t)\|_{L^p_x} \leq C_1 t^{-N(1-1/N-1/p)}\,, \qquad t > 0\,, \tag{1.8}$$

where $C_1 = C_1(\|(1+|x|^2)\phi\|_{L^1_{x,v}}, \|\phi\|_{L^\infty_{x,v}})$ is a constant depending on $\|(1+|x|^2)\phi\|_{L^1_{x,v}}$ and $\|\phi\|_{L^\infty_{x,v}}$.

Finally we fix some notation. The function spaces $L^p_{x,v}$ and L^p_x mean $L^p(\mathbb{R}^N \times \mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ with usual norms $\|\cdot\|_{L^p_{x,v}}$ and $\|\cdot\|_{L^p_x}$ for $1 \leq p \leq \infty$, respectively. Positive constants will be denoted by C and will change from line to line.

2 Proof

We first state the well-known convolution inequality (see for instance [13]).

Lemma 2.1 (Hardy–Littlewood–Sobolev inequality)
Let $0 < \lambda < N$ *and* $1 < q < p < \infty$. *Then*

$$\||x|^{-\lambda} * f(x)\|_{L^p_x} \le C\|f\|_{L^q_x} \qquad \text{for } f \in L^q_x$$

with $1 + 1/p = \lambda/N + 1/q$.

The following proposition plays an important role in the proof of Theorem 1.1.

Proposition 2.2

$$(1) \qquad \frac{d}{dt}\|E(t)\|^2_{L^2_x} = -2\gamma \int E \cdot j \, dx \,, \qquad j = \int vf \, dv$$

$$(2) \qquad \frac{N-2}{2\gamma}\|E(t)\|^2_{L^2_x} = \int x \cdot E\rho \, dx \,, \qquad \rho = \int f \, dv$$

Proof. (1) Using (1.2) and integrating by parts, we observe that

$$\begin{aligned}\frac{d}{dt}\|E(t)\|^2_{L^2_x} &= \frac{d}{dt}\int |\nabla_x U|^2 dx = -2\int U\Delta U_t \, dx \\ &= -2\gamma \int U \partial_t \rho \, dx = 2\gamma \int U \nabla_x \cdot j \, dx \\ &= 2\gamma \int \nabla_x U \cdot j \, dx = -2\gamma \int E \cdot j \, dx \,,\end{aligned}$$

where we used the fact $\partial_t \rho + \nabla_x \cdot j = 0$, indeed, $\partial_t \rho = \int \partial_t f \, dv = -\int (v \cdot \nabla_x f + E \cdot \nabla_v f) \, dv = -\nabla \cdot \int vf \, dv = -\nabla_x \cdot j$.

(2) Using (1.2) and (1.4) and integrating by parts, we observe that

$$\begin{aligned}\int x \cdot E\rho \, dv &= \frac{1}{\gamma}\int x \cdot \nabla_x U \Delta_x U \, dx = \frac{1}{\gamma}\sum_{k,j}\int x_k U_{x_k} U_{x_j x_j} \, dx \\ &= -\frac{1}{\gamma}\sum_{k,j}\int \partial_{x_j}(x_k U_{x_k}) U_{x_j} \, dx \\ &= -\frac{1}{\gamma}\left(\int |\nabla_x U|^2 \, dx + \frac{1}{2}\sum_{k,j}\int x_k \partial_{x_k}(U^2_{x_j}) \, dx\right) \\ &= -\frac{1}{\gamma}\left(\|E(t)\|^2_{L^2_x} - \frac{1}{2}\sum_{k,j}\int U^2_{x_j} \, dx\right) \\ &= -\frac{1}{\gamma}\left(\|E(t)\|^2_{L^2_x} - \frac{N}{2}\int |\nabla_x U|^2 \, dx\right) = \frac{N-2}{2\gamma}\|E(t)\|^2_{L^2_x} \,.\end{aligned}$$

Proof of Theorem 1.1 Using the Vlasov–Poisson system and integrating by parts, we observe that

$$\begin{aligned}
&\frac{d}{dt}\||x-tv|^2 f\|_{L^1_{x,v}} \\
&= -2\iint (x-tv)\cdot vf\,dvdx - \iint |x-tv|^2(v\cdot\nabla_x f + E\cdot\nabla_v f)\,dvdx \\
&= -\iint |x-tv|^2 E\cdot\nabla_v f\,dvdx = -2t\iint (x-tv)\cdot Ef\,dvdx \\
&= -2t\int x\cdot E\rho\,dx + 2t^2\int E\cdot j\,dx\,,
\end{aligned}$$

where $\rho = \int f\,dv$ and $j = \int vf\,dv$.

From Proposition 2.2, we have

$$\frac{d}{dt}\||x-tv|^2 f\|_{L^1_{x,v}} = -\frac{N-2}{\gamma}t\|E(t)\|^2_{L^2_x} - \frac{1}{\gamma}t^2\frac{d}{dt}\|E(t)\|^2_{L^2_x}$$

or

$$\frac{d}{dt}\left\{\||x-tv|^2 f\|_{L^1_{x,v}} + \frac{1}{\gamma}t^2\frac{d}{dt}\|E(t)\|^2_{L^2_x}\right\} = -\frac{N-4}{\gamma}t\|E(t)\|^2_{L^2_x}\,.$$

When $\gamma > 0$ and $N \geq 4$, we see

$$\||x-tv|^2 f\|_{L^1_{x,v}} + \frac{1}{\gamma}t^2\frac{d}{dt}\|E(t)\|^2_{L^2_x} \leq \||x|^2\phi\|_{L^1_{x,v}}$$

or

$$\||x/t-v|^2 f\|_{L^1_{x,v}} + \frac{1}{\gamma}\frac{d}{dt}\|E(t)\|^2_{L^2_x} \leq \||x|^2\phi\|_{L^1_{x,v}}t^{-2}\,, \qquad t>0\,, \tag{2.1}$$

which gives the estimate (1.6).

For $a \geq 1$ and $R > 0$, we observe

$$\begin{aligned}
&\int f\,dv \\
&\leq \int_{|x/t-v|\leq R} f\,dv + \int_{|x/t-v|\geq R}(R^{-2}|x/t-v|^2 f)^{1/a} f^{1-1/a}\,dv \\
&\leq CR^N\|f\|_{L^\infty_{x,v}} + R^{-2/a}\left(\int |x/t-v|^2 f\,dv\right)^{1/a}\left(\int f\,dv\right)^{1-1/a}\,.
\end{aligned}$$

Optimizing the above estimate in R, that is, taking

$$R^{N+2/a} = \left(C\|f\|_{L^\infty_{x,v}}\right)^{-1}\left(\int |x/t-v|^2 f\,dv\right)^{1/a}\left(\int f\,dv\right)^{1-1/a}\,,$$

we have that

$$
\begin{aligned}
&\int f\,dv \\
&\leq C\left(\|f\|_{L^\infty_{x,v}}^{-1}\left(\int |x/t-v|^2 f\,dv\right)^{1/a}\left(\int f\,dv\right)^{1-1/a}\right)^{aN/(aN+2)} \|f\|_{L^\infty_{x,v}}\,,
\end{aligned}
$$

and from the Hölder inequality,

$$
\begin{aligned}
&\|\int f\,dv\|_{L_x^{(aN+2)/(aN)}} \\
&\leq C\|f\|_{L^\infty_{x,v}}^{2/(aN+2)}\left(\int\left(\int |x/t-v|^2 f\,dv\right)^{1/a}\left(\int f\,dv\right)^{1-1/a} dx\right)^{aN/(aN+2)} \\
&\leq C\left(\|f\|_{L^\infty_{x,v}}^{2/(aN)}\||x/t-v|^2 f\|_{L^1_{x,v}}^{1/a}\|f\|_{L^1_{x,v}}^{1-1/a}\right)^{aN/(aN+2)}.
\end{aligned}
$$

Putting $q=(aN+2)/(aN)$ (i.e. $a=2/(N(q-1))$), we obtain that for $1\leq q\leq 1+2/N$,

$$
\|\int f\,dv\|_{L_x^q}\leq C\left(\|f\|_{L^\infty_{x,v}}^{q-1}\||x/t-v|^2 f\|_{L^1_{x,v}}^{\frac{N}{2}(q-1)}\|f\|_{L^1_{x,v}}^{1-\frac{N}{2}(q-1)}\right)^{1/q}.
$$

Here, we note that $\|f\|_{L^1_{x,v}}=\|\phi\|_{L^1_{x,v}}$, indeed, $\frac{d}{dt}\|f\|_{L^1_{x,v}}=\iint \partial_t f\,dvdx=-\iint(v\cdot\nabla_x f+E\cdot\nabla_v f)\,dvdx=0$. And, f is a constant along characteristics, indeed, since f is an integral of the system of ordinary differential equations

$$
\dot{X}=V\,,\quad \dot{V}=E(X,t)\,,\quad t\geq 0\,,
$$

f satisfies that

$$
f(X(t),V(t),t)=f(X(0),V(0),0)=\phi(X(0),V(0))\,,\quad t\geq 0\,,
$$

and hence, $\|f\|_{L^\infty_{x,v}}\leq\|\phi\|_{L^\infty_{x,v}}$ (see [9], [10]).

Thus, we have that for $1\leq q\leq 1+2/N$,

$$
\|\int f\,dv\|_{L_x^q}\leq C_1\||x/t-v|^2 f\|_{L^1_{x,v}}^{\frac{N}{2}(1-1/q)}\,,
$$

and from (2.1),

$$
\|\rho(t)\|_{L_x^q}=\|\int f\,dv\|_{L_x^q}\leq C_1 t^{-N(1-1/q)}\,,\quad t>0\,,
$$

which implies the estimate (1.7), where $C_1=C_1(\|(1+|x|^2)\phi\|_{L^1_{x,v}}\,,\|\phi\|_{L^\infty_{x,v}})$ is a constant depending on $\|(1+|x|^2)\phi\|_{L^1_{x,v}}$ and $\|\phi\|_{L^\infty_{x,v}}$.

Moreover, using Lemma 2.1 with $\lambda = N - 1$, we obtain

$$\begin{aligned}\|E(t)\|_{L^p_x} &\le C\|\frac{x}{|x|^N} * \rho(t)\|_{L^p_x} \le C\|\rho(t)\|_{L^q_x} \\ &\le C_1 t^{-N(1-1/N-1/p)}\,, \quad t > 0\end{aligned}$$

with $1/p = 1/q - 1/N$, $1 < q \le (N+2)/N$, i.e. $N/(N-1) < p \le N(N+2)/(N^2 - N - 2)$, which implies the estimate (1.8). □

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