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The abc Conjecture and Square Free Parts of Fibonacci Numbers

By

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Abstract

In the paper [11], the second author considered a conjecture on the fundamental units of certain family of real quadratic fields related to Fibonacci numbers. In this paper, we shall investigate this conjecture more precisely in section 3, using the constant terms of the abc conjecture. We also prove the conjecture in section 4 for some special cases, using the integer points of several elliptic curves.

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1 Introduction

The well known abc conjecture of Masser-Oesterlé states that

The abc Conjecture. (cf. [15], [22]) For any $\varepsilon > 0$, there exists a constant $K(\varepsilon)$ depending only on ε such that if

$$(*) \qquad a + b = c$$

where a, b and c are coprime positive integers, then the following inequality holds

$$c \leq K(\varepsilon) r^{1+\varepsilon}.$$

Here r is defined by putting $\prod_{p|abc} p$ and called the *radical* of abc.

The following diophantine equations are called the simultaneous Pell equations

$$(1) \begin{cases} x^2 - az^2 = 1 \\ y^2 - bz^2 = 1 \end{cases},$$

where a, b are distinct positive integers such that a, b and ab are not perfect squares. Recently considerable works have been done on the number of positive integer solutions of simultaneous Pell equations by various mathematicians (see for instance [1], [2], [25] and [26]). It was proved the number of positive integer solutions to be at most two in general and was proved at most one for several families of simultaneous Pell equations (see for example [26]).
It is easy to see the positive integer solution (x, y, z) of (1) determine two units $x + z\sqrt{a}$ and $y + z\sqrt{b}$ of real quadratic fields $\mathbb{Q}(\sqrt{a})$ and $\mathbb{Q}(\sqrt{b})$ respectively. Let ε_a and ε_b be the fundamental units of $\mathbb{Q}(\sqrt{a})$ and $\mathbb{Q}(\sqrt{b})$. It is a natural problem to investigate the group indices $e(a)$ and $e(b)$ which is determined by $e(a) = [\langle -1, \varepsilon_a \rangle : \langle -1, x + z\sqrt{a} \rangle]$ and $e(b) = [\langle -1, \varepsilon_b \rangle : \langle -1, y + z\sqrt{b} \rangle]$. Though there have been many progress concerning the simultaneous Pell equations, these properties were treated only in [25] for the special cases $a = 2d$ and $b = d$. In [25], G. Walsh has proved $e(2d) = 1$ in general, but $e(d)$ was not treated explicitly. In our previous papers [10], [12] and [13], we have investigated more general cases and shown $e(a) = e(b) = 1$ under the abc conjecture. It should be noted our results include the index $e(d) = 1$ of [25] as a special case. However in those papers, we have utilized the results on the square free part of binary recurrence sequences of P. Ribemboim and G. Walsh [23] depending on the abc conjecture. Since their results are asymptotic, our conclusions obtained in our previous papers are also asymptotic.
In this note, we shall show more explicit and precise conclusions for the following special family of simultaneous Pell equations using the abc conjecture directly.

$$(2) \begin{cases} x^2 - 5dz^2 = 1 \\ y^2 - dz^2 = 1 \end{cases}, \quad (3) \begin{cases} x^2 - 5dz^2 = -1 \\ y^2 - dz^2 = -1 \end{cases}$$

We note that our conclusions on the indices $e(5d)$ and $e(d)$ contain the abc constant $K(\varepsilon)$ explicitly. Our methods may work for more general cases which we have treated in [12] and [13], but here we restrict ourselves to the above two special cases for the sake of simplicity. We also note that the latter system of simultaneous Pell equations (3) which corresponds to the Pell equations with norm -1 have been rarely covered by recent papers. Finally we quote that S. Mochizuki has announced that he had proved the abc conjecture in his preprints [17], [18], [19] and [20] in 2012. Hence it will be of some interest to investigate the above simultaneous Pell equations and give explicit results containing the abc constant term $K(\varepsilon)$. In the next section, we will investigate the asymptotic behavior of the square free part of Fibonacci numbers and prepare several preliminary propositions in the first four sections. We shall show those explicit results on our simultaneous Pell equations in section 5. At last, we shall report the numerical data which suggests our results on the fundamental units of $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{5d})$, where d is the positive square free integer in the simultaneous Pell equations (2) and (3).

2 Square Free Part of Fibonacci Numbers

Let δ be a fixed positive constant such that $0 < \delta < 1$. Take a positive number ε which satisfies $0 < \varepsilon < \frac{\delta}{4-\delta}$. Then, from the assumption $0 < \delta < 1$, we know that $\varepsilon < \frac{1}{3}$. For any positive integer m, we shall write $s(m)$ to be the square free part of m and $q(m)^2$ to be the perfect square part of m, that is, $m = s(m)q(m)^2$. Let F_n and L_n be nth Fibonacci number and Lucas number, respectively. Though the following property on the square free part of Fibonacci numbers has been proved in [23], we shall give a simple and direct proof as follows.

Proposition 2.1 *Under the abc conjecture, there exists a positive constant $N(\delta, \varepsilon)$ which depends only on δ and ε, such that, for any $0 < \delta < 1$ and $n > N(\delta, \varepsilon)$*

$$F_n^{1-\delta} \leq s(F_n) \leq F_n.$$

Proof. Since $F_n = s(F_n)q(F_n)^2$, we know

$$F_n^{1-\delta} \leq s(F_n) \Longleftrightarrow s(F_n)^{1-\delta}q(F_n)^{2-2\delta} \leq s(F_n) \Longleftrightarrow q(F_n)^{2-2\delta} \leq s(F_n)^{\delta}.$$

Suppose on the contrary $s(F_n)^{\delta} < q(F_n)^{2-2\delta}$. Take $0 < \varepsilon$ such as $0 < \varepsilon < \frac{\delta}{4-\delta}$. Applying the abc conjecture to the equation $L_n^2 - 5F_n^2 = 4(-1)^n$, we get $5F_n^2 \leq K(\varepsilon)r^{1+\varepsilon}$. From the assumption $s(F_n)^{\delta} < q(F_n)^{2-2\delta}$, we know

$$s(F_n)q(F_n) = s(F_n)^{1-\frac{\delta}{2}}s(F_n)^{\frac{\delta}{2}}q(F_n) < s(F_n)^{1-\frac{\delta}{2}}q(F_n)^{2-\delta} = F_n^{1-\frac{\delta}{2}}.$$

Therefore, combining the fact $L_n^2 \leq 5F_n^2 + 4 < 10F_n^2$, the radical r satisfies the inequality $r \leq 10L_n s(F_n)q(F_n) \leq 10\sqrt{10}F_n^{2-\frac{\delta}{2}}$. Thus we have

$$5F_n^2 < K(\varepsilon)(10\sqrt{10}F_n^{2-\frac{\delta}{2}})^{1+\varepsilon} = K(\varepsilon)2^{\frac{3+3\varepsilon}{2}}5^{\frac{3+3\varepsilon}{2}}F_n^{2+2\varepsilon-\frac{\delta+\delta\varepsilon}{2}}$$

and then $F_n^{\frac{\delta-(4-\delta)\varepsilon}{2}} < K(\varepsilon)2^{\frac{3+3\varepsilon}{2}}5^{\frac{3+3\varepsilon}{2}}$. Put $\delta_0 = \frac{\delta-(4-\delta)\varepsilon}{2}$. From the assumption $0 < \varepsilon < \frac{\delta}{4-\delta}$, we see $\delta_0 > 0$. Hence we can show the following inequality

$$\frac{\varphi^{n-1}}{\sqrt{5}} < F_n < (K(\varepsilon)2^{\frac{3+3\varepsilon}{2}}5^{\frac{3+3\varepsilon}{2}})^{\frac{1}{\delta_0}},$$

where φ is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. Taking the log of the above inequality, we have

$$(n-1)\log\varphi < \frac{2}{\delta-(4-\delta)\varepsilon}\log(K(\varepsilon)) + \frac{3(1+\varepsilon)}{\delta-(4-\delta)\varepsilon}\log 2 + \frac{(2+\delta)(1+\varepsilon)}{2(\delta-(4-\delta)\varepsilon)}\log 5.$$

We will denote $N(\delta,\varepsilon)$

$$= (\frac{2}{\delta-(4-\delta)\varepsilon}\log(K(\varepsilon)) + \frac{3(1+\varepsilon)}{\delta-(4-\delta)\varepsilon}\log 2 + \frac{(2+\delta)(1+\varepsilon)}{2(\delta-(4-\delta)\varepsilon)}\log 5 + \log\varphi)/\log\varphi.$$

Thus we have shown that the assumption $s(F_n)^\delta < q(F_n)^{2-2\delta}$ implies $n < N(\delta,\varepsilon)$. Therefore, for any $n \geq N(\delta,\varepsilon)$, $s(F_n)^\delta \geq q(F_n)^{2-2\delta}$, i.e., $F_n^{1-\delta} \leq s(F_n) \leq F_n$, under the abc conjecture, which completes the proof.

In section 5, we shall use this proposition to show the growth of the sequence of d which has the positive integer solutions of the simultaneous Pell equations (2) or (3).

3 Explicit Bound

Here we shall notice that the simultaneous Pell equations (2) and (3) imply $x^2 - 5y^2 = \mp 4$, that is, $x = L_n$ and $y = F_n$, where L_n and F_n are nth Lucas number and nth Fibonacci number, as before. Combining the fact $F_n^2 + (-1)^n = F_{n-1}F_{n+1}$ and $(F_{2n}, F_{2n+2}) = 1$, we have $d = s(F_{2n+1}^2 - 1) = s(F_{2n})s(F_{2n+2})$ in (2). Moreover $x = L_{2n+1}$, $y = F_{2n+1}$ and $z = q(F_{2n})q(F_{2n+2})$. Similarly in (3), $d = s(F_{2n}^2 + 1) = s(F_{2n-1})s(F_{2n+1})$, $x = L_{2n}$, $y = F_{2n}$ and $z = q(F_{2n-1})q(F_{2n+1})$.
In the following, we shall consider the cases $2|F_n$ and $2\not|F_n$ separately, because we should apply the abc conjecture to the different equations according to the conditions $2|F_n$ or $2\not|F_n$. We shall obtain the following constant containing the constant term $K(\frac{1}{5})$ in the abc conjecture

$$N_F = \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 30\log 3 + 3\log 5 - 12\log 2 + \log\varphi}{2\log\varphi}.$$

We shall show the following proposition.

Proposition 3.1 *If $n > N_F$ then we have*

$$s(F_n)^2 > 2q(F_n), \text{ under the abc conjecture.}$$

Similarly we shall determine the following constant N_L and prove the following proposition.

$$N_L = \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 30\log 3 + 7\log 5 - 12\log 2 + \log\varphi}{2\log\varphi},$$

Proposition 3.2 *If $n > N_L$ then we have*

$$s(F_n)^2 > 10\sqrt{5}q(F_n), \text{ under the abc conjecture.}$$

Here we shall give a table of small Fibonacci and Lucas numbers for the readers who are not familiar with the properties of these numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322

Since the period length of Fibonacci number mod 2 is 3, that is, $F_{n+3} \equiv F_3 \mod 2$, we see that
$F_{2n} \equiv 0 \mod 2 \Longleftrightarrow n \equiv 0 \mod 3$, and $F_{2n+1} \equiv 0 \mod 2 \Longleftrightarrow n \equiv 1 \mod 3$.

Case 1. We shall consider the case $n \not\equiv 1 \mod 3$. Since $2 \not| F_{2n+1}$, we know $(L_{2n+1}, 5F_{2n+1}) = 1$. Then we can apply the abc conjecture to the following equality

$$L_{2n+1}^2 - 5F_{2n+1}^2 = -4.$$

We note that $L_{2n+1}^2 < L_{2n+1}^2 + 4 = 5F_{2n+1}^2$. Suppose $s(F_{2n+1})^2 \leq 2q(F_{2n+1})$ under the abc conjecture. Taking $\varepsilon = 1/5$ in the abc conjecture, we have

$$5F_{2n+1}^2 \leq K(1/5)r^{1+\frac{1}{5}},$$

where r is the radical of $10F_{2n+1}L_{2n+1}$. Hence

$$r \leq 10L_{2n+1}s(F_{2n+1})q(F_{2n+1}).$$

Since $L_{2n+1} \leq \sqrt{5}F_{2n+1}$ and $s(F_{2n+1})^{\frac{2}{5}} \leq 2^{\frac{1}{5}}q(F_{2n+1})^{\frac{1}{5}}$, we know

$$r < 2 \times 5 \times (\sqrt{5}F_{2n+1}) \times (s(F_{2n+1})q^2(F_{2n+1}))^{\frac{3}{5}} \times 2^{\frac{1}{5}} = 2^{\frac{6}{5}} \times 5^{\frac{3}{2}} \times F_{2n+1}^{\frac{8}{5}}.$$

Thus

$$5F_{2n+1}^2 < K(1/5)(2^{\frac{6}{5}} \times 5^{\frac{3}{2}} \times F_{2n+1}^{\frac{8}{5}})^{\frac{6}{5}}.$$

Hence

$$F_{2n+1}^{\frac{2}{25}} < K(1/5) \times 2^{\frac{36}{25}} \times 5^{\frac{4}{5}}.$$

Then

$$\frac{\varphi^{2n+1}}{\sqrt{5}} < F_{2n+1} < K(1/5)^{\frac{25}{2}} \times 2^{18} \times 5^{10},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. Taking the log of the above inequality

$$(2n+1)\log\varphi < \frac{25}{2}\log(K(1/5)) + 18\log 2 + \frac{21}{2}\log 5.$$

Hence we can conclude

$$n < \frac{\frac{25}{2}\log(K(1/5)) + 18\log 2 + \frac{21}{2}\log 5 - \log\varphi}{2\log\varphi}$$

Case 2. Consider the case $n \equiv 1 \mod 3$. Then $2|F_{2n+1}$ and $(L_{2n+1}/2, 5F_{2n+1}/2) = 1$ in the following equality

$$(L_{2n+1}/2)^2 - 5(F_{2n+1}/2)^2 = -1.$$

In the same way as above, the assumption $s(F_{2n+1})^2 \le 2q(F_{2n+1})$ and the abc conjecture implies

$$n < \frac{\frac{25}{2}\log(K(\frac{1}{5})) - 2\log 2 + \frac{21}{2}\log 5 - \log\varphi}{2\log\varphi}$$

Case 3. Consider the case $n \not\equiv 0 \mod 3$. Then $2 \nmid F_{2n}$ and $(L_{2n}, 5F_{2n}) = 1$ in the following equality

$$L_{2n}^2 - 5F_{2n}^2 = 4.$$

Thus $L_{2n}^2 < 5F_{2n}^2 + 4 \le \left(\frac{9}{4}F_{2n}\right)^2$ for $n \ge 3$. Suppose $s(F_{2n})^2 \le 2q(F_{2n})$. Applying the abc conjecture to the above equality with $\varepsilon = 1/5$, we have

$$5F_{2n}^2 \le K(1/5)r^{\frac{6}{5}},$$

where r is the radical of $10F_{2n}L_{2n}$. Thus

$$r \le 2 \times 5 \times L_{2n}s(F_{2n})q(F_{2n}).$$

Since $L_{2n} < \frac{9}{4}F_{2n}$ and $s(F_{2n})^{\frac{2}{5}} \le 2^{\frac{1}{5}}q(F_{2n})^{\frac{1}{5}}$, we know

$$r < 2 \times 5\left(\frac{9}{4}F_{2n}\right)(s(F_{2n})q^2(F_{2n}))^{\frac{3}{5}} \times 2^{\frac{1}{5}} = 2^{-\frac{4}{5}} \times 3^2 \times 5F_{2n}^{\frac{8}{5}}.$$

Thus

$$5F_{2n}^2 < K(1/5)(2^{-\frac{4}{5}} \times 3^2 \times 5 \times F_{2n}^{\frac{8}{5}})^{\frac{6}{5}}.$$

Hence

$$F_{2n}^{\frac{2}{25}} < K(1/5) \times 2^{-\frac{24}{25}} \times 3^{\frac{12}{5}} \times 5^{\frac{1}{5}}.$$

Then

$$\frac{\varphi^{2n-1}}{\sqrt{5}} < F_{2n-1} < F_{2n} < K(1/5)^{\frac{25}{2}} \times 3^{30} \times 2^{-12} \times 5^3.$$

Hence

$$(2n-1)\log\varphi < \frac{25}{2}\log(K(1/5)) + 30\log 3 - 12\log 2 + \frac{5}{2}\log 5.$$

Then we can conclude

$$n < \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 30\log 3 - 12\log 2 + \frac{5}{2}\log 5 + \log\varphi}{2\log\varphi}$$

Case 4. Consider the case $n \equiv 0 \mod 3$. Then $2|F_{2n}$ and $(L_{2n}/2, 5F_{2n}/2) = 1$ in the following equality

$$(L_{2n}/2)^2 - 5(F_{2n}/2)^2 = 1.$$

In the same way as above, the assumption $s(F_{2n})^2 \leq 2q(F_{2n})$ implies

$$n < \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 30\log 3 - 32\log 2 + 3\log 5 + \log\varphi}{2\log\varphi}$$

Put

$$N_1 = \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 18\log 2 + \frac{21}{2}\log 5 - \log\varphi}{2\log\varphi},$$

$$N_2 = \frac{\frac{25}{2}\log(K(\frac{1}{5})) - 2\log 2 + \frac{21}{2}\log 5 - \log\varphi}{2\log\varphi},$$

$$N_3 = \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 30\log 3 - 12\log 2 + \frac{5}{2}\log 5 + \log\varphi}{2\log\varphi},$$

$$N_4 = \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 30\log 3 - 32\log 2 + 3\log 5 + \log\varphi}{2\log\varphi}.$$

Since $\max(N_1, N_2, N_3, N_4) = N_3$, put

$$N_F = N_3 = \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 30\log 3 - 12\log 2 + \frac{5}{2}\log 5 + \log\varphi}{2\log\varphi}.$$

Then, for the case $n \geq N_F$, the following inequality holds

$$s(F_n)^2 > 2q(F_n).$$

Thus we have proved Proposition 3.1.

Now we shall show the explicit unit $F_n + \sqrt{F_n^2 + (-1)^n}$ is not a higher power of the other unit $\dfrac{t + \sqrt{t^2 + (-1)^n 4}}{2}$ of the quadratic field $\mathbb{Q}(\sqrt{F_n^2 + (-1)^n}) = \mathbb{Q}(\sqrt{t^2 + (-1)^n 4})$.

We consider the cases $F_{2n+1} + \sqrt{F_{2n+1}^2 - 1}$ and $F_{2n} + \sqrt{F_{2n}^2 + 1}$ separately.

Suppose $F_{2n+1} + \sqrt{F_{2n+1}^2 - 1} = \left(\dfrac{t + \sqrt{t^2 - 4}}{2}\right)^k$ for $k \geq 5$. Then $\left(\dfrac{t + \sqrt{t^2 - 4}}{2}\right)^k = \dfrac{v_k(t) + u_k(t)\sqrt{t^2 - 4}}{2}$, where $u_k(t)$ is *the Lucas sequences associated to the pair* $(k, 1)$ and $v_k(t)$ is *the companion Lucas sequences associated to the pair* $(k, 1)$. Thus u_k and v_k are the binary recurrence sequences which satisfy

$$u_{k+1} = tu_k - u_{k-1}, v_{k+1} = tv_k - v_{k-1}$$

with initial terms $u_0 = 1, u_1 = 1$ and $v_0 = 2, v_1 = t$ $(t \geq 3)$. Hence we have $F_{2n+1}^2 - 1 = \dfrac{u_k(t)^2(t^2 - 4)}{4}$. Thus we have $s(F_{2n+1}^2 - 1) \leq t^2 - 4$, and

$$q(F_{2n+1}^2 - 1) \geq \frac{u_k(t)}{2} \geq \frac{u_5(t)}{2} = \frac{t^4 - 3t^2 + 1}{2}.$$

Then from the condition $t \geq 3$ we have

$$4q(F_{2n+1}^2 - 1) \geq 2(t^4 - 3t^2 + 1) > (t^2 - 4)^2 \geq s(F_{2n+1}^2 - 1)^2.$$

On the other hand, from Proposition 3.1, we have $s(F_n)^2 > q(F_n)$ for $n \geq N_F$. Since $(F_{2n+2}, F_{2n}) = 1$, we have

$$s(F_{2n+1}^2 - 1)^2 = s(F_{2n}F_{2n+2})^2 = s(F_{2n})^2 s(F_{2n+2})^2 > 4q(F_{2n})q(F_{2n+2}) = 4q(F_{2n+1}^2 - 1),$$

which contradicts the above inequality. Hence we can conclude that if $F_{2n+1} + \sqrt{F_{2n+1}^2 - 1}$ is the power of some unit $\left(\dfrac{t + \sqrt{t^2 - 4}}{2}\right)^k$, then $k \leq 4$ under the abc conjecture.

Now suppose $F_{2n} + \sqrt{F_{2n}^2 + 1} = \left(\dfrac{t + \sqrt{t^2 + 4}}{2}\right)^k$ for $k \geq 5$. Then

$$\left(\frac{t + \sqrt{t^2 + 4}}{2}\right)^k = \frac{v_k(t) + u_k(t)\sqrt{t^2 + 4}}{2},$$

where $u_k(t)$ is the Lucas sequences associated to the pair $(k, -1)$ and $v_k(t)$ is the companion Lusas sequences associated to the pair $(k, -1)$. Thus u_k and v_k are the binary recurrence sequences which satisfy

$$u_{k+1} = tu_k + u_{k-1}, v_{k+1} = tv_k + v_{k-1}$$

with initial terms $u_0 = 1, u_1 = 1$ and $v_0 = 2, v_1 = t$ $(t \geq 3)$. Hence we have $F_{2n}^2 + 1 = \dfrac{u_k(t)^2(t^2+4)}{4}$. Thus we have

$$s(F_{2n}^2 + 1) \leq t^2 + 4,$$

and

$$q(F_{2n}^2 + 1) \geq \frac{u_k(t)}{2} \geq \frac{u_5(t)}{2} = \frac{t^4 + 3t^2 + 1}{2}$$

Then we have

$$4q(F_{2n}^2 + 1) \geq 2(t^4 + 3t^2 + 1) > (t^2 + 4)^2 \geq s(F_{2n}^2 + 1)^2$$

for the case $t \geq 3$. We shall consider the cases $t = 1$ and 2 in later. On the other hand, from Proposition 3.1, we have $s(F_n)^2 > q(F_n)$ for $n \geq N_F$. Since $(F_{2n+1}, F_{2n-1}) = 1$, we have

$$s(F_{2n}^2+1)^2 = s(F_{2n+1}F_{2n-1})^2 = s(F_{2n+1})^2 s(F_{2n-1})^2 > 4q(F_{2n+1})(F_{2n-1}) = 4q(F_{2n}^2+1),$$

which contradicts the above inequality,
Now we shall consider the exceptional case $t = 1$. Here we use the symbol $a = \Box$ if the integer a is a perfect square. Then

$$F_{2n} + \sqrt{F_{2n}^2 + 1} = \left(\frac{1+\sqrt{5}}{2}\right)^k$$

for some k implies $F_{2n}^2 + 1 = F_{2n-1}F_{2n+1} = 5\Box$. It was proved by J. H. E. Cohn in [3] that when $n > 0$, $F_n = \Box \iff n = 1, 2, 12$ (see for details Proposition 5.1). One can easily examine no such case occurs.
Now we shall consider the exceptional case $t = 2$. Then

$$F_{2n} + \sqrt{F_{2n}^2 + 1} = (1 + \sqrt{2})^k$$

for some k implies $F_{2n}^2 + 1 = F_{2n-1}F_{2n+1} = 2\Box$. From Cohn's result we must have $n = 1$ and $F_2 + \sqrt{F_2^2 + 1}$ is the fundamental unit $1 + \sqrt{2}$ for this case. Thus under the abc conjecture, if $n \geq N_F$, $F_{2n} + \sqrt{F_{2n}^2 + 1}$ is the power of some unit $\left(\dfrac{t + \sqrt{t^2+4}}{2}\right)^k$, then k is also satisfies $k \leq 4$.

Proposition 3.3 (*Assuming the abc conjecture*) *In the case* $n \geq N_F$, $F_n + \sqrt{F_n^2 + (-1)^n} = \left(\frac{t + \sqrt{t + (-1)^n 4}}{2}\right)^k$ *implies* $k \leq 4$.

We can determine N_L in the same way as above. We must consider the following 4 cases separately,
Case 1) Consider the case $n \not\equiv 1 \mod 3$ and suppose $s(F_{2n+1})^2 \leq 10\sqrt{5}q(F_{2n+1})$. Then we have

$$n < N_1' = \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 18\log 2 + 15\log 5 - \log\varphi}{2\log\varphi}$$

Case 2) Consider the case $n \equiv 1 \mod 3$ and suppose $s(F_{2n+1})^2 \leq 10\sqrt{5}q(F_{2n+1})$. Then we have

$$n < N_2' = \frac{\frac{25}{2}\log(K(\frac{1}{5})) - 2\log 2 + 15\log 5 - \log\varphi}{2\log\varphi}$$

Case 3) Consider the case $n \not\equiv 0 \mod 3$ and suppose $s(F_{2n})^2 \leq 10\sqrt{5}q(F_{2n})$. We shall write down this case explicitly. Then $s(L_{2n}^2 + 1)s(5F_{2n-1}F_{2n+1}) \geq \frac{s(F_{2n-1}F_{2n+1})}{5}$. In the case $n \geq 3$, we know $L_{2n} \leq \frac{9}{4}F_{2n}$. From the assumption $s(F_{2n})^2 \leq 10\sqrt{5}q(F_{2n})$, applying the abc conjecture to the equation $L_{2n}^2 - 5F_{2n}^2 = 4$ we have the inequality $5F_{2n}^2 \leq r^{\frac{6}{5}}$. From the assumption we obtain

$$r \leq 10L_{2n}s(F_{2n})q(F_{2n}) \leq 2 \times 5\left(\frac{9}{4}F_{2n}\right)s(F_{2n})^{\frac{3}{5}}(10\sqrt{5})^{\frac{1}{5}}q(F_{2n})^{\frac{6}{5}}.$$

Hence the radical satisfies $r < 2^{-\frac{4}{5}} \times 3^2 \times 5^{\frac{13}{10}}F_{2n}^{\frac{8}{5}}$. Then

$$5F_{2n}^2 < K(1/5) \times 2^{-\frac{24}{25}} \times 3^{\frac{12}{5}} \times 5^{\frac{39}{25}}F_{2n}^{\frac{48}{25}}$$

Thus we have

$$F_{2n}^{\frac{2}{25}} < K(1/5) \times 2^{-\frac{24}{25}} \times 3^{\frac{12}{5}} \times 5^{\frac{14}{25}}.$$

Hence

$$\frac{\varphi^{2n-1}}{\sqrt{5}} < F_{2n-1} < F_{2n} < K(1/5)^{\frac{25}{2}} \times 2^{-12} \times 3^{30} \times 5^7.$$

Taking the log of the both hand side, we have

$$n < N_3' = \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 30\log 3 - 12\log 2 + \frac{15}{2}\log 5 + \log\varphi}{2\log\varphi}.$$

Case 4) Consider the case $n \equiv 0 \mod 3$ and suppose $s(F_{2n})^2 \leq 10\sqrt{5}q(F_{2n})$. Then we have

$$n < N_4' = \frac{\frac{25}{2}\log(K(\frac{1}{5})) + 30\log 3 - 32\log 2 + \frac{15}{2}\log 5 + \log\varphi}{2\log\varphi}.$$

N_L denotes $\max(N_1', N_2', N_3', N_4') = N_3'$. Then under the abc conjecture, $s(F_n)^2 > 10\sqrt{5}q(F_n)$ for $n \geq N_L$. Thus we have proved Proposition 3.2.

Now we shall show the explicit unit $L_n + \sqrt{L_n^2 + (-1)^n}$ is not a higher power of the other unit $\dfrac{t + \sqrt{t^2 + (-1)^n 4}}{2}$ of the quadratic field $\mathbb{Q}(\sqrt{L_n^2 + (-1)^n}) = \mathbb{Q}(\sqrt{t^2 + (-1)^n 4})$.
Since $s(F_n)^2 > 10\sqrt{5}q(F_n)$ for $n > N_L$, we know

$$s(L_n^2+(-1)^n)^2 = s(5F_{n-1}F_{n+1})^2 \geq \frac{s(F_{n-1})^2 s(F_{n+1})^2}{5^2} > \frac{10^2 \times 5q(F_{n-1})q(F_{n+1})}{5^2}$$

$$= 4(5q(F_{n-1})q(F_{n+1})) \geq 4q(5F_{n-1}F_{n+1}) = 4q(L^2+(-1)^n).$$

Hence we have $s(L_n^2 + (-1)^n) > 4q(L^2 + (-1)^n)$, for $n \geq N_L$. If $L_n + \sqrt{L_n^2 + (-1)^n} = \left(\dfrac{t + \sqrt{t^2 + (-1)^n 4}}{2}\right)^k$ for $k \geq 5$, we have $s(L_n^2 + (-1)^n) \leq 4q(L^2 + (-1)^n)$ in the same way as above for the case $t \geq 3$. Thus $L_n + \sqrt{L_n^2 + (-1)^n} = \left(\dfrac{t + \sqrt{t^2 + (-1)^n 4}}{2}\right)^k$ implies $k \leq 4$ for the case $t \geq 3$. Now we shall take care of the following exceptional cases $t = 1$ and $t = 2$, that is, $L_{2n} + \sqrt{L_{2n}^2 + 1} = \left(\dfrac{1 + \sqrt{5}}{2}\right)^k$, and $L_{2n} + \sqrt{L_{2n}^2 + 1} = (1 + \sqrt{2})^k$.
Since $L_{2n}^2 + 1 = 5F_{2n-1}F_{2n+1}$, the above equalities imply one of F_{2n-1} and F_{2n+1} is square or 2 times square. Using Cohn's results $F_n = \square \iff n = 1, 2, 12$ and $F_n = 2\square \iff n = 3, 6$. we can conclude that there are only two cases

$$L_0 + \sqrt{L_0^2 + 1} = 2 + \sqrt{5} = \left(\frac{1 + \sqrt{5}}{2}\right)^3, L_4 + \sqrt{L_4^2 + 1} = 7 + \sqrt{50} = (1 + \sqrt{2})^3.$$

Combining these, if $L_n + \sqrt{L_n^2 + (-1)^n} = \left(\dfrac{t + \sqrt{t^2 + (-1)^n 4}}{2}\right)^k$ for some k, then $k \leq 4$.

Proposition 3.4 (*Assuming the abc conjecture*) *In the case $n > N_L$,*

$$L_n + \sqrt{L_n^2 + (-1)^n} = \left(\frac{t + \sqrt{t + (-1)^n 4}}{2}\right)^k \textit{ implies } k \leq 4.$$

4 Cube Power and Elliptic Curves

In this section, we shall determine the exceptional n and t such that

$$F_{2n+1}+\sqrt{F_{2n+1}^2-1}=\left(\frac{t+\sqrt{t^2-4}}{2}\right)^3 \quad \text{for some } t\geq 3,$$

$$F_{2n}+\sqrt{F_{2n}^2+1}=\left(\frac{t+\sqrt{t^2+4}}{2}\right)^3 \quad \text{for some } t\geq 1,$$

$$L_{2n+1}+\sqrt{L_{2n+1}^2-1}=\left(\frac{t+\sqrt{t^2-4}}{2}\right)^3 \quad \text{for some } t\geq 3,$$

or

$$L_{2n}+\sqrt{L_{2n}^2+1}=\left(\frac{t+\sqrt{t^2+4}}{2}\right)^3 \quad \text{for some } t\geq 1.$$

Our strategy is to change these problems to the determination of integer points on corresponding elliptic curves. Let us consider the first case

$$\left(\frac{t+\sqrt{t^2-4}}{2}\right)^3=\frac{t(t^2-3)+(t^2-1)\sqrt{t^2-4}}{2}.$$

Then $F_{2n+1}=\dfrac{t(t^2-3)}{2}$ for some $t\geq 3$. Since $L_{2n+1}^2-5F_{2n+1}^2=-4$, we have

$$(10L_{2n+1})^2=(5t^2)(5t^2-15)^2-400.$$

Put $X=5t^2-10$ and $Y=10L_{2n+1}$. Then we obtain an elliptic curve

$$E: Y^2=X^3-75X-150.$$

Then the solutions of $F_{2n+1}=\dfrac{t(t^2-3)}{2}$ correspond to the integer points on the above elliptic curve. The discriminant of this elliptic curve is $\Delta(E)=2^6\cdot 3^2\cdot 5^4$ and Nagell-Lutz's theorem states that $E_{tor}(\mathbb{Q})=\{O\}$, i.e., trivial. Moreover the conductor of E is 10800 and the Mordell-Weil rank of E is one. Actually this curve is called $10800bt1$ in Cremona's table [4]. We can show $E(\mathbb{Q})\cong\mathbb{Z}=\langle P=(-5,10)\rangle$. Then we must verify when nP is integer points. Using the methods developed in [5] with the help of LLL-reduction, n is bounded up to 10. We have calculated all the cases $1\leq n\leq 10$ and verified $nP\notin E(\mathbb{Z})$ for $|n|\geq 3$, that is,

$$E(\mathbb{Z})=\{\pm P,\pm 2P\}=\{(-5,\pm 10),(10,\pm 10)\}.$$

$X = 5t^2 - 10 = -5$ and 10 imply $t = 1$ and 2. Since for $t \geq 3$, there exists no case such that $F_{2n+1} = \dfrac{t(t^2-3)}{2}$.

Proposition 4.1 *There exists no n and $t \geq 3$ which satisfy*

$$F_{2n+1} + \sqrt{F_{2n+1}^2 - 1} = \left(\frac{t + \sqrt{t^2-4}}{2}\right)^3.$$

We can treat the following case similarly

$$F_{2n} + \sqrt{F_{2n}^2 + 1} = \left(\frac{t + \sqrt{t^2+4}}{2}\right)^3.$$

Then

$$\left(\frac{t + \sqrt{t^2+4}}{2}\right)^3 = \frac{t(t^2+3) + (t^2+1)\sqrt{t^2+4}}{2}.$$

Suppose $F_{2n} = \dfrac{t(t^2+3)}{2}$ for some $t \geq 1$ in the equality $L_{2n}^2 - 5F_{2n}^2 = 4$. Then we have

$$(10L_{2n})^2 = (5t^2)(5t^2+15)^2 + 400.$$

Putting $X = 5t^2 + 10$ and $Y = 10L_{2n}$, we get an elliptic curve

$$E : Y^2 = X^3 - 75X + 150.$$

The solutions of $F_{2n} = \dfrac{t(t^2+3)}{2}$ correspond to the integer points on the above elliptic curve. The discriminant of this elliptic curve is also $\Delta(E) = 2^6 \cdot 3^2 \cdot 5^4$ and we can verify $E_{tor}(\mathbb{Q}) = \{O\}$, i.e., trivial. The conductor of E is 5400 and the Mordell-Weil rank of E is one. This curve is called $5400bj1$ in Cremona's table and $E(\mathbb{Q}) \cong \mathbb{Z} = \langle P = (-5, 20)\rangle$. In the same way as above, we have examined $nP \notin E(\mathbb{Z})$ for $|n| \geq 3$. Hence we can conclude

$$E(\mathbb{Z}) = \{\pm P, \pm 2P\} = \{(-5, \pm 20), (10, \pm 20)\}.$$

Thus $X = 5t^2 + 10 = -5$ and 10 imply $t = 0$. Since for $t \geq 1$, there exists no case such that $F_{2n} = \dfrac{t(t^2+3)}{2}$.

Proposition 4.2 *There exists no n and $t \geq 1$ which satisfy*

$$F_{2n} + \sqrt{F_{2n}^2 + 1} = \left(\frac{t + \sqrt{t^2+4}}{2}\right)^3.$$

Now we can treat the following case similarly

$$L_{2n+1} + \sqrt{L_{2n+1}^2 - 1} = \left(\frac{t + \sqrt{t^2 - 4}}{2} \right)^3, \text{ for some } t \geq 3.$$

Putting $X = 5t^2 - 10$ and $Y = 50F_{2n+1}$, we have obtained the following elliptic curve

$$E : Y^2 = X^3 - 75X + 2250.$$

The solutions of $L_{2n+1} = \dfrac{t(t^2 - 3)}{2}$ correspond to the integer points on the above elliptic curve. The discriminant of this elliptic curve is $\Delta(E) = -2^{10} \cdot 3^3 \cdot 5^7$ and $E_{tor}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} = \langle Q = (-15, 0) \rangle$. This curve is called $3600d1$ in Cremona's table and the Mordell-Weil rank is one. We can show

$$E(\mathbb{Q}) \cong \mathbb{Z}/\mathbb{Z} \times \mathbb{Z} = \langle Q = (-15, 0) \rangle \times \langle P = (-5, 50) \rangle.$$

In the same way as above, we have examined $mQ + nP \notin E(\mathbb{Z})$ for $|n| \geq 5$. Calculating these cases

$$E(\mathbb{Z}) = \{Q, \pm P, \pm 2P, Q \pm P.Q \pm 2P, Q \pm 4P\}$$

$$= \{(-15, 0), (-5, \pm 50), (10, \pm 50), (45, \pm 300), (9, \pm 48), (9585, \pm 938400)\}.$$

$X = 5t^2 - 10$ implies $t = 1$ or 2. Since for $t \geq 3$, there exists no case such that $L_{2n+1} = \dfrac{t(t^2 - 3)}{2}$.

Proposition 4.3 *There exists no n and $t \geq 3$ which satisfy*

$$L_{2n+1} + \sqrt{L_{2n+1}^2 - 1} = \left(\frac{t + \sqrt{t^2 - 4}}{2} \right)^3.$$

Finally we shall consider the following case similarly

$$L_{2n} + \sqrt{L_{2n}^2 + 1} = \left(\frac{t + \sqrt{t^2 + 4}}{2} \right)^3, \text{ for some } t \geq 1.$$

Putting $X = 5t^2 + 10$ and $Y = 50F_{2n}$, we have obtained an elliptic curve

$$E : Y^2 = X^3 - 75X - 2250.$$

The solutions of $L_{2n} = \dfrac{t(t^2 + 3)}{2}$ correspond to the integer points on the above elliptic curve. The discriminant of this elliptic curve is $\Delta(E) = -2^{10} \cdot 3^3 \cdot 5^7$

and $E_{tor}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} = \langle Q = (15, 0)\rangle$. This curve is called 1800b1 in Cremona's table and the Mordell-Weil rank is one. We can show

$$E(\mathbb{Q}) \cong \mathbb{Z}/\mathbb{Z} \times \mathbb{Z} = \langle Q = (15, 0)\rangle \times \langle P = (30, 150)\rangle.$$

In the same way as above, we have examined $mQ + nP \notin E(\mathbb{Z})$ for $|n| \geq 3$. Hence

$$E(\mathbb{Z}) = \{Q, \pm P, Q\pm P, Q\pm 2P\} = \{(15, 0), (30, \pm 150), (55, \pm 400), (399, \pm 7968)\}.$$

Then $X = 5t^2 + 10$ implies $t = 1, 2$ or 3. $t = 1$ corresponds to the case $L_0 = 2$ and

$$L_0 + \sqrt{L_0^2 + 1} = 2 + \sqrt{5} = \left(\frac{1 + \sqrt{5}}{2}\right)^3.$$

$t = 2$ corresponds to the case $L_4 = 7$ and

$$L_4 + \sqrt{L_4^2 + 1} = 7 + 5\sqrt{2} = (1 + \sqrt{2})^3.$$

$t = 3$ corresponds to the case $L_6 = 18$ and

$$L_6 + \sqrt{L_6^2 + 1} = 18 + 5\sqrt{13} = \left(\frac{3 + \sqrt{13}}{2}\right)^3.$$

Hence we have shown the following proposition,

Proposition 4.4

$$L_{2n} + \sqrt{L_{2n}^2 + 1} = \left(\frac{t + \sqrt{t^2 + 4}}{2}\right)^3, \quad \textit{for some } t \geq 1$$

if and only if $n = 0, 2$ *and* 3.

Now we shall quote some related problem which is called Eisenstein's problem. Let D be a positive integer congruent to 5 mod 8. In 1844 Eisenstein asked when the following equation has the odd solutions X and Y;

$$(5)\ X^2 - DY^2 = 4.$$

Let ℓ and ℓ^* be the length of the continued fraction expansions of $\sqrt{D}$ and $\frac{\sqrt{D}+1}{2}$, respectively. In [8], P. Kaplan and K. S. Williams have given a necessary and sufficient condition when the equation $x^2 - Dy^2 = -1$ is solvable, i.e., $\ell \equiv \ell^* \equiv 1 \mod 2$ as follows.

Proposition 4.5 ([8]) *The above equation* (5) *is solvable if and only if* $\ell \equiv \ell^*$

mod 4.

More general conditions for the solvability of Eisenstein's problem have been investigated in [7] and [16]. We can verify that $L_n^2 + (-1)^n$ and hence $s(L_n^2 + (-1)^n)$ is congruent to 5 mod 8 for the case $n = 6m$. Put $D_m = s(L_{6m}^2 + 1)$. Though we could not use the above criterion for D_m, because we could not express the exact value D_m explicitly. But the existence of the explicit unit $L_{6m} + \sqrt{L_{6m}^2 + 1}$ of $\mathbb{Q}(\sqrt{D_m})$ will play as a key ingredient as follows. Suppose that there exists an odd solution X and Y for the equation $X^2 - D_mY^2 = 4$. Then the fundamental unit ε of $\mathbb{Q}(\sqrt{D_m})$ must be written in the form $\dfrac{t_0 + \sqrt{t_0^2 + 4}}{2}$ with odd t_0. Then the explicit unit $L_{6m} + \sqrt{L_{6m}^2 + 1}$ is expressed as ε^{3k} for some $k \geq 1$. From Proposition 4.4, we see that it occurs if and only if $m = 1$. Thus we have proved the following theorem.

Theorem 4.1 *With the notations, the following equation has the odd solutions X and Y if and only if $m = 1$,*

$$X^2 - D_mY^2 = 4.$$

Let ℓ_m and ${\ell_m}^*$ be the length of the continued fraction expansions of $\sqrt{D_m}$ and $\frac{\sqrt{D_m}+1}{2}$, respectively. Then combining Proposition 4.5 and the above theorem, we can state the following corollary.

Corollary 4.1 *With the above notations*

$$\ell_m \equiv {\ell_m}^* \mod 4 \Longleftrightarrow m = 1.$$

Numerical examples. Since $L_6^2 + 1 = 325 = 5^2 \cdot 13$, we see $D_1 = 13$ and the length of the continued fraction expansions of $\sqrt{13}$ and $\frac{\sqrt{13}+1}{2}$ are 5 and 1 and hence $\ell_1 = 5 \equiv 1 = \ell_1^* \mod 4$. In the case $D_2 = 103685 = 5 \cdot 89 \cdot 233$, the length of the continued fraction expansions of $\sqrt{103685}$ and $\frac{\sqrt{103685}+1}{2}$ are 1 and 3 and hence $\ell_2 = 1 \not\equiv 3 = \ell_2^* \mod 4$.

5 Simultaneous Pell Equations

We shall recall our previous results in [12] which determine all n such that

$$F_{2n+1} + \sqrt{F_{2n+1}^2 - 1} = \left(\frac{t + \sqrt{t^2 + 4}}{2}\right)^2 \text{ for some } t \geq 1, \text{ or } \left(\frac{t + \sqrt{t^2 - 4}}{2}\right)^2$$

for some $t \geq 3$, and

$$L_{2n+1} + \sqrt{L_{2n+1}^2 - 1} = \left(\frac{t+\sqrt{t^2+4}}{2}\right)^2 \text{ for some } t \geq 1, \text{ or } \left(\frac{t+\sqrt{t^2-4}}{2}\right)^2$$

for some $t \geq 3$.
For the sake of completeness, we shall give the sketch of proofs of our results as follows. We shall prepare the following elementary lemma on the properties on Fibonacci and Lucas numbers.

Lemma 5.1

$$\begin{array}{ll} F_{2n+1}L_{2n} = F_{4n+1} + 1, & F_{2n}L_{2n+1} = F_{4n+1} - 1, \\ F_{2n-1}L_{2n} = F_{4n-1} + 1, & F_{2n}L_{2n-1} = F_{4n-1} - 1, \\ 5F_{2n-1}F_{2n} = L_{4n-1} + 1, & 5F_{2n+1}F_{2n} = L_{4n+1} - 1, \\ L_{2n}L_{2n+1} = L_{4n+1} + 1, & L_{2n-1}L_{2n} = L_{4n+1} - 1. \end{array}$$

We can verify the first case of this lemma, using Binet's formula as follows. Let us denote $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$. Then

$$F_{2n+1}L_{2n} = \frac{(\varphi^{2n+1} - \bar{\varphi}^{2n+1})(\varphi^{2n} + \bar{\varphi}^{2n})}{\sqrt{5}} = \frac{\varphi^{4n+1} - \bar{\varphi}^{4n+1} + \varphi - \bar{\varphi}}{\sqrt{5}} = F_{4n+1}+1.$$

One can easily verify other cases similarly. Here we shall recall J. H. E. Cohn's results in [3] which we have already mentioned.

Proposition 5.1 ([3], [21]) *F_n and L_n $(n \geq 0)$ satisfy*

$$\begin{array}{llll} F_n = \Box & \Longleftrightarrow\; n = 0, 1, 2 \textit{ or } 12 & \Longleftrightarrow & F_n = F_0 = 0, F_1 = F_2 = 1 \textit{ or } F_{12} = 12^2, \\ F_n = 2\Box & \Longleftrightarrow\; n = 0, 3 \textit{ or } 6 & \Longleftrightarrow & F_n = F_0 = 0, F_3 = 2 \textit{ or } F_6 = 2 \cdot 2^2, \\ L_n = \Box & \Longleftrightarrow\; n = 1 \textit{ or } 3 & \Longleftrightarrow & L_n = L_1 = 1 \textit{ or } L_2 = 2^2, \\ F_n = 2\Box & \Longleftrightarrow\; n = 0 \textit{ or } 4 & \Longleftrightarrow & L_n = L_0 = 2 \textit{ or } L_6 = 2 \cdot 3^2. \end{array}$$

Here we shall recall the case

$$F_{4n+1} + \sqrt{F_{4n+1}^2 - 1} = \left(\frac{t+\sqrt{t^2+4}}{2}\right)^2 \text{ for some } t \geq 1, \text{ or } \left(\frac{t+\sqrt{t^2-4}}{2}\right)^2.$$

Since

$$\left(\frac{t+\sqrt{t^2 \pm 4}}{2}\right)^2 = \frac{t^2 \pm 2 + \sqrt{(t^2 \pm 2)^2 - 4}}{2},$$

the above equality shows $F_{4n+1} \mp 1 = 2\square$, that is, $F_{2n}L_{2n+1} = 2\square$ or $F_{2n+1}L_{2n} = 2\square$. Thus at least $L_{2n} = \square, 2\square$ or $L_{2n+1} = \square, 2\square$. From Proposition 5.2, $L_0 = 2, L_1 = 1, L_3 = 2$ or $L_6 = 18$, and $F_1 + 1 = 2$, i.e., $n = 0$ is the only case. Since $F_1^2 - 1 = 0$, it is not the case. In the same way, one can check finitely many candidates and there is no n except the case $L_{11} + \sqrt{L_{11}^2 - 1} = 199 + 20\sqrt{99} = (10 + 3\sqrt{11})^2$.

Proposition 5.2 *With the above notations, there exists no* n

$$F_{2n+1} + \sqrt{F_{2n+1}^2 - 1} = \left(\frac{t + \sqrt{t^2+4}}{2}\right)^2 \text{ (resp. } \left(\frac{t + \sqrt{t^2-4}}{2}\right)^2 \text{) for some } t \geq 1$$

(*resp.* $t \geq 3$), *and*

$$L_{2n+1} + \sqrt{L_{2n+1}^2 - 1} = \left(\frac{t + \sqrt{t^2+4}}{2}\right)^2 \text{ (resp. } \left(\frac{t + \sqrt{t^2-4}}{2}\right)^2 \text{) for some } t \geq 1$$

(*resp.* $t \geq 3$) $\iff n = 5$.

Now we shall recall our conclusion in section 3 and 4. In section 3, we have proved, for the case $n \geq N_F$,

$$F_n + \sqrt{F_n^2 + (-1)^n} = \left(\frac{t + \sqrt{t^2 + (-1)^n 4}}{2}\right)^k \quad \text{implies } k \leq 4,$$

under the abc conjecture. In section 4, we have examined,

$$F_n + \sqrt{F_n^2 + (-1)^n} \neq \left(\frac{t + \sqrt{t^2 + (-1)^n 4}}{2}\right)^3.$$

Since $\left(\frac{t + \sqrt{t^2 \pm 4}}{2}\right)^4 = \left(\frac{t^2 \pm 2 + \sqrt{(t^2 \pm 2)^2 - 4}}{2}\right)^2$, we have verified

$$F_n + \sqrt{F_n^2 + (-1)^n} \neq \left(\frac{t + \sqrt{t^2 + (-1)^n 4}}{2}\right)^k \quad \text{for any } k \geq 2.$$

Theorem 5.1 (*Assuming the abc conjecture*) *In the case* $n \geq N_F$,

$$F_n + \sqrt{F_n^2 + (-1)^n} \neq \left(\frac{t + \sqrt{t^2 + (-1)^m 4}}{2}\right)^k \quad \textit{for any } t, m \textit{ and } k (\geq 2).$$

In the same way as above, from the fact $N_L \gg 6$, we can show the following similar result.

Theorem 5.2 (*Assuming the abc conjecture*) *In the case* $n \geq N_L$,

$$L_n + \sqrt{L_n^2 + (-1)^n} \neq \left(\frac{t + \sqrt{t^2 + (-1)^m 4}}{2} \right)^k \quad \text{for any } t, m \text{ and } k (\geq 2).$$

Let us denote the set of positive integers $\{D_n = s(F_n^2 + (-1)^n) \mid n \geq 2\}$ by $\mathbf{D}$ and $\max\{D_n \in \mathbf{D} \mid n \leq N_F\}$ by D_0. Then as a corollary of this theorem, we can show the precise description of the existence of positive integer solutions of our simultaneous Pell equations (2) and (3) as follows.

Theorem 5.3 (*Assuming the abc conjecture*) *The simultaneous Pell equations*

$$(2) \begin{cases} x^2 - 5dz^2 = 1 \\ y^2 - dz^2 = 1 \end{cases}, \quad (3) \begin{cases} x^2 - 5dz^2 = -1 \\ y^2 - dz^2 = -1 \end{cases}$$

have positive integer solutions if and only if $d \in \mathbf{D}$. *The number of positive integer solutions of* (2) *or* (3) *is at most one for the case* $d > D_0$, *and* d *is expressed as* $d = s(F_n^2 + (-1)^n)$ *for some* $n > N_F$. *Moreover the unique positive integer solution* (x, y, z) *is given by* $x = L_n, y = F_n$ *and* $z = q(F_n^2 + (-1)^n)$.

Finally we shall show that $d \in \mathbf{D}$ grows exponentially under the abc conjecture. Analogous result has been obtained by G. Walsh in [25]. By virtue of the Binet's formula, it is easy to show

$$\frac{\varphi^n}{3} < F_n = \frac{\varphi^n - (1/\varphi)^n}{\sqrt{5}} < \frac{\varphi^n}{2} \quad \text{for } n \geq 4.$$

On the other hand, assuming the abc conjecture, we have proved that, for any δ such that $0 < \delta < 1$, the following inequality holds for sufficiently large n in Preposition 2.1.

$$F_n^{1-\delta} \leq s(F_n) \leq F_n.$$

Since $D_n = s(F_n^2 + (-1)^n) = s(F_{n-1}F_{n+1}) = s(F_{n-1})s(F_{n+1})$, one can see

$$\left(\frac{1}{9}\right)^{1-\delta} \varphi^{2n} < D_n < \frac{1}{4}\varphi^{2n}.$$

Denote $C(\delta) = (1/9)^{1-\delta}$. Then we have obtained the following exponential growth of D_n.

Theorem 5.4 (*Assuming the abc conjecture*) *Let D_n denote in increasing order $d \in \mathbf{D}$, that is, d with* (2) *or (3) has a solution. Then, for any $0 < \delta < 1$, there exists a constant $C(\delta)$ which satisfies*

$$C(\delta)\varphi^{2n} < D_n < \frac{1}{4}\varphi^{2n}.$$

6 Biquadratic Fields

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ be a real biquadratic field. k_1, k_2 and k_3 denote the real quadratic subfield $\mathbb{Q}(\sqrt{d_1}), \mathbb{Q}(\sqrt{d_2})$ and $\mathbb{Q}(\sqrt{d_1 d_2})$, respectively. Let $\varepsilon_1, \varepsilon_2$ and ε_3 be the fundamental units of k_1, k_2 and k_3. E_K denotes the unit group of K. Then the Hasse unit index Q_K of $E = \langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ in E_K is defined by the group index $[E_K : E]$. We note here that Q_K is known to be $1, 2$ or 4 (see for example [24]). Let h_{k_i} be the class number of the real quadratic field k_i and h_K be the class number of the biquadratic field K. Then the classical Dirichlet's class number formula states that

$$h_K = \frac{Q_K}{4} h_{k_1} h_{k_2} h_{k_3}.$$

Now we shall recall our previous results on $K = \mathbb{Q}(\sqrt{5}, \sqrt{F_2^2 + (-1)^n})$. Then $k_1 = \mathbb{Q}(\sqrt{5})$, $k_2 = \mathbb{Q}(\sqrt{F_n^2 + (-1)^n})$ and $k_3 = \mathbb{Q}(\sqrt{L_n^2 + (-1)^n})$ and $\varepsilon_1 = \frac{1+\sqrt{5}}{2} = \varphi$. We verified $Q_K = 2$ for the cases $n = 2, 3, 4, 5, 6, 11$ and 13 and $Q_K = 1$ for other cases.

Proposition 6.1

$$E_K = \begin{cases} \langle -1, \varepsilon_1, \sqrt{\varepsilon_2}, \varepsilon_3 \rangle & \text{for } n = 11, \\ \langle -1, \varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3} \rangle & \text{for } n = 5, \\ \langle -1, \varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_2 \varepsilon_3} \rangle & \text{for } n = 3 \text{ or } 13, \\ \langle -1, \varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_1 \varepsilon_2 \varepsilon_3} \rangle & \text{for } n = 2, 4 \text{ or } 6, \\ \langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle & \text{otherwise.} \end{cases}$$

Put $h_2 = h_F$ and $h_3 = h_L$. Then assuming the abc conjecture, we have the following theorem.

Theorem 6.1 *For any $n \geq N_L$, we have*
$E_K = \langle -1, \dfrac{1+\sqrt{5}}{2}, F_n + \sqrt{F_n^2 + (-1)^n}, L_n + \sqrt{L_n^2 + (-1)^n} \rangle$ *and* $h_K = \dfrac{h_F h_L}{4}$.

7 Numerical Calculations

In our previous paper [11], we have given the following conjecture.

Conjecture. For any $n \geq 2$, $F_n + \sqrt{F_n^2 + (-1)^n}$ is the fundamental unit of the real quadratic fields $\mathbb{Q}(\sqrt{F_n^2 + (-1)^n})$.

In our previous paper, we have calculated the fundamental unit using the continued fraction expansion of $\sqrt{s(F_n^2 + (-1)^n)}$ and verified this conjecture for small $n \leq 225$. We also checked the square free parts of Fibonacci numbers and verified this conjecture for $2 \leq n \leq 702$. We have stopped to verify this conjecture at that time, because F_{703} is the smallest Fibonacci number which is not completely factorized at that time. From the proof of Proposition 3.1 and Proposition 4.1, 4.2 and 5.2, to verify the above conjecture, one only need to show $s(F_n)^2 > 2q(F_n)$. Using the new table of the prime factorizations of Fibonacci numbers of B. Kelly [14], we can easily verify that $s(F_n)^2 > 2q(F_n)$ holds for any $13 \leq n \leq 1270$. Now F_{1271} is the smallest Fibonacci number which is not completely factorized so far. Thus we have verified the following proposition.

Proposition 7.1 *For any* $2 \leq n \leq 1270$*,* $F_n + \sqrt{F_n^2 + (-1)^n}$ *is the fundamental unit of the real quadratic fields* $\mathbb{Q}(\sqrt{F_n^2 + (-1)^n})$*.*

In the same way as above, we have verified the square free parts of Fibonacci numbers satisfy $s(F_n)^2 > 10\sqrt{10}q(F_n)$ for $13 \leq n \leq 1270$. From the proof of Proposition 3.2 we know each $L_n + \sqrt{L_n^2 + (-1)^n}$ is at most 4th power of the fundamental unit. Combining Proposition 4.3, 4.4 and 5.2, we know $L_n + \sqrt{L_n^2 + (-1)^n}$ is the fundamental unit except for the cases $n = 4, 6$ and 11. Thus we have obtained the following proposition.

Proposition 7.2 *For* $n = 2, 3, 5, 7, 8, 9, 10$ *and* $12 \leq n \leq 1270$*,* $L_n + \sqrt{L_n^2 + (-1)^n}$ *is the fundamental unit of the real quadratic fields* $\mathbb{Q}(\sqrt{L_n^2 + (-1)^n})$*. For the case* $n = 4, 6$ *and* 11*,*

$$L_4 + \sqrt{L_4^2 + 1} = 7 + 5\sqrt{2} = (1 + \sqrt{2})^3,$$

$$L_6 + \sqrt{L_6^2 + 1} = 18 + 5\sqrt{13} = \left(\frac{3 + \sqrt{13}}{2}\right)^3,$$

$$L_{11} + \sqrt{L_{11}^2 - 1} = 199 + 20\sqrt{99} = (10 + 3\sqrt{11})^2.$$

Let K be the biquadratic field $\mathbb{Q}(\sqrt{5}, \sqrt{F_n^2 + (-1)^n}) = \mathbb{Q}(\sqrt{F_n^2 + (-1)^n}, \sqrt{L_n^2 + (-1)^n})$ as above. Then $k_1 = \mathbb{Q}(\sqrt{5})$, $k_2 = \mathbb{Q}(\sqrt{F_n^2 + (-1)^n})$ and $k_3 = \mathbb{Q}(\sqrt{L_n^2 + (-1)^n})$ and $\varepsilon_1 = \frac{1+\sqrt{5}}{2} = \varphi$. We also denote the fundamental units of k_2 and k_3 by ε_2 and ε_3. Here we shall quote the numerical data on the unit group

E_K for small n. Let U_K be the subgroup of unit group E_K generated by $\varepsilon_1, F_n + \sqrt{F_n^2 + (-1)^n}$ and $L_n + \sqrt{L_n^2 + (-1)^n}$. We define the group index I_n by putting

$$I_n = [E_K : \langle -1, \varepsilon = \frac{1+\sqrt{5}}{2}, F_n + \sqrt{F_n^2 + (-1)^n}, L_n + \sqrt{L_n^2 + (-1)^n}\rangle].$$

Then $I_n = 1$ for $2 \leq n \leq 1270$ except for $n = 2, 3, 4, 5, 6, 11$ and 13. $I_n = 2$ for the cases $n = 2, 3, 5$ or 13 and $I_n = 4$ for the case $n = 11$ and $I_n = 6$ for the cases $n = 4$ or 6. More precisely, we shall give the following table: Unit groups with exceptional group indices I_n.

n	I_n	E_K	U_K
2	2	$\langle -1, \varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\rangle$	$\langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3\rangle$
3	2	$\langle -1, \varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_2\varepsilon_3}\rangle$	$\langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3\rangle$
4	6	$\langle -1, \varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\rangle$	$\langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3^3\rangle$
5	2	$\langle -1, \varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\rangle$	$\langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3\rangle$
6	6	$\langle -1, \varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\rangle$	$\langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3^3\rangle$
11	4	$\langle -1, \varepsilon_1, \sqrt{\varepsilon_2}, \varepsilon_3\rangle$	$\langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3^2\rangle$
13	2	$\langle -1, \varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_2\varepsilon_3}\rangle$	$\langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3\rangle$

Combining these data on I_n for $13 \leq n \leq 1270$ and Theorem 6.1, now we shall state a new conjecture which generalize the above conjecture,

Conjecture. For any $n \geq 13$, $I_n = 1$, that is, the set of the units $\{\varphi, F_n + \sqrt{F_n^2 + (-1)^n}, L_n + \sqrt{L_n^2 + (-1)^n}\}$ is the fundamental system of units of the biquadratic field $\mathbb{Q}(\sqrt{F_n^2 + (-1)^n}, \sqrt{L_n^2 + (-1)^n})$.

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