q-Stokes Phenomenon of a Basic Hypergeometric Series ${}_1\phi_1(0; a; q, x)$

By

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Abstract

We show a connection formula of a linear q-differential equation satisfied by ${}_1\phi_1(0; a; q, x)$. The basic hypergeometric series ${}_1\phi_1(0; a; q, x)$ represents the Hahn-Exton q-Bessel function. Since the q-differential equation has a divergent series solution, a q-analogue of the Stokes phenomenon appears. We give a resummation procedure of the divergent series by means of the q-Borel-Laplace transformation of order 1/2.

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Introduction

We study the following q-difference equation

$$ay(qz) + [z - (a + q)]y(z) + qy(z/q) = 0, \tag{1}$$

which has a solution $y(z) = {}_1\phi_1(0; a; q, z)$. We assume that $a \neq 0$. The basic hypergeometric series ${}_1\phi_1(0; a; q, z)$ is related to the Hahn-Exton q-Bessel function [15], one of the three different types of Jackson's q-analogue of the Bessel function. We solve the connection problem of (1), which gives relations between solutions around the origin and solutions around the infinity. Since (1) has a solution represented by a divergent power series around the infinity, a q-analogue of the Stokes phenomenon appears when we give a resummation of the divergent series. We show a resummation of the divergent solution by means of the q-Borel-Laplace transformation of order 1/2, which is studied by Dreyfus and Eloy [1, 2].

It is known that there exist three different types of q-analogues of the Bessel function. Jackson defines his first q-analogue of the Bessel functions in [6], and the second q-Bessel function is introduced in [7]. Following the modern notation by Ismail [5], we denote

$$\begin{aligned} J_\nu^{(1)}(x;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_2\phi_1\left(0,0;q^{\nu+1};q,-\frac{x^2}{4}\right), \\ J_\nu^{(2)}(x;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_0\phi_1\left(-;q^{\nu+1};q,-\frac{q^{\nu+1}x^2}{4}\right), \\ J_\nu^{(3)}(x;q) &= \frac{(q^{2\nu+2};q^2)_\infty}{(q^2;q^2)_\infty} x^\nu {}_1\phi_1\left(0;q^{2\nu+2};q^2,x^2q^2\right). \end{aligned}$$

Since the third one is found by Hahn [4] and Exton [3], it is called the Hahn-Exton q-Bessel function, which satisfies the q-difference equation

$$f(xq^2) + q^{-\nu}(x^2q^2 - 1 - q^{2\nu})f(xq) + f(x) = 0. \tag{2}$$

Any linear q-difference equation has two singular points, the origin and the infinity. Local solutions around each singular point are represented by a product of theta functions and a formal power series. A connection problem of a linear q-difference equation is to give a relation between the system of local solutions at the origin and the infinity. When the power series is divergent, the the q-Stokes phenomenon appears by a resummation procedure.

A connection formula of the first Jackson q-Bessel function is shown by Zhang [19]. Since the second Jackson q-Bessel function is related to the first Jackson q-Bessel function

$$J_\nu^{(2)}(x;q) = (-x^2/4;q)_\infty \cdot J_\nu^{(1)}(x;q),$$

the connection formula of $J_\nu^{(2)}(x;q)$ follows from the connection formula of the first Jackson q-Bessel function.

But the connection problem for the third Jackson q-Bessel function has not been solved completely. Solutions of the Hahn-Bessel equation (2) has two independent solutions represented by $J_\nu^{(3)}(x;q)$ around the origin. One local solution around the infinity is represented by a convergent power series, and the other is represented by a divergent power series. The asymptotic behavior of $J_\nu^{(3)}(x;q)$ around the infinity is studied by Olde Daalhuis [13], but the connection problem is not treated. One connection formula only for the convergent series around the infinity has been shown by Morita [11]. But the q-Stokes phenomenon of the divergent power series solution is not studied.

In section two we show a q-difference equation satisfied by ${}_1\phi_1(0;a;q,x)$. In section three we review the q-Borel transformation and q-Laplace transformation. In section four we give a resummation of the divergent solution of (1). For divergent power series which satisfy q-difference equations, the q-Borel-Laplace transformation is a powerful tool to give a q-summation procedure [14]. The Newton diagram of (1) has two segments at the infinity. The slopes of two segments are 1 and -1. Since the difference of the two slopes are two, we need the q-Borel resummation of order 1/2 [1, 2]. For q-difference linear equations, the Stokes region is not an angle domain, but an open dense set $\mathbb{C}^* \setminus \lambda q^{\mathbb{Z}}$ for $\lambda \in \mathbb{C}^*$ [18, 14]. By using the q-Borel-Laplace resummation method of order 1/2, we show the q-Stokes phenomenon of the divergent series solution of (1) and the q-Stokes region is not outside of a q-spiral $\lambda q^{\mathbb{Z}}$ but outside of a $\sqrt{q}$-spiral $\lambda \sqrt{q}^{\mathbb{Z}}$. We remark that a Borel transformation for q-series is also studied by Jackson [8].

In section five we show a connection formula for the convergent solution of (1) around the infinity. This formula is essentially shown in [11]. Thus we obtain a complete connection formula of (1).

In the case $a = -q$, (1) reduces to the q-Airy equation studied by Hamamoto, Kajiwara and Witte [10]. The q-Stokes phenomenon of the q-Airy equation is also studied by Morita [12]. Our results contains the connection formula for the q-Airy equation.

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1 Notations and Preliminary

In the following we assume that $q \in \mathbb{C}^*$ and $0 < |q| < 1$. For $n = 0, 1, 2, ...$, we set the q-shifted factorial

$$(a;q)_n = \prod_{j=0}^{n-1}(1 - aq^j), \quad (a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j).$$

We set $(a_1, a_2, ..., a_m; q)_n = \prod_{j=1}^{m}(a_j; q)_n$ for $n = 0, 1, 2, ..$ or $n = \infty$.

We set the theta function

$$\theta_q(x) := \theta(x) = \sum_{k\in\mathbb{Z}} q^{k(k-1)/2} x^k = \left(q, -x, -q/x; q\right)_\infty .$$

The theta function satisfies

$$\theta(q^k x) = q^{-k(k-1)/2} x^{-k} \theta(x) \quad (k \in \mathbb{Z}),$$
$$x\theta(1/x) = \theta(x), \quad \theta(1/x) = \theta(qx).$$

It is easy to show the following lemma on relations between different bases q.

Lemma 1. *We have*

$$
\begin{aligned}
(x;q)_\infty &= (x;q^2)_\infty (xq;q^2)_\infty, \\
(x;q)_\infty(-x;q)_\infty &= (x^2;q^2)_\infty, \\
(q^2;q^2)_\infty \theta_q(x) &= (q;q^2)_\infty \theta_{q^2}(x)\theta_{q^2}(xq), \\
(-q;q)_\infty \theta_q(x)\theta_q(-x) &= (q;q)_\infty \theta_{q^2}(-x^2).
\end{aligned}
$$

1.1 Transformation of q-difference equation

The q-difference operator σ_q is given by $\sigma_q[f(t)] = f(tq)$. We use the following lemma frequently in this paper. The proof is evident.

Lemma 2. *We transform a second order q-difference equation*

$$\left[a(z)\sigma_q + b(z) + c(z)\sigma_q^{-1}\right] y(z) = 0.$$

(1) We set $t = 1/z$ and $v(t) = y(1/t)$. Then $v(t)$ satisfies

$$\left[c(1/t)\sigma_q + b(1/t) + a(1/t)\sigma_q^{-1}\right] v(t) = 0.$$

(2) We set $y(z) = \theta(rz)y_1(z)$. Then $y_1(z)$ satisfies

$$\left[\frac{a(z)}{rz}\sigma_q + b(z) + \frac{rzc(z)}{q}\sigma_q^{-1}\right] y_1(z) = 0.$$

(3) We set $y(z) = (rz;q)_\infty y_2(z)$. Then $y_2(z)$ satisfies

$$\left[\frac{a(z)}{1-rz}\sigma_q + b(z) + (1-rz/q)c(z)\sigma_q^{-1}\right] y_2(z) = 0.$$

1.2 Basic hypergeometric series

The basic hypergeometric series [9] is defined by

$$
\begin{aligned}
&{}_r\phi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,x) \\
&:= \sum_{n\geq 0} \frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s;q)_n(q;q)_n}\left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{1+s-r} x^n.
\end{aligned}
$$

Heine's basic hypergeometric series ${}_2\phi_1(a,b;c;q,z)$ satisfies the equation

$$\left[(c-abqz)\sigma_q^2 - (c+q-(a+b)qz)\sigma_q + q(1-z)\right] {}_2\phi_1(a,b;c;q,z) = 0.$$

A connection formula of ${}_2\phi_1(a,b;c;q,z)$ is shown by Thomae [16] and Watson [17] :

$$
\begin{aligned}
{}_2\phi_1\left(a,b;c;q;x\right) &= \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta(-ax)}{\theta(-x)}{}_2\phi_1\left(a,aq/c;aq/b;q,/cq/abx\right) \\
&+ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{\theta(-bx)}{\theta(-x)}{}_2\phi_1\left(b,bq/c;bq/a;q,cq/abx\right). \qquad (3)
\end{aligned}
$$

It is known that there exist many relations between hypergeometric series. The relation

$$ {}_0\phi_1(-;a^2q;q^2,a^2qx^2) = (x;q)_\infty \cdot {}_2\phi_1(a,-a;a^2;q,x) \tag{4} $$

is shown in [19].

1.3 Formal q-Borel transformation

We review the q-Borel transformation and the q-Laplace transformation. See [14, 18, 20] for detail.

The q-Borel transformation $\mathcal{B}_q^{\pm} : \mathbb{C}[[t]] \to \mathbb{C}[[\tau]]$ is defined by

$$ \mathcal{B}_q^{\pm}\left[\sum_{n=0}^{\infty} a_n t^n\right] := \sum_{n=0}^{\infty} a_n q^{\pm n(n-1)/2}\tau^n. $$

In usual we identify a germ of holomorphic functions at the origin $\mathcal{O}_{\mathbb{C},0}$ as a subset of $\mathbb{C}[[t]]$. As a linear operator on $\mathbb{C}[[t]]$, we have

$$ \mathcal{B}_q^{\pm}(t^m\sigma_q^n f) = q^{\pm m(m-1)/2}\tau^m\sigma_q^{n+m}\mathcal{B}_q^{\pm}(f). $$

The q-Laplace transform of $\varphi(\tau)$ is given by the Jackson integral

$$ L_{q;1}^{[\lambda]}\varphi(t) := \frac{1}{1-q}\int_0^{\lambda\infty} \frac{\varphi(\tau)}{\theta_q(\tau/x)}\frac{d_q\tau}{\tau} = \sum_{n\in\mathbb{Z}}^{\infty} \frac{\varphi(q^n\lambda)}{\theta_q(q^n\lambda/x)}. $$

When $f(t) \in \mathbb{C}[[t]]$ is a convergent power series,

$$ L_{q;1}^{[\lambda]} \circ \mathcal{B}_q^{+}(f) = f. $$

In this sense, $L_q^{[\lambda]}$ is a formal inverse of $\mathcal{B}_q^{+}$.

The following lemma is useful to calculate the q-Laplace transform. We can prove by direct calculations.

Lemma 3. *1) Assume that*

$$ \varphi(\xi) = \frac{\theta(a\xi)}{\theta(b\xi)}\sum_{m\geq 0} c_m\xi^{-m}. $$

Then

$$ L_{q;1}^{[\lambda]}\varphi(x) := \frac{\theta(a\lambda)\theta(qax/b\lambda)}{\theta(b\lambda)\theta(qx/\lambda)}\sum_{m\geq 0} c_m q^{-m(m-1)/2}(b/aqx)^m. $$

In the case $a = b$, we obtain

$$ L_{q;1}^{[\lambda]}\left[\sum_{m\geq 0} c_m\xi^{-m}\right] = \sum_{m\geq 0} c_m q^{-(-m)(-m-1)/2}x^{-m}, $$

which gives a formal q-Borel transformation $\mathcal{B}_q^-$.

2) Assume that

$$\varphi(\xi) = \frac{\theta(a\xi)}{\theta(b_1\xi)\theta(b_2\xi)} \sum_{m\geq 0} c_m \xi^{-2m}.$$

Then

$$L_{q;1}^{[\lambda]}\varphi(x) := \frac{\theta_q(a\lambda)\theta_{q^2}(aq^2x/b_1b_2\lambda^2)}{\theta_q(b_1\lambda)\theta_q(b_2\lambda)\theta_q(qx/\lambda)} \sum_{m\geq 0} c_m q^{-m(m-1)}(b_1b_2/aq^2x)^m.$$

2 q-difference equation satisfied by ${}_1\phi_1(0; a; q, z)$

We assume $a \neq 0$. We consider the following q-difference equation

$$ay(qz) + [z-(a+q)]y(z) + qy(z/q) = 0, \tag{5}$$

which has a solution $y(z) = {}_1\phi_1(0; a; q, z)$. Since the degree of the coefficients of (1) is up to one, we study (1) instead of the Hahn-Exton equation (2).

We set $t = 1/z$ and $v(t) = y(1/t)$. Then $v(t)$ satisfies

$$qtv(tq) + [1-(a+q)t]v(t) + atv(t/q) = 0. \tag{6}$$

In the following, we study a connection problem and the q-Stokes phenomenon of (6). Since (6) has a divergent series solution around the origin, the q-Stokes phenomenon appears when we give a resummation of the divergent series.

Local solutions of (6) around $t = \infty$ are

$$v_1^{(\infty)}(t) = {}_1\phi_1(0; a; q, 1/t), \quad v_2^{(\infty)}(t) = \frac{\theta(-qt)}{\theta(-at)}\, {}_1\phi_1(0; q^2/a; q, q/at).$$

Local (formal) solutions of (6) around $t = 0$ are

$$v_1^{(0)}(t) = \theta(-qt)\sum_{m=0}^{\infty} b_m t^m, \quad v_2^{(0)}(t) = \frac{1}{\theta(-aqt)}\sum_{m=0}^{\infty} c_m t^m.$$

We assume that $b_0 = 1$ and $c_0 = 1$. Here $u_1(t) = \sum b_m t^m$ is divergent and $u_2(t) = \sum c_m t^m$ is convergent. The q-Borel transforms of $u_1(t)$ and $u_2(t)$ are given by

$$\begin{aligned} \mathcal{B}_q^+(u_1)(\tau) &= (-a\tau; q)_\infty(-q\tau; q)_\infty, \\ \mathcal{B}_q^-(u_2)(\tau) &= \frac{1}{(-q^2\tau; q)_\infty(-aq\tau; q)_\infty}. \end{aligned} \tag{7}$$

3 q-Stokes phenomenon

We give a resummation procedure of the divergent power series $u_1(t)$ by the q-Laplace transformation and study the q-Stokes phenomenon. We set $v(t) = \theta(-qt)u(t)$ in (6). Then $u(t)$ satisfies

$$\{\sigma_q - [1-(a+q)t] + at^2\sigma_q^{-1}\}u(t) = 0. \tag{8}$$

The series $u_1(t)$ is a unique formal power series solution of (8) around the origin with $b_0 = 1$.

The q-Borel transform of $u_1(t)$ is given by

$$\mathcal{B}_q^+(u_1)(\tau) = (-a\tau, -q\tau; q)_\infty.$$

But the q-Laplace transform of $(-a\tau, -q\tau; q)_\infty$ is divergent. We apply a q-analogue of Borel transform of order 1/2 studied in [1, 2] in order to obtain a resummation of $u_1(t)$.

We set $p^2 = q$. We consider the p-Borel-Laplace transform of $u_1(t)$

$$f_p(t,\lambda) = L_{p;1}^{[\lambda]} \circ \mathcal{B}_p^+(u_1)(t).$$

The two choices of p give the different p-Borel-Laplace transforms. Since $p^2 = q$, $L_{p;1}^{[\lambda]}$ is considered as the p-Borel-Laplace transform of order 1/2.

Our main result is as follows.

Theorem 4. *The p-Borel-Laplace transform $f_p(t,\lambda)$ is a meromorphic function on $\mathbb{C}^*$ and has at most a simple pole on $t = -\lambda p^{\mathbb{Z}}$:*

$$f_p(t,\lambda) = \frac{\theta_q(a\lambda)\theta_q(ap\lambda)}{(q/a;q)_\infty \theta_q(-ap\lambda^2)} \frac{\theta_q(-pt/\lambda^2)}{\theta_q(pt/\lambda)\theta_q(qt/\lambda)} {}_1\phi_1(0; a; q, 1/t)$$
$$+ \frac{\theta_q(q\lambda)\theta_q(qp\lambda)}{(a/q;q)_\infty \theta_q(-ap\lambda^2)} \frac{\theta_q(-pqt/a\lambda^2)}{\theta_q(pt/\lambda)\theta_q(qt/\lambda)} {}_1\phi_1(0; q^2/a; q, q/at).$$

Proof. The divergent series $u_1(t)$ satisfies the p-difference equation

$$\{\sigma_p^2 - [1-(a+p^2)t] + at^2\sigma_p^{-2}\}u_1(t) = 0.$$

The p-Borel transform of $\varphi(\tau) = \mathcal{B}_p^+(u_1)(\tau)$ satisfies

$$\{\sigma_p^2 + (a+p^2)\tau\sigma_p - (1-ap\tau^2)\}\varphi(\tau) = 0. \tag{9}$$

The power series $\varphi(\tau)$ give a unique holomorphic solution around the origin of (9) with $\varphi(0) = 0$. We set $c^2 = ap$. Then $g(\tau) = (-c\tau; p)_\infty \varphi(\tau)$ satisfies

$$\{(1+cp\tau)\sigma_p^2 + \left(c^2/p + p^2\right)\tau\sigma_p - (1-c\tau)\}g(\tau) = 0,$$

which has a solution $g(\tau) = {}_2\phi_1\left(-c/p, -p^2/c; -p; p, c\tau\right)$. Therefore we have

$$\varphi(\tau) = \frac{1}{(-c\tau;p)_\infty}\, {}_2\phi_1\left(-c/p, -p^2/c; -p; p, c\tau\right).$$

We study the asymptotic behavior of $\varphi(\tau)$ around the infinity. It is evident that

$$\frac{1}{(-c\tau;p)_\infty} = \frac{(p;p)_\infty}{\theta_p(-c\tau;p)}(-p/c\tau;p)_\infty.$$

By the connection formula (3), we have

$$\begin{aligned} &{}_2\phi_1\left(-c/p, -p^2/c; -p; p, c\tau\right) \\ &= \frac{(p^2/c, -p^2/c;p)_\infty}{(-p, p^3/c^2;p)_\infty}\frac{\theta_p(c^2\tau/p)}{\theta_p(-c\tau)}{}_2\phi_1\left(c/p, -c/p; c^2/p^2; p, -p/c\tau\right) \\ &+ \frac{(c/p, -c/p;p)_\infty}{(-p, c^2/p^3;p)_\infty}\frac{\theta_p(p^2\tau)}{\theta_p(-c\tau)}{}_2\phi_1\left(p^2/c, -p^2/c; p^4/c^2; p, -p/c\tau\right). \end{aligned} \tag{10}$$

By (4), we have

$$(-p/c\tau;p)_\infty\, {}_2\phi_1\left(c/p, -c/p; c^2/p^2; p, -p/c\tau\right) = {}_0\phi_1\left(-; c^2/p; p^2, p/\tau^2\right).$$

$$(-p/c\tau;p)_\infty\, {}_2\phi_1\left(p^2/c, -p^2/c; p^4/c^2; p, -p/c\tau\right) = {}_0\phi_1\left(-; p^5/c^2; p^2, p^7/c^4\tau^2\right).$$

Therefore the behavior of $\varphi(\tau)$ at the infinity is as follows.

$$\begin{aligned} \varphi(\tau) &= \frac{(p, p^2/c, -p^2/c;p)_\infty}{(-p, p^3/c^2;p)_\infty}\frac{\theta_p(c^2\tau/p)}{\theta_p(c\tau)\theta_p(-c\tau)}{}_0\phi_1\left(-; c^2/p; p^2, p/\tau^2\right) \\ &+ \frac{(p, c/p, -c/p;p)_\infty}{(-p, c^2/p^3;p)_\infty}\frac{\theta_p(p^2\tau)}{\theta_p(c\tau)\theta_p(-c\tau)}{}_0\phi_1\left(-; p^5/c^2; p^2, p^7/c^4\tau^2\right). \end{aligned}$$

We calculate the p-Laplace transform $f_p(t,\lambda)$ of $\varphi(\tau)$ by Lemma 3:

$$\begin{aligned} f_p(t,\lambda) &= \frac{(p, p^2/c, -p^2/c;p)_\infty}{(-p, p^3/c^2;p)_\infty}\frac{\theta_p(c^2\lambda/p)\theta_{p^2}(-pt/\lambda^2)}{\theta_p(c\lambda)\theta_p(-c\lambda)\theta_p(pt/\lambda)}\, {}_1\phi_1\left(0; c^2/p; p^2, 1/t\right) \\ &+ \frac{(p, c/p, -c/p;p)_\infty}{(-p, c^2/p^3;p)_\infty}\frac{\theta_p(p^2\lambda)\theta_{p^2}(-p^4t/c^2\lambda^2)}{\theta_p(c\lambda)\theta_p(-c\lambda)\theta_p(pt/\lambda)}\, {}_1\phi_1(0; p^5/c^2; p^2, p^3/c^2t). \end{aligned}$$

By Lemma 1, we have

$$\frac{(p;p)_\infty}{(-p;p)_\infty}\frac{1}{\theta_p(c\lambda)\theta_p(-c\lambda)} = \frac{1}{\theta_{p^2}(-c^2\lambda^2)} = \frac{1}{\theta_q(-ap\lambda^2)},$$

$$\frac{(p^2/c, -p^2/c;p)_\infty}{(p^3/c^2;p)_\infty} = \frac{(p^4/c^2;p^2)_\infty}{(p^3/c^2;p)_\infty} = \frac{1}{(p^3/c^2;p^2)_\infty} = \frac{1}{(q/a;q)_\infty}.$$

And

$$\frac{\theta_p(a\lambda)\theta_q(-pt/\lambda^2)}{\theta_q(-ap\lambda^2)\theta_p(pt/\lambda)} = \frac{\theta_q(a\lambda)\theta_q(ap\lambda)\theta_q(-pt/\lambda^2)}{\theta_q(-ap\lambda^2)\theta_q(pt/\lambda)\theta_q(qt/\lambda)}.$$

Applying Lemma 1 to the second term, we obtain Theorem 4. □

4 Connection formula of convergent series

We show a connection formula between $v_2^{(0)}(t)$ and solutions of (6) around the infinity.

Theorem 5. *The solution $v_2^{(0)}(t)$ is written by the sum of $v_1^{(\infty)}(t)$ and $v_2^{(\infty)}(t)$ on $t \in \mathbb{C}^*$:*

$$v_2^{(0)}(t) = \frac{1}{(q;q)_\infty (q/a;q)_\infty} v_1^{(\infty)}(t) + \frac{q}{a \cdot (q;q)_\infty (a/q;q)_\infty} v_2^{(\infty)}(t).$$

Remark. This relation is essentially obtained by Morita [11].

Proof. By (7), $u_2(t)$ has an integral representation

$$u_2(t) = \frac{1}{2\pi i} \int_{|\tau|=\varepsilon} \frac{1}{(-q^2\tau;q)_\infty (-aq\tau;q)_\infty} \theta_q(t/\tau) \frac{d\tau}{\tau},$$

by the residue calculus around the origin. Here ε is sufficiently small so that $(-q^2t;q)_\infty(-aqt;q)_\infty$ does not have zeros in $|\tau| \leqq \varepsilon$.

If we take R so that the circle $|z| = R$ does not pass through the poles, we have

$$\frac{1}{2\pi i} \int_{|\tau|=R} \frac{1}{(-q^2t;q)_\infty (-aqt;q)_\infty} \theta_q(t/\tau) \frac{d\tau}{\tau} \to 0$$

when $R \to \infty$. Therefore

$$u_2(t) = -\sum_{n=0}^{\infty} \operatorname{Res}\left\{ \frac{1}{(-q^2t;q)_\infty (-aqt;q)_\infty} \theta_q(t/\tau) \frac{d\tau}{\tau} \ : \ \tau = -q^{-n-2} \right\}$$

$$-\sum_{n=0}^{\infty} \operatorname{Res}\left\{ \frac{1}{(-q^2t;q)_\infty (-aqt;q)_\infty} \theta_q(t/\tau) \frac{d\tau}{\tau} \ : \ \tau = -q^{-n-1}/a \right\}.$$

We can calculate the residues by the following lemma [18].

Lemma 6. *We assume that $b, c \in \mathbb{C}^*$, $c \notin q^{\mathbb{Z}}$ and $n = 0, 1, 2, 3, \ldots$. Then we have*

$$\operatorname{Res}\left\{ \frac{1}{(bz;q)_\infty} \frac{dz}{z} \ : \ z = q^{-n}/b \right\} = \frac{(-1)^{-n+1} q^{n(n+1)/2}}{(q;q)_\infty (q;q)_n},$$

$$\theta_q(bq^n t) = q^{-n(n-1)/2} b^{-n} t^{-n} \theta_q(bt),$$

$$\frac{1}{(cq^{-n};q)_\infty} = \frac{(-c)^{-n} q^{n(n+1)/2}}{(c;q)_\infty (q/c;q)_n}.$$

By the lemma above we have

$$u_2(t) = \frac{\theta_q(-q^2t)}{(q;q)_\infty (a/q;q)_\infty} {}_1\phi_1(0; a/q; q, q/at) + \frac{\theta_q(-aqt)}{(q;q)_\infty (q/a;q)_\infty} {}_1\phi_1(0; a; q, 1/t).$$

Since $u_2(t) = \theta(-aqt) v_2^{(0)}(t)$, we obtain Theorem 5. □

5 Summary

We have shown a connection formula of a second order q-difference equation whose solution is represented by ${}_1\phi_1(0;a;q,t)$:

$$qtv(tq) + [1-(a+q)t]v(t) + atv\left(t/q\right) = 0.$$

Local solutions around $t=\infty$ are

$$v_1^{(\infty)}(t) = {}_1\phi_1(0;a;q,1/t), \quad v_2^{(\infty)}(t) = \frac{\theta(-qt)}{\theta(-at)}\,{}_1\phi_1(0;q^2/a;q,q/at).$$

Local (formal) solutions around $t=0$ are

$$v_1^{(0)}(t) = \theta(-qt)\sum_{m=0}^{\infty} b_m t^m, \quad v_2^{(0)}(t) = \frac{1}{\theta(-aqt)}\sum_{m=0}^{\infty} c_m t^m.$$

Here $\sum b_m t^m$ is divergent, $\sum c_m t^m$ is convergent. We assume that $b_0 = 1$, $c_0 = 1$. We set $\tilde{v}_1^{(0)}(t,\lambda;p) = \theta(-qt)f_p(t,\lambda)$ for $p^2 = q$. Here $f_p(t,\lambda)$ is a resummation

$$f_p(t,\lambda) = L_{p;1}^{[\lambda]} \circ \mathcal{B}_p^+ \left[\sum_{m=0}^{\infty} b_m t^m\right].$$

Theorem 7. *The connection formulae between $\tilde{v}_1^{(0)}(t,\lambda;p), v_2^{(0)}(t)$ and $v_1^{(\infty)}(t)$, $v_2^{(\infty)}(t)$ are given as follows.*

$$\begin{aligned}\tilde{v}_1^{(0)}(t,\lambda;p) = &\frac{\theta_q(a\lambda)\theta_q(ap\lambda)}{(q/a;q)_\infty\theta_q(-ap\lambda^2)}\frac{\theta(-qt)\theta_q(-pt/\lambda^2)}{\theta_q(pt/\lambda)\theta_q(qt/\lambda)}v_1^{(\infty)}(t)\\ &+\frac{\theta_q(q\lambda)\theta_q(qp\lambda)}{(a/q;q)_\infty\theta_q(-ap\lambda^2)}\frac{\theta(-at)\theta_q(-pqt/a\lambda^2)}{\theta_q(pt/\lambda)\theta_q(qt/\lambda)}v_2^{(\infty)}(t),\end{aligned}$$

$$v_2^{(0)}(t) = \frac{1}{(q;q)_\infty(q/a;q)_\infty}v_1^{(\infty)}(t) + \frac{q}{a\cdot(q;q)_\infty(a/q;q)_\infty}v_2^{(\infty)}(t).$$

The second connection formula is already shown by Morita [11]. The case $a=-q$ is obtained in [12]. A connection formula of the Hahn-Exton q-Bessel equation is derived from the theorem above by simple calculations.

The q-Laplace transform of order $1/2$ is shown in [1, 2] is necessary to determine the q-Stokes coefficients. Our results is the first example to calculate the q-Stokes coefficients when the slope of the Newton diagram is two in q-difference equations. Our method would be useful to study the q-Stokes phenomenon of other q-difference equations with slopes higher than two.

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