LETTER

A Computation of Bifurcation Parameter Values for Limit Cycles

Tetsushi UETA[†], ***Member,*** **Masafumi TSUEIKE**[†], ***Student Member,*** **Hiroshi KAWAKAMI**[†], **Tetsuya YOSHINAGA**[†], ***and*** **Yuuji KATSUTA**[††], ***Members***

SUMMARY This letter describes a new computational method to obtain the bifurcation parameter value of a limit cycle in nonlinear autonomous systems. The method can calculate a parameter value at which local bifurcations; tangent, period-doubling and Neimark-Sacker bifurcations are occurred by using properties of the characteristic equation for a fixed point of the Poincaré mapping. Conventionally a period of the limit cycle is not used explicitly since the Poincaré mapping needs only whether the orbit reaches a cross-section or not. In our method, the period is treated as an independent variable for Newton's method, so an accurate location of the fixed point, its period and the bifurcation parameter value can be calculated simultaneously. Although the number of variables increases, the Jacobian matrix becomes simple and the recurrence procedure converges rapidly compared with conventional methods.

key words: *limit cycle, bifurcation, characteristic equation, Newton's method*

1. Introduction

In high-dimensional nonlinear autonomous systems, e.g., the coupled electric circuits, coupled neural oscillators, and cellular neural networks, a limit cycle is frequently generated by changing their system parameter. As is well known, quasi-periodic solution, chaotic attractors and other complicated phenomena are directly caused via bifurcations. The atlas of bifurcation sets in the parameter plane — the bifurcation diagram has important information to understand quickly the changing of qualitative properties for limit cycles.

To calculate the bifurcation parameter values, we should provide a cross-section called the Poincaré section and define the corresponding Poincaré mapping. The bifurcation parameter value and the location of the fixed point on the local coordinate derived from the Poincaré mapping are solved simultaneously [4]. This method implicitly requires the accurate location of the point at which the periodic orbit started from the cross-section reaches. Thus the Jacobian matrix of Newton's method becomes too complicated since the period is a dependent variable according to the initial point on the Poincaré section and the method cannot converge when the period changes widely.

In this letter we propose a new method to calculate the bifurcation parameter value. The method treats the period of the limit cycle as an independent variable for Newton's method. Its Jacobian matrix becomes simple and recurrence formula converge rapidly against the conventional method.

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[†] The authors are with the Faculty of Engineering, Tokushima University, Tokushima-shi, 770 Japan.
[††] The author is with the Department of Intelligent System Engineering, Ube National College of Technology, Ube-shi, 755 Japan.

2. Limit Cycle and the Poincaré Mapping

We consider an n-dimensional nonlinear autonomous system described as:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) \tag{1}$$

where, $t \in \boldsymbol{R}$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}^n$. Let $\boldsymbol{f} : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a C^∞ mapping for any states and parameters. A solution of Eq. (1) with an initial condition is written as follows:

$$\boldsymbol{x}(t) = \boldsymbol{\varphi}(t, \boldsymbol{x}_0), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 = \boldsymbol{\varphi}(0, \boldsymbol{x}_0) \tag{2}$$

Assume that Eq. (1) has a limit cycle with the period L. Then we define a cross-section Π which is transversal to the orbit of the limit cycle.

$$\Pi = \{\boldsymbol{x} \in \boldsymbol{R}^n \mid q(\boldsymbol{x}) = 0\} \tag{3}$$

The section Π is an $(n-1)$-dimensional hypersurface described by:

$$\begin{aligned} &q : \boldsymbol{R}^n \to \boldsymbol{R} \\ &q(\boldsymbol{x}) = 0 \end{aligned} \tag{4}$$

Since the cross-section is transversal to the periodic orbit, thus we have

$$\frac{\partial q}{\partial \boldsymbol{x}} \cdot \boldsymbol{f} \neq 0 \quad \text{for all } \boldsymbol{x} \in \Pi \tag{5}$$

We choose a suitable section which is parallel with an orthogonal coordinate of $\boldsymbol{R}^n$. Let $\tilde{\boldsymbol{x}} \in \hat{\Pi} \subset \Pi$ be the neighborhood of a point $\boldsymbol{x}$ on the cross-section. Then the Poincaré mapping T is described as:

$$\begin{aligned} T : \hat{\Pi} &\to \Pi \\ \tilde{\boldsymbol{x}} &\mapsto \boldsymbol{\varphi}(\tau(\tilde{\boldsymbol{x}}), \tilde{\boldsymbol{x}}) \end{aligned} \tag{6}$$

Let $\tau(\tilde{\boldsymbol{x}})$ be the return time which is spent during the orbit started from $\tilde{\boldsymbol{x}} \in \hat{\Pi}$ intersects to the Π again. The fixed point of the mapping is as:

$$T(\boldsymbol{x}_0) = \boldsymbol{x}_0 \tag{7}$$

Then the return time is coincident to the period of the limit cycle.

To investigate stability of the fixed point (7), we consider variations for the neighborhood of the fixed point. Let $\tilde{\boldsymbol{x}}$ be the neighborhood of the fixed point and $\boldsymbol{\xi}$ be the variation.

$$\boldsymbol{\xi}_0 = \tilde{\boldsymbol{x}} - \boldsymbol{x}_0 \tag{8}$$

By substituting this relations to Eq. (6) and applying the Taylor-expansion, we have

$$\boldsymbol{\xi}_1 = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} \boldsymbol{\xi}_0 \tag{9}$$

The Jacobian matrix $\partial\boldsymbol{\varphi}/\partial\boldsymbol{x}_0$ is a principal matrix solution obtained by integrating the following variational equation from $t = 0$ to $t = \tau(\boldsymbol{x}_0)$:

$$\frac{d}{dt}\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} \tag{10}$$

$$\left.\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0}\right|_{t=0} = I_n \tag{11}$$

where I_n is the $n \times n$ identical matrix. Stability of the fixed point depends on the roots of the characteristic equation:

$$\chi_\mu = \left| \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} - \mu I_n \right| = 0 \tag{12}$$

We attach an $(n-1)$-dimensional local coordinate Σ to the cross-section. A projection $\Pi \to \Sigma$ is defined as:

$$\begin{array}{rcl} \Pi & = & \{\boldsymbol{x} \in \boldsymbol{R}^n \mid q(\boldsymbol{x}) = 0\} \\ h : \Pi & \to & \Sigma \subset \boldsymbol{R}^{n-1} \end{array} \tag{13}$$

This projection h is often called local coordinate of Π. The embedding map h^{-1} is written as follows:

$$h^{-1} : \Sigma \to \Pi \subset \boldsymbol{R}^n \tag{14}$$

Suppose that $\boldsymbol{u} \in \Sigma \subset \boldsymbol{R}^{n-1}$ is a location on the local coordinate, then the relation $h(\boldsymbol{x}_0) = \boldsymbol{u}_0$ is held. Let $\hat{\Sigma}$ be the neighborhood of $\boldsymbol{u}_0 \in \Sigma$, $\boldsymbol{u}_1$ be a point on $\hat{\Sigma}$, and $\boldsymbol{\varphi}(t, \boldsymbol{x}_1)$ be the solution starting in $h^{-1}(\boldsymbol{u}_1) = \boldsymbol{x}_1 \in \hat{\Pi}$. Let also the $\boldsymbol{x}_2 \in \Pi$ be a point at which $\boldsymbol{\varphi}(t, \boldsymbol{x}_1)$ intersects with the return time $\tau(\boldsymbol{x}_1)$:

$$\boldsymbol{x}_2 = \boldsymbol{\varphi}(\tau(\boldsymbol{x}_1), \boldsymbol{x}_1) \tag{15}$$

Then we define the Poincaré mapping on the local coordinate system:

$$\begin{array}{rcl} T_\ell & : & \hat{\Sigma} \to \Sigma \\ \boldsymbol{u}_1 & \mapsto & \boldsymbol{u}_2 = h(\boldsymbol{\varphi}(\tau(h^{-1}(\boldsymbol{u}_1)), h^{-1}(\boldsymbol{u}_1))) \\ & & = h \circ T \circ h^{-1}(\boldsymbol{u}_1) \end{array} \tag{16}$$

The fixed point of the mapping T_ℓ is given by:

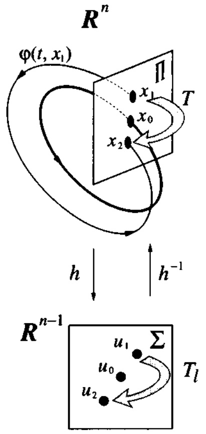

Fig. 1 Cross-section and the Poincaré mapping T_ℓ.

$$T_\ell(\boldsymbol{u}_0) = \boldsymbol{u}_0 \tag{17}$$

Note that the corresponding point $\boldsymbol{x}_0 = h^{-1}(\boldsymbol{u}_0)$ is the fixed point of mapping T. Note also that the following relationship is held.

$$q(\boldsymbol{\varphi}(\tau(h^{-1}(\boldsymbol{u}_0)), h^{-1}((\boldsymbol{u}_0))) = 0 \tag{18}$$

3. Computation of Bifurcation Parameter Value

3.1 Recurrence Formula for Newton's Method

In the case of nonautonomous systems, the bifurcation parameter value of a periodic solution is calculated easily since a period of the Poincaré mapping is invariant and identical to the period of the forcing term. While in the case of autonomous systems, the period of the limit cycle is changed as the parameter changes. Conventional method [4] has to detect an accurate location of the cross point at which the trajectory starting from Σ returns again. Usually bisection method and Runge-Kutta method are applied to calculate this location. But a lot of iterations are spent to calculate the location by the bisection method, moreover, the period is a dependent variable according to the initial point $\boldsymbol{x}_0$ on Π, so that the derivative of the Poincaré mapping becomes too complicated such that

$$DT(\boldsymbol{x}) = [I_n - (Dq(\boldsymbol{x}) \cdot \boldsymbol{f})^{-1} \cdot \boldsymbol{f} \cdot Dq(\boldsymbol{x})] \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} \tag{19}$$

The Poincaré mapping does not include information about the period explicitly. This fact affects the solvability or convergency of Newton's method, especially the method sometimes fails when the period changes widely as the system parameter varies little.

Kawakami et al. showed that the fixed point and its period can be solved simultaneously by using Newton's method [2]. Kuznetsov introduces some similar methods in Ref. [3]. In these references, the period τ

is chosen as an independent variable, so that the dependency of the initial state $\boldsymbol{x}_0$ for the Jacobian matrix Eq. (19) is simplified as:

$$DT(\boldsymbol{x}) = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} \tag{20}$$

In this letter, we apply this method to bifurcation problems. The fixed point $\boldsymbol{x}_0$, period τ_0 and bifurcation parameter value λ_0 are obtained by solving the following equation:

$$F(\boldsymbol{u}, \tau, \lambda) = \begin{pmatrix} T_\ell(\boldsymbol{u}) - \boldsymbol{u} \\ q(\boldsymbol{\varphi}(\tau, h^{-1}(\boldsymbol{u}))) \\ \chi_\mu(h^{-1}(\boldsymbol{u})) \end{pmatrix} = \boldsymbol{0} \tag{21}$$

The third equation of Eq. (21) is the characteristic equation derived from the local coordinate system by using the embedding map h^{-1}.

The Jacobian matrix of Eq. (21) are needed for Newton's method:

$$\begin{aligned} &DF(\boldsymbol{u}, \tau, \lambda) \\ &= \begin{pmatrix} DT_\ell(\boldsymbol{u}) - I_{n-1} & DT_\ell(\tau) & DT_\ell(\lambda) \\ Dq(\boldsymbol{u}) & Dq(\tau) & Dq(\lambda) \\ D\chi_\mu(\boldsymbol{u}) & D\chi_\mu(\tau) & D\chi_\mu(\lambda) \end{pmatrix} \end{aligned} \tag{22}$$

where

$$DT_\ell(\boldsymbol{u}) = Dh(\boldsymbol{x}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} \cdot Dh^{-1}(\boldsymbol{u}) \tag{23}$$

$$DT_\ell(\tau) = Dh(\boldsymbol{x}) \cdot \boldsymbol{f} \tag{24}$$

$$DT_\ell(\lambda) = Dh(\boldsymbol{x}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial \lambda} \tag{25}$$

$$Dq(\boldsymbol{u}) = Dq(\boldsymbol{x}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} \cdot Dh^{-1}(\boldsymbol{u}) \tag{26}$$

$$Dq(\tau) = Dq(\boldsymbol{x})\boldsymbol{f} \tag{27}$$

$$Dq(\lambda) = Dq(\boldsymbol{x}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial \lambda} \tag{28}$$

3.2 Properties of the Characteristic Equation and Its Derivatives

The characteristic equation $\chi_\mu(h^{-1}(\boldsymbol{u}))$ can be expanded as a polynomial such that

$$\begin{aligned} \chi_\mu = \mu^n + a_1\mu^{n-1} + a_2\mu^{n-2} + \cdots \\ \cdots + a_{n-1}\mu + a_n = 0 \end{aligned} \tag{29}$$

Since the trajectory is periodic, Eq. (29) must have at least a root whose value is unity. Thus Eq. (29) can be resolved into the factors.

$$\begin{aligned} \chi_\mu = (\mu - 1)(\mu^{n-1} + b_1\mu^{n-2} + b_2\mu^{n-3} \cdots \\ \cdots + b_{n-2}\mu + b_{n-1}) = 0 \end{aligned} \tag{30}$$

Consequently one can choose an $(n-1)$-dimensional polynomial $\bar{\chi}_\mu$ as a condition instead of Eq. (29).

$$\bar{\chi}_\mu = \mu^{n-1} + b_1\mu^{n-2} + \cdots + b_{n-2}\mu + b_{n-1} = 0$$

$$b_k = 1 + a_1 + a_2 + \cdots + a_k = 1 + \sum_{i=1}^{k} a_i \tag{31}$$

Mathematically, Eq. (31) is a universal condition to obtain codimension-one bifurcation parameter value because corresponding Jacobian matrix (22) is always non-singular even if Eq. (29) has a doubly unity root, i.e., the tangent bifurcation is occurred.

However, in high-dimensional systems, coefficients of the polynomial (29) become too complicated combinations of the variables. Newton's method needs the derivatives of all conditions for all variables, so that it is very troublesome to calculate each element in the Jacobian matrix.

After all, we propose to use the matrix form of the original characteristic equation $\chi_\mu(h^{-1}(\boldsymbol{u})) = 0$ instead of Eq. (31). For instance, the derivative of χ_μ by x_i is given by the following equation:

$$D\chi_\mu(x_i) = \sum_{k=1}^{n} \begin{vmatrix} \dfrac{\partial \varphi_1}{\partial x_1} - \mu & \dfrac{\partial \varphi_1}{\partial x_2} & \cdots & \dfrac{\partial^2 \varphi_1}{\partial x_k \partial x_i} & \cdots & \dfrac{\partial \varphi_1}{\partial x_n} \\ \dfrac{\partial \varphi_2}{\partial x_1} & \dfrac{\partial \varphi_2}{\partial x_2} - \mu & \cdots & \dfrac{\partial^2 \varphi_2}{\partial x_k \partial x_i} & \cdots & \dfrac{\partial \varphi_2}{\partial x_n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \dfrac{\partial \varphi_n}{\partial x_1} & \dfrac{\partial \varphi_n}{\partial x_2} & \cdots & \dfrac{\partial^2 \varphi_n}{\partial x_k \partial x_i} & \cdots & \dfrac{\partial \varphi_n}{\partial x_n} - \mu \end{vmatrix} \tag{32}$$

All columns of the determinant are given by solving the variational and second variational equations. Equation (32) indicates that the determinant of Jacobian matrix is calculated from only matrix operations, i.e., expansion of the characteristic Eq. (29) is not necessary. Therefore this method is quite suitable for computational algorithm.

3.3 Period-Doubling Bifurcation

Period-doubling bifurcation is obtained by solving Eq. (21) for $\boldsymbol{u}$, τ, and λ with $\mu = -1$.

3.4 Tangent Bifurcation

In the case of tangent bifurcation, The characteristic equation $\chi_\mu(h^{-1}(\boldsymbol{u})) = 0$ has a multiple root whose value is unity, thus Newton's method fails because the Jacobian matrix DF for Eq. (21) becomes singular. Then we use the following conditions:

$$F_t = \begin{pmatrix} T_\ell(\boldsymbol{u}) - \boldsymbol{u} \\ q(\boldsymbol{\varphi}(\tau, h^{-1}(\boldsymbol{u}))) \\ \dfrac{\partial \chi_\mu}{\partial \mu}(h^{-1}(\boldsymbol{u})) \end{pmatrix} = \boldsymbol{0} \tag{33}$$

Table 1 Bifurcation of the limit cycle in Eq. (35).

Bifurcation	c	x_0	z_0	τ	μ_1	μ_2	μ_3
Period doubling	5.42787	3.50257	3.06504	6.50433	−1.00002	0.0	1.0
Tangent bifurcation	9.84900	5.57657	3.77715	13.63128	0.99991	0.0	1.0

with $\mu = 1$. Each factor of the Jacobian matrix of Newton's method is given by solving variational equations. $\partial D\chi_\mu(\boldsymbol{x})/\partial\mu$ is a derivative of the characteristic equation and it can be obtained as matrix form like Eq. (32).

4. An Example

As an illustrated example of the method, we consider a Rössler equation described by the following equation:

$$\begin{aligned} \frac{dx}{dt} &= -y - z \\ \frac{dy}{dt} &= x + 0.4y \\ \frac{dz}{dt} &= x - cz + xz \end{aligned} \tag{34}$$

Now we show briefly numerical results of a limit cycle observed in Eq. (35). Assume that we can choose a cross-section $y = 0$ for a limit cycle:

$$\Pi = \{\boldsymbol{x} \in \boldsymbol{R}^3 \mid q(\boldsymbol{x}) = y = 0\} \tag{35}$$

By solving Eqs. (21) and (33), we obtain parameter values of the period doubling and tangent bifurcation of the limit cycle. Table 1 shows their values. Iteration of these Newton's recurrent formula is a few times and accuracy is given by their value of μ_i, $i = 1, 2, 3$. We would like to report about their convergence abilities in future.

5. Conclusions

We proposed a method to obtain the bifurcation parameter value and investigate its mathematical preliminaries. Similar method is introduced in Ref. [3], however, the method uses a periodic boundary-value problem which needs some analytical considerations. Our method is simple and universal for any nonlinear autonomous systems given by Eq. (1) without analytical information.

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