# On Partitions and $k-$Polygons

By

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**Abstract**

Let $C$ be a circle divided into $n$ parts equally. The set of the ends of these parts on $C$ are denoted by $S = \{P_0, P_1, \ldots, P_{n-1}\}$. Let $C_k(n)$ be the number of incongruent $k-$polygons inscribed in $C$, where the vertices of $k-$polygons are chosen from $S$. In this note, we shall investigate the generating functions of $C_4(n)$ and $C_5(n)$.

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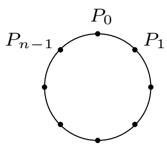
# Introduction

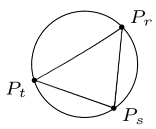
Let $C$ be a circle divided into $n$ parts equally and the ends of parts be labeled $S = \{P_0, P_1, \ldots, P_{n-1}\}$ as in the following Figure 1:

**Figure 1**



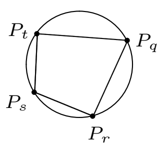

$S = \{P_0, P_1, \ldots, P_{n-1}\}$

**Figure 2**




$P_r, P_s, P_t \in S$

**Figure 3**




$P_q, P_r, P_s, P_t \in S$

Let $C_3(n)$ be the number of incongruent triangles $\triangle P_rP_sP_t$ with vertices $P_r, P_s, P_t$ chosen from $S = \{P_0, P_1, \ldots, P_{n-1}\}$ as in Figure 2. In our prvious paper [4], we have proved

$$C_3(n) = p(n-3, 3),$$

where $p(n, m)$ denotes the number of partitions of $n$ with each part $\leq m$, that is

$$p(n, m) = p(n \mid \text{parts in } \{1, 2, \ldots, m\}).$$

It is well known that $p(n, 1) = 1$ and $p(n, 2) = \left[\dfrac{n}{2}\right] + 1$, where $[x]$ denotes the greatest integer $\leq x$. In the case $m = 3$, it has been known that(see for example [3] Chapter 3)

$$p(n, 3) = \left\{\frac{(n+3)^2}{12}\right\},$$

where $\{x\}$ denotes the nearest integer to $x$. One can easily verify that $p(n, m)$ has the following generating function:

$$\sum_{n=0}^{\infty} p(n, m)q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

Let $C_k(n)$ be the number of incongruent $k-$polygons $P_{i_1}P_{i_2}\ldots P_{i_k}$ inscribed in $C$, where the vertices $\{P_{i_1}, P_{i_2}, \ldots, P_{i_k}\}$ $(1 \leq k \leq n)$ are chosen from $S$. In our previous paper [4], we have shown:

$$C_k(n) = p(n-k, k) \quad \text{for } 1 \leq k \leq 3.$$

and

$$C_4(n) \neq p(n-4, 4), \text{ and } C_4 \geq p(n-4, 4) \quad \text{for small } n.$$

In this paper, we shall show

$$C_4(n) > p(n-4, 4) \quad \text{for } n \geq 6,$$

and more precisely,

$$C_4(n) = p(n-4, 4) + p(n-6, 4) + p(n-8, 4),$$

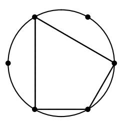
where $p(n, m)$ are defined to be 0 for $n < 0$. Moreover, we shall show $C_5(n)$ has much complicated representation in the last section.

# Quadratics

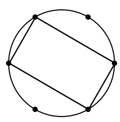
Here we shall introduce several notations. Let $S = \{P_0, P_1, \ldots, P_{n-1}\}$ be as above. Let $\wp_k$ be the set of $k-$polygons where the vertices $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ $(0 \leq$

$i_1 < i_2 < \cdots < i_k \leq n-1$) are took from $S$. Put $a_1 = i_2 - i_1, a_2 = i_3 - i_2, \ldots, a_{k-1} = i_k - i_{k-1}$ and $a_k = n + i_1 - i_k$. Let $O$ be the center of the given circle $C$. Then the central angles of $k-$polygon $P_{i_1}P_{i_2}\cdots P_{i_k}$ are $\angle P_{i_j}OP_{i_{j+1}} = \dfrac{2\pi a_j}{n}$ $(1 \leq j \leq k-1)$ and $\angle P_{j_k}OP_{i_1} = \dfrac{2\pi a_k}{n}$.
Then $a_1, a_2, \ldots, a_k$ satisfies $a_1, a_2, \ldots, a_k \geq 1$ and $a_1 + a_2 + \cdots + a_k = n$. In the following, we shall denote the $k-$polygon $P_0P_{a_1}P_{a_1+a_2}\cdots P_{a_1+a_2+\cdots+a_{k-1}}$ by $P(a_1, a_2, \ldots, a_k)$. Then any $k-$polygon $P_{i_1}P_{i_2}\cdots P_{i_k}$ is congruent to the $k-$polygon $P(a_1, a_2, \ldots, a_k)$, where $a_1 = i_2 - i_1, \ldots, a_{k-1} = i_k - i_{k-1}$ and $a_k = n + i_1 - i_k$ as above.
We shall use the notation $\cong$ when two $k-$polygons $P(a_1, \ldots, a_k)$ and $P(b_1, \ldots, b_k)$ are congruent.

In our previous paper [4], we noted that $P(a_1, a_2, a_3) \cong P(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$ for any permutation $\sigma \in S_3$. Here we note that these properties do not hold for the cases $k \geq 4$. For example, let us verify the case $n = 6$ and $k = 4$. Then a partition $6 = 2 + 2 + 1 + 1$, corresponds to the following two incongruent quadrangles $P(2, 2, 1, 1)$ and $P(2, 1, 2, 1)$.



$P(2, 2, 1, 1)$



$P(2, 1, 2, 1)$

Hence, in the case of quadrangles, we must consider the congruent classes of quadrangles which correspond to the same partition $n = a + b + c + d$. Thus we shall consider the following 8 cases of the partitions of $n$ separately.

(1) $P(n \mid n = a + b + c + d,$ with $a > b > c > d \geq 1)$,
(2) $P(n \mid n = 2a + b + c,$ with $a > b > c \geq 1)$,
(3) $P(n \mid n = a + 2b + c,$ with $a > b > c \geq 1)$,
(4) $P(n \mid n = a + b + 2c,$ with $a > b > c \geq 1)$,
(5) $P(n \mid n = 2a + 2b,$ with $a > b \geq 1)$,
(6) $P(n \mid n = 3a + b,$ with $a > b \geq 1)$,
(7) $P(n \mid n = a + 3b,$ with $a > b \geq 1)$,
(8) $P(n \mid n = 4a,$ with $a \geq 1)$.

Firstly, we shall consider the case (1).
It is easy to see that each partition $n = a + b + c + d$ corresponds to exactly 3 incongruent quadrangles $P(a, b, c, d), P(a, c, d, b)$ and $P(a, d, b, c)$. On the other hand, we have

$$\begin{aligned} &P(n \mid n = a + b + c + d, \text{ with } a > b > c > d \geq 1) \\ &\quad = P(n - 10 \mid n - 10 = (a-4) + (b-3) + (c-2) + (d-1), \\ &\qquad\qquad \text{with } a - 4 \geq b - 3 \geq c - 2 \geq d - 1 \geq 0) \end{aligned}$$

$$
\begin{aligned}
&= P(n-10 \mid n-10 = 4x+3y+2z+w, \text{ with } x,y,z,w \geq 0) \\
&= P(n-10 \mid \text{parts in } \{1,2,3,4\}).
\end{aligned}
$$

Thus the generating function of the partition $P(n \mid n = a+b+c+d, \text{ with } a > b > c > d \geq 1)$ is

$$\sum_{n=0}^{\infty} P(n \mid n = a+b+c+d, \text{ with } a > b > c > d \geq 1) q^n = \frac{q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)}.$$

Now we denote the subset of congruent classes of quadrangles $\wp_4/\cong$ which correspond to this case (1), by $\overline{\wp_4}(a,b,c,d)$. We also denote the number of congruent classes $\overline{\wp_4}(a,b,c,d)$ by $C_4(a,b,c,d)$. Since $C_4(a,b,c,d) = 3P(n-10 \mid \text{parts in } \{1,2,3,4\})$, we have

$$\sum_{n=0}^{\infty} C_4(a,b,c,d) q^n = \frac{3q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)}.$$

Now we shall consider the case (2), similarly.
It is easy to see that each partition $n = 2a+b+c$ corresponds to exactly 2 incongruent quadrangles $P(a,a,b,c)$ and $P(a,b,a,c)$. Moreover we have

$$
\begin{aligned}
P(n \mid n = 2a+b+c, \text{ with } a > b > c \geq 1) \\
&= P(n-9 \mid n-9 = 2(a-3)+(b-2)+(c-1), \\
&\qquad \text{with } a-3 \geq b-2 \geq c-1 \geq 0) \\
&= P(n-9 \mid \text{parts in } \{2,3,4\}).
\end{aligned}
$$

Thus the generating function of the partition $P(n \mid n = 2a+b+c, \text{ with } a > b > c \geq 1)$ is

$$\sum_{n=0}^{\infty} P(n \mid n = 2a+b+c, \text{ with } a > b > c \geq 1) q^n = \frac{q^9}{(1-q^2)(1-q^3)(1-q^4)}.$$

We denote the subset of congruent classes of quadrangles $\wp_4/\cong$ which correspond to this case (2), by $\overline{\wp_4}(a,a,b,c)$. We also denote the number of congruent classes of $\overline{\wp_4}(a,a,b,c)$ by $C_4(a,a,b,c)$. Since $C_4(a,a,b,c) = 2P(n-9 \mid \text{parts in } \{2,3,4\})$, we have

$$\sum_{n=0}^{\infty} C_4(a,a,b,c) q^n = \frac{2q^9}{(1-q^2)(1-q^3)(1-q^4)}.$$

Now we shall consider the case (3), similarly and define the symbols $\overline{\wp_4}(a,b,b,c) = C_4(a,a,b,c)$ as above. Then each partition $n = a+2b+c$ corresponds to exactly 2 incongruent quadrangles $P(a,b,b,c)$ and $P(a,b,c,b)$. The generating function of the partition $P(n \mid n = a+2b+c, \text{ with } a > b > c \geq 1)$ is

$$\sum_{n=0}^{\infty} P(n \mid n = a+2b+c, \text{ with } a > b > c \geq 1) q^n = \frac{q^8}{(1-q)(1-q^3)(1-q^4)}.$$

Since $C_4(a,b,b,c) = 2P(n-8 \mid \text{parts in } \{1,3,4\})$, we have

$$\sum_{n=0}^{\infty} C_4(a,b,b,c)q^n = \frac{2q^8}{(1-q)(1-q^3)(1-q^4)}.$$

In the case (4), we shall use the similar definition of the symbols $\overline{\wp_4}(a,b,c,c) = C_4(a,b,c,c)$ as above. Since each partition $n = a+b+2c$ corresponds to exactly 2 incongruent quadrangles $P(a,b,c,c)$ and $P(a,c,b,c)$, we have $C_4(a,b,c,c) = 2P(n-7 \mid \text{parts in } \{1,2,4\})$. Hence

$$\sum_{n=0}^{\infty} C_4(a,b,c,c)q^n = \frac{2q^7}{(1-q)(1-q^2)(1-q^4)}.$$

In the case (5), we shall use the similar definition of the symbols $\overline{\wp_4}(a,a,b,b) = C_4(a,a,b,b)$ as above. Since each partition $n = 2a+2b$ corresponds to exactly 2 incongruent quadrangles $P(a,a,b,b)$ and $P(a,b,a,b)$, we have $C_4(a,a,b,b) = 2P(n-6 \mid \text{parts in } \{2,4\})$. Thus we have

$$\sum_{n=0}^{\infty} C_4(a,a,b,b)q^n = \frac{2q^6}{(1-q^2)(1-q^4)}.$$

In the case (6), we shall use the similar definition of the symbols $\overline{\wp_4}(a.a,a,b) = C_4(a,a,a,b)$ as above. Since each partition $n = 3a+b$ corresponds to exactly one quadrangle $P(a,a,a,b)$, we have $C_4(a,a,a,b) = P(n-7 \mid \text{parts in } \{3,4\})$. Thus we have

$$\sum_{n=0}^{\infty} C_4(a,a,a,b)q^n = \frac{q^7}{(1-q^3)(1-q^4)}.$$

In the case (7), we shall use the similar definition of the symbols $\overline{\wp_4}(a.b,b,b) = C_4(a,b,b,b)$ as above. Then each partition $n = a+3b$ corresponds to exactly one quadrangle $P(a,b,b,b)$ and $C_4(a,b,b,b) = P(n-5 \mid \text{parts in } \{1,4\})$. Thus we have

$$\sum_{n=0}^{\infty} C_4(a,b,b,b)q^n = \frac{q^5}{(1-q)(1-q^4)}.$$

In the case (8), we shall use the similar definition of the symbols $\overline{\wp_4}(a,a,a,a) = C_4(a,a,a,a)$ as above. Then each partition $n = 4a$ corresponds to exactly one quadrangle $P(a,a,a,a)$ and $C_4(a,a,a,a) = P(n-4 \mid \text{part in } \{4\})$. Thus we have

$$\sum_{n=0}^{\infty} C_4(a,a,a,a)q^n = \frac{q^4}{1-q^4}.$$

From the definition of the symbols, we have

$$\begin{aligned}\textstyle\sum_{n=0}^{\infty} C_4(n)q^n &= \textstyle\sum_{n=0}^{\infty} C_4(a,b,c,d)q^n + \sum_{n=0}^{\infty} C_4(a,a,b,c)q^n \\ &\quad \textstyle + \sum_{n=0}^{\infty} C_4(a,b,b,c)q^n + \sum_{n=0}^{\infty} C_4(a,b,c,c)q^n \\ &\quad \textstyle + \sum_{n=0}^{\infty} C_4(a,a,b,b)q^n + \sum_{n=0}^{\infty} C_4(a,a,a,b)q^n \\ &\quad \textstyle + \sum_{n=0}^{\infty} C_4(a,b,b,b)q^n + \sum_{n=0}^{\infty} C_4(a,a,a,a)q^n.\end{aligned}$$

Thus we have

$$\begin{aligned}&\textstyle\sum_{n=0}^{\infty} C_4(n)q^n \\ &= \frac{3q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \frac{2q^9}{(1-q^2)(1-q^3)(1-q^4)} \\ &\quad + \frac{2q^8}{(1-q)(1-q^3)(1-q^4)} + \frac{2q^7}{(1-q)(1-q^2)(1-q^4)} \\ &\quad + \frac{2q^6}{(1-q^2)(1-q^4)} + \frac{q^7}{(1-q^3)(1-q^4)} + \frac{q^5}{(1-q)(1-q^4)} + \frac{q^4}{1-q^4} \\ &= \frac{q^4}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \frac{2q^{10}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\ &\quad + \frac{q^9}{(1-q^2)(1-q^3)(1-q^4)} + \frac{q^8}{(1-q)(1-q^3)(1-q^4)} \\ &\quad + \frac{q^7}{(1-q)(1-q^2)(1-q^4)} + \frac{q^6}{(1-q^2)(1-q^4)} \\ &= \frac{q^4+q^6+q^8}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\ &= \sum_{n=0}^{\infty}(p(n-4,4)+p(n-6,4)+p(n-8,4))q^n,\end{aligned}$$

where $p(m,4)=0$ for $m<0$.

Hence we have shown the following theorem:

**Theorem 1**. *With the above notation, we have*

$$C_4(n) = p(n-4,4)+p(n-6,4)+p(n-8,4),$$

$$\sum_{n=0}^{\infty} C_4(n)q^n = \frac{q^4+q^6+q^8}{(1-q)(1-q^2)(1-q^3)(1-q^4)}.$$

## Pentagons

In this section we shall briefly show the generating function of $C_5(n)$.

In the case of pentagons, we must also consider the congruent classes of pentagons which correspond to the same partition $n = a + b + c + d + e$. For the sake of simplisity, we shall borrow the name of poker hands. Thus we shall treat the following 16 cases of the partitions of $n$ separately.

(1) High card
$P(n \mid n = a + b + c + d + e, \text{ with } a > b > c > d > e \geq 1)$,

(2) One pair
(2-1) $P(n \mid n = 2a + b + c + d, \text{ with } a > b > c > d \geq 1)$,
(2-2) $P(n \mid n = a + 2b + c + d, \text{ with } a > b > c > d \geq 1)$,
(2-3) $P(n \mid n = a + b + 2c + d, \text{ with } a > b > c > d \geq 1)$,
(2-4) $P(n \mid n = a + b + c + 2d, \text{ with } a > b > c > d \geq 1)$,

(3) Two pair
(3-1) $P(n \mid n = 2a + 2b + c, \text{ with } a > b > c \geq 1)$,
(3-2) $P(n \mid n = a + 2b + 2c, \text{ with } a > b > c \geq 1)$,
(3-3) $P(n \mid n = 2a + b + 2c, \text{ with } a > b > c \geq 1)$,

(4) Three card
(4-1) $P(n \mid n = 3a + b + c, \text{ with } a > b > c \geq 1)$,
(4-2) $P(n \mid n = a + 3b + c, \text{ with } a > b > c \geq 1)$,
(4-3) $P(n \mid n = a + b + 3c, \text{ with } a > b > c \geq 1)$,

(5) Full house
(5-1) $P(n \mid n = 3a + 2b, \text{ with } a > b \geq 1)$,
(5-2) $P(n \mid n = 2a + 3b, \text{ with } a > b \geq 1)$,

(6) Four card
(6-1) $P(n \mid n = 4a + b, \text{ with } a > b \geq 1)$,
(6-2) $P(n \mid n = a + 4b, \text{ with } a > b \geq 1)$,

(7) Five card
$P(n \mid n = 5a, \text{ with } a \geq 1)$.

Now we shall verify that the number of incongruent pentagons which correspond to the same partition $n = a + b + c + d + e$.

In the case of high card (1), we see there are 12 incongruent pentagons

$$P(a,b,c,d,e), P(a,b,d,c,e), P(a,b,d,e,c), P(a,b,e,d,c),$$
$$P(a,b,e,c,d), P(a,b,c,e,d), P(a,c,b,d,e), P(a,c,d,b,e),$$
$$P(a,c,b,e,d), P(a,c,e,b,d), P(a,d,b,c,e), P(a,d,c,b,e).$$

In the case of one pair (2), we see there are 6 incongruent pentagons for the same partition. Her we shall list up for the case $(2-1)$, i.e., the case $n = 2a + b + c + d$. Then there are 6 incongruent pentagons

$$P(a, a, b, c, d), P(a, a, c, d, b), P(a, a, d, b, c), P(a, b, a, c, d),$$
$$P(a, c, a, b, d), P(a, d, a, b, c).$$

In the case of two pair (3), we see there are 4 incongruent pentagons for the same partition. Her we shall list up for the case $(3-1)$, i.e., the case $n = 2a + 2b + c$. Then there are 4 incongruent pentagons

$$P(c, a, a, b, b), P(c, a, b, a, b), P(c, a, b, b, a), P(c, b, a, a, b).$$

In the case of three card (4), we see there are 2 incongruent pentagons for the same partition. Her we shall list up for the case $(4-1)$, that is for the case $n = 3a + b + c$. Then there are 2 incongruent pentagons

$$P(a, a, a, b, c), P(a, a, b, a, c).$$

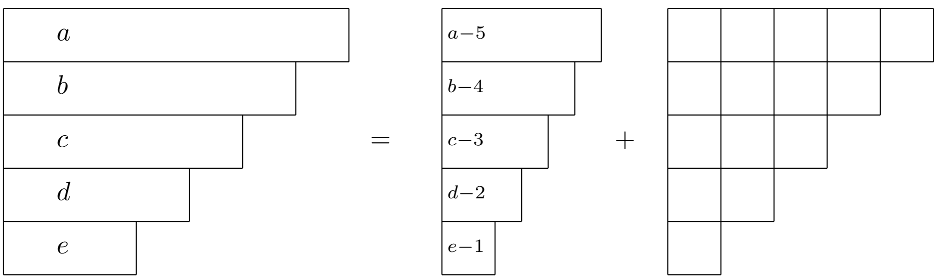
In the case of full house (5), we see there are 2 incongruent pentagons for the same partition. Her we shall list up for the case $(5-1)$, that is for the case $n = 3a + 2b$. Then there are 2 incongruent pentagons

$$P(a, a, a, b, b), P(a, a, b, a, b).$$

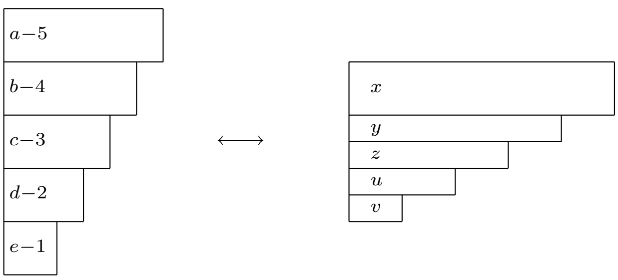
In the cases four card and five card, there is exactly one pentagon which corresponds to each partition.

**The case (1)**

Firstly we shall consider the case of high card. Then the Young diagram of the partition $n = a + b + c + d + e$ with $a > b > c > d > e \geq 1$ satisfies the following relations.




Put $x = e - 1, y = d - e - 1, z = c - d - 1, u = b - c - 1, v = a - b - 1$. Then $x, y, z, u, v \geq 0$ and the following conjugate of Young diagrams impies:



Thus we know that

$$
\begin{aligned}
&P(n \mid n = a+b+c+d+e,\ \text{with } a>b>c>d>e \geq 1)\\
&\quad = P(n-15 \mid n-15 = (a-5)+(b-4)+(c-3)+(d-2)+(e-1),\\
&\qquad\qquad \text{with } a-5 \geq b-4 \geq c-3 \geq d-2 \geq e-1 \geq 0)\\
&\quad = P(n-15 \mid n-15 = 5x+4y+3z+2u+v,\ \text{with } x,y,z,u,v \geq 0)\\
&\quad = P(n-15 \mid \text{parts in } \{1,2,3,4,5\}).
\end{aligned}
$$

Hence the generating function of the partition
$P(n \mid n = a+b+c+d+e,\ \text{with } a>b>c>d>e \geq 1)$ is

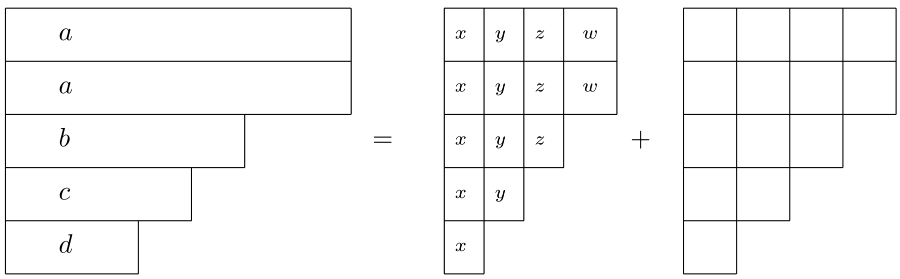
$$
\begin{aligned}
&\sum_{n=0}^{\infty} P(n \mid n = a+b+c+d+e,\ \text{with } a>b>c>d>e \geq 1)q^n\\
&\quad = \frac{q^{15}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}.
\end{aligned}
$$

We denote the number of incongruent pentagons which corresponds to the partition $n = a+b+c+d+e$ by $C_5(a,b,c,d,e)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a,b,c,d,e)q^n = \frac{12q^{15}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}.$$

**The case (2-1)**

Now we shall consider the case of one pair (2-1). We know the partition $n = 2a+b+c+d$ with $a>b>c>d \geq 1$ corresponds to




Here $x = d-1, y = c-d-1, z = b-c-1, w = a-b-1$. Then $x,y,z,w \geq 0$ and the conjugation of Young diagrams imply that $n-14 = 5x+4y+3z+2w$ with $x,y,z,w \geq 0$.
Thus we know that

$$
\begin{aligned}
&P(n \mid n = 2a+b+c+d,\ \text{with } a>b>c>d \geq 1)\\
&\quad = P(n-14 \mid n-14 = 2(a-4)+(b-3)+(c-2)+(d-1),\\
&\qquad\qquad \text{with } a-4 \geq b-3 \geq c-2 \geq d-1 \geq 0)\\
&\quad = P(n-14 \mid n-14 = 5x+4y+3z+2w,\ \text{with } x,y,z,w \geq 0)\\
&\quad = P(n-14 \mid \text{parts in } \{2,3,4,5\}).
\end{aligned}
$$

Hence the generating function of the partition $P(n \mid n = 2a+b+c+d,\ \text{with } a > b > c > d \geq 1)$ is

$$\sum_{n=0}^{\infty} P(n \mid n = 2a+b+c+d,\ \text{with } a > b > c > d \geq 1)q^n$$
$$= \frac{q^{14}}{(1-q^2)(1-q^3)(1-q^4)(1-q^5)}.$$

We denote the number of incongruent pentagons which corresponds to the partition $n = 2a+b+c+d$ by $C_5(a,a,b,c,d)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a,a,b,c,d)q^n = \frac{6q^{14}}{(1-q^2)(1-q^3)(1-q^4)(1-q^5)}.$$

**The case (2-2)**

For the case of one pair (2-2), we see that $n = a+2b+c+d$ with $a > b > c > d \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a+2b+c+d$ by $C_5(a,b,b,c,d)$. Then similary as above, we have

$$\sum_{n=0}^{\infty} C_5(a,b,b,c,d)q^n = \frac{6q^{13}}{(1-q)(1-q^3)(1-q^4)(1-q^5)}.$$

**The case (2-3)**

Consider the case of one pair (2-3) similarly. Then $n = a+b+2c+d$ with $a > b > c > d \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a+b+2c+d$ by $C_5(a,b,c,c,d)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a,b,c,c,d)q^n = \frac{6q^{12}}{(1-q)(1-q^2)(1-q^4)(1-q^5)}.$$

**The case (2-4)**

Consider the case of one pair (2-4). Then $n = a+b+c+2d$ with $a > b > c > d \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a+b+c+2d$ by $C_5(a,b,c,d,d)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a,b,c,d,d)q^n = \frac{6q^{11}}{(1-q)(1-q^2)(1-q^3)(1-q^5)}.$$

**The case (3-1)**

Consider the case of two pair (3-1). Then $n = 2a+2b+c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 2a+2b+c$ by $C_5(a,a,b,b,c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a,a,b,b,c)q^n = \frac{4q^{11}}{(1-q^2)(1-q^4)(1-q^5)}.$$

**The case (3-2)**

Consider the case of two pair (3-2), similarly. Then $n = 2a + b + 2c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 2a + b + 2c$ by $C_5(a, a, b, c, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, b, c, c)q^n = \frac{4q^{10}}{(1-q^2)(1-q^3)(1-q^5)}.$$

**The case (3-3)**

Consider the case of two pair (3-3). Then $n = a + 2b + 2c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + 2b + 2c$ by $C_5(a, b, b, c, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, b, c, c)q^n = \frac{4q^9}{(1-q)(1-q^3)(1-q^5)}.$$

**The case (4-1)**

Now we shall consider the case of three card (4-1). Then $n = 3a + b + c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 3a + b + c$ by $C_5(a, a, a, b, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, a, b, c)q^n = \frac{2q^{12}}{(1-q^3)(1-q^4)(1-q^5)}.$$

**The case (4-2)**

Consider the case of three card (4-2). Then $n = a + 3b + c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + 3b + c$ by $C_5(a, b, b, b, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, b, b, c)q^n = \frac{2q^{10}}{(1-q)(1-q^4)(1-q^5)}.$$

**The case (4-3)**

Consider the case of three card (4-3). Then $n = a + b + 3c$ with $a > b > c \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + b + 3c$ by $C_5(a, b, c, c, c)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, c, c, c)q^n = \frac{2q^8}{(1-q)(1-q^2)(1-q^5)}.$$

**The case (5-1)**

Now we shall consider the case of full house (5-1). Then $n = 3a + 2b$ with $a > b \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 3a + 2b$ by $C_5(a, a, a, b, b)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, a, b, b)q^n = \frac{2q^8}{(1 - q^3)(1 - q^5)}.$$

**The case (5-2)**

Consider the case of full house (5-2). Then $n = 2a + 3b$ with $a > b \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 2a + 3b$ by $C_5(a, a, b, b, b)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, b, b, b)q^n = \frac{2q^7}{(1 - q^2)(1 - q^5)}.$$

**The case (6-1)**

Now we shall consider the case of four card (6-1). Then $n = 4a + b$ with $a > b \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 4a + b$ by $C_5(a, a, a, a, b)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, a, a, b)q^n = \frac{q^9}{(1 - q^4)(1 - q^5)}.$$

**The case (6-2)**

Consider the case of four card (6-2). Then $n = a+4b$ with $a > b \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = a + 4b$ by $C_5(a, b, b, b, b)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, b, b, b, b)q^n = \frac{q^6}{(1 - q)(1 - q^5)}.$$

**The case (7)**

Now we shall consider the case of five card (7). Then $n = 5a$ with $a \geq 1$. Denote the number of incongruent pentagons which corresponds to the partition $n = 5a$ by $C_5(a, a, a, a, a)$. Then we have

$$\sum_{n=0}^{\infty} C_5(a, a, a, a, a)q^n = \frac{q^5}{(1 - q^5)}.$$

From the definition of the symbols, we have

$$
\begin{aligned}
&\textstyle\sum_{n=0}^{\infty} C_5(n)q^n \\
&= \frac{12q^{15}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)} + \frac{6q^{14}}{(1-q^2)(1-q^3)(1-q^4)(1-q^5)} \\
&\quad + \frac{6q^{13}}{(1-q)(1-q^3)(1-q^4)(1-q^5)} + \frac{6q^{12}}{(1-q)(1-q^2)(1-q^4)(1-q^5)} \\
&\quad + \frac{6q^{11}}{(1-q)(1-q^2)(1-q^3)(1-q^5)} + \frac{4q^{11}}{(1-q^2)(1-q^4)(1-q^5)} \\
&\quad + \frac{4q^{10}}{(1-q^2)(1-q^3)(1-q^5)} + \frac{4q^{9}}{(1-q)(1-q^3)(1-q^5)} \\
&\quad + \frac{2q^{12}}{(1-q^3)(1-q^4)(1-q^5)} + \frac{2q^{10}}{(1-q)(1-q^4)(1-q^5)} \\
&\quad + \frac{2q^{8}}{(1-q)(1-q^2)(1-q^5)} + \frac{2q^{8}}{(1-q^3)(1-q^5)} + \frac{2q^{7}}{(1-q^2)(1-q^5)} \\
&\quad + \frac{q^{9}}{(1-q^4)(1-q^5)} + \frac{q^{6}}{(1-q)(1-q^5)} + \frac{q^{5}}{1-q^5} \\
&= \frac{q^5+q^7+q^8+2q^9+2q^{10}+2q^{11}+q^{12}+q^{13}+q^{15}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}
\end{aligned}
$$

Hence we have shown the following theorem:

**Theorem 2**. *With the above notation, we have the formulae*

$$\sum_{n=0}^{\infty} C_5(n)q^n = \frac{q^5+q^7+q^8+2q^9+2q^{10}+2q^{11}+q^{12}+q^{13}+q^{15}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}.$$

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