J. Math. Univ. Tokushima
Vol.38(2004), 9 −16

New Notions of Convergence of Directed Families of Points and Convergence of Filters

By

Yoshifumi Ito

Faculty of Integrated Arts and Sciences
The University of Tokushima
1-1, Minamijosanjima-cho
Tokushima-shi 770-8502, Japan
e-mail : y-ito@ias.tokushima-u.ac.jp
(Received November 4, 2003)

Abstract

In this article, we introduce new notions of convergence of directed families of points and convergence of filters in a general topological space where we do not necessarily assume any seperation axiom. Then we mention some new properties of them for a Hausdorff topological space.

2000 Mathematics Subject Classification. Primary 54A20.

Introduction

In a Hausdorff topological space which satisfies the first countability axiom, we can also define the topology by way of defining the limit of sequence of countable points. In a general topological space which does not necessarily satisfy the first countability axiom, we cannot define the topology by using sequences of countable points as above. In order to improve this, Moore-Smith indroduced the notion of limit of directed families of points which are not necessarily countable [1],[6],[7],[8]. On the other hand, H. Cartan and N. Bourbaki introduced the notion of limit of filters [2],[3],[4],[7]. Both define the equivalent topology.

Nevertheless, in the topological space where we do not assume any separation axiom, the meanings of both notions of limit in the sense of Moore-Smith and of limit of filters in the sense of Cartan-Bourbaki, are ambiguous. We cannot define their convergence to a certain point clearly. It occurs that they converge to more than one points simultaneously.

Therefore, in this article, we try to improve the notions of limit of directed families of points and of limit of filters so that these notions have reasonable meaning in the topological space where we do not necessarily assume any separation axiom.

Even only for a Hausdorff topological space, we have some new results of convergence.

1. Convergence of sequences of points

Let X be a topological space. For a set S in X, $\overline{S}$ denotes the closure of S. Here we give the definition of new notion of convergence of sequences of points.

Definition 1.1. Let X be a topological space. For a sequence $\{a_n\}$ in X and a nonempty set A of accumulation points of $\{a_n\}$, we say that the sequence $\{a_n\}$ converges to A if A satisfies the following conditions :

(1) For any neighborhood U of A, there exists some natural number n_0 such that, for every $n \geq n_0$, $a_n \in U$ holds.

(2) A is the maximum one which satisfies the condition (1).

We say that A is the limit set of $\{a_n\}$ or simply the limit and we denote it as $\lim\limits_{n\to\infty} a_n = A$ or $a_n \to A(n \to \infty)$. We say that every point in A is a limit point of $\{a_n\}$. Then we have the following.

Corollary 1.2. *We use the notation in Definition 1.1. Then the set A is calculated by the relation*

$$\bigcap_{n=1}^{\infty} \overline{\bigcup_{m\geq n} \{a_m\}} = A.$$

Thus the limit set can be calculated by the set operation. If $A = \{a\}$, the limit point of $\{a_n\}$ is the same as considered until now. In this case, we say that $\{a_n\}$ converges to a in the narrow sense or mearly $\{a_n\}$ converges to a. But if A is composed of more than one points, then the notion of the limit of $\{a_n\}$ is of a new case. In this case, we say that $\{a_n\}$ converges to A in the wide sense. In any way, if the limit set A is composed of one or more than one points, we say that $\{a_n\}$ converges to A. After all the limit set A is the set of all accumulation points of $\{a_n\}$. But, conversely, the set of all accumulation points of $\{a_n\}$ is not necesarily the limit set of $\{a_n\}$. For example, we have the following.

Example 1 (Asaoka). If we put $a_n = 0$ for odd n and $a_n = n$ for even n, then the set of all accumulation points of $\{a_n\}$ is $\{0\}$. Then $\{a_n\}$ does not converges to $\{0\}$.

Here we have another example.

Example 2 (Ito). Let $\{a_n\}$ be the sequence obtained by lining all rational numbers up. Then $a_n \to \boldsymbol{R}$.

In this case, the limit set is not compact.

Then we have the following.

Proposition 1.3. *A sequence of real numbers converges to a nonempty bounded closed set if and only if the sequence is bounded.*

As for the convergence of sequence of real numbers in the narrow sense, we have the following.

Theorem 1.4 (Cauchy's Criterion of Convergence). *A sequence $\{a_n\}$ of real numbers converges to a certain real number in the narrow sense if and only if, for any positive number ε, there exists some natural number N such that, for every $m, n > N$, the inequality $|a_m - a_n| < \varepsilon$ holds.*

Let X be a topological space. We say that a sequence $\{a_n\}$ of points in X is precompact if the closure of the set of points $\{a_n; n \geq 1\}$ is a compact set in X.

Then we have the following.

Theorem 1.5. *Let X be a Hausdorff topological space. Then a sequence $\{a_n\}$ of points in X converges to a certain nonempty compact set A if and only if $\{a_n\}$ is a precompact sequence. Then the limit set A is given by the relation*

$$\bigcap_{n=1}^{\infty} \overline{\bigcup_{m \geq n} \{a_m\}} = A.$$

In the above theorem 1.5, the limit set A is not empty because the family of sets in X $\overline{\{a_m; m \geq n\}}, (n = 1, 2, \cdots)$ has the finite intersection property.

Then we have the following.

Corollary 1.6. *Let X be a compact Hausdorff topological space. Then every sequence $\{a_n\}$ of points in X converges to a certain nonempty compact set.*

Proposition 1.7. *In a Hausdorff topological space X, a sequence $\{a_n\}$ of points in X converges to a point a if, for any neighborhood U of a, there exists some natural number n_0 such that we have $a_n \in U$ for all n such as $n \geq n_0$.*

Theorem 1.8. *For the convergence of sequences in a topological space X, we have the following properties. We use the above notation.*

(S1) *If $a_n = a$ for all n, we have $a_n \to \overline{\{a\}}$.*

(S2) *If $a_n \to A$ for the set A of all accumulation points of $\{a_n\}$, then, for any convergent subsequence $\{a_{k_m}\}$ of $\{a_n\}$, we have*

$$\bigcup_{\{k_m\}} \bigcap_{n=1}^{\infty} \overline{\bigcup_{m \geq n} \{a_{k_m}\}} = A.$$

Here $\bigcup_{\{k_m\}}$ means the union for all subsequences $\{k_m\}$ of the sequence of all natural numbers such that $\{a_{k_m}\}$ converges.

(S3) *Let $\{a_n\}$ be a sequence in X and A the set of all accumulation points of $\{a_n\}$. If, for any subsequence $\{a_{k_m}\}$ of $\{a_n\}$, we have $a_{k_m} \to A$, we have $a_n \to A$.*

For a topological space X which is not Hausdorff, we have the following.

Example 3 (Asaoka). Let X be a set $\{0, 1\}$ of 0 and 1. Assume that all open sets in X are $\emptyset$, $\{1\}$ and $X = \{0, 1\}$. Then X becomes a topological space which is not Hausdorff. In X, all closed sets are $\emptyset$, $\{0\}$ and X. Then if we consider a sequence $\{a_n\}$ such as $a_n = 1$ for all $n \geq 1$, the set of all accumulation points of $\{a_n\}$ is X and $\{a_n\}$ converges to X. In this case, we have,

$$\bigcap_{n=1}^{\infty} \overline{\bigcup_{m \geq n} \{a_m\}} = \{0, 1\} = X$$

If we consider a sequence $\{a_n\}$ such as $a_n = 0$ for all $n \geq 1$, the set of all accumulation points of $\{a_n\}$ is $\{0\}$ and $\{a_n\}$ converges to $\{0\}$. In this case, we have

$$\bigcap_{n=1}^{\infty} \overline{\bigcup_{m \geq n} \{a_m\}} = \overline{\{0\}} = \{0\}.$$

But the limit set is not the intersection of all neighborhoods of the limit set.

Let X and Y be two topological spaces. In general, we assume that f is a mapping from a certain subset D of X to Y. Let A be a certain set of accumulation points of D. For a certain set B of accumulation points of the set $\{f(x); x \in D\}$, we say that $\lim_{x \to A} f(x) = B$ or $f(x) \to B$ as $x \to A$ if, for any neighborhood V of B, there exists some neighborhood U of A such that we have $f((U \cap D) \setminus A) \subset V$. $\lim_{x \to A} f(x) = B$ is equivalent to say that, for any sequence $\{a_n\}$ in $D \setminus A$ such as $a_n \to A$, B is the union of all limit sets $\lim_{n \to \infty} f(a_n)$. If we have $B = \{b\}$ for a given f and $A = \{a\}$ in X, b is said to be the limit value of $f(x)$ as $x \to a$ or the limit of $f(x)$ as $x \to a$.

In Definition 1.1, we give the definition of new notion of convergence of sequences of points for a general topological space.

2. Cnvergence of directed families of points

In this section, we consider the new notions of convergence of directed families of points.

Definition 2.1. Let X be a topological space, $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ a directed family of points in X and A a nonempty set of accumulation points of $\{x_\alpha\}_{\alpha \in \mathcal{A}}$. Then we say that $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ converges to A if A satisfies the following conditions :

(1) For any neighborhood U of A, there exists some $\alpha_0 \in \mathcal{A}$ such that, for every $\alpha \geq \alpha_0$, $x_\alpha \in U$ holds.

(2) A is the maximum one which satisfies the condition (1).

Then we denote this as $x_\alpha \to A (\alpha \in \mathcal{A})$ or simply $x_\alpha \to A$. Then we have the following.

Corollary 2.2. *We use the notation of Definition 2.1. Then the set A is calculated by the relation*

$$\bigcap_{\alpha_0 \in \mathcal{A}} \overline{\bigcup_{\alpha \geq \alpha_0} \{x_\alpha\}} = A.$$

When $\mathcal{A} = \boldsymbol{N}$, this is nothing but the convergence of a sequence of points $\{x_n\}$ in X. The limit set A is the certain set of accumulation points of $\{x_\alpha\}_{\alpha\in\mathcal{A}}$. But, conversely, any set of accumulation points of $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is not necesarily the limit set of $\{x_\alpha\}_{\alpha\in\mathcal{A}}$.

Then we have the following.

Theorem 2.3. *Let X be a Hausdorff topological space. Then a directed family of points $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ in X converges to a certain nonempty compact set A if there exists some $\alpha_0 \in \mathcal{A}$ such that the closure of the set of points $\{x_\alpha; \alpha \geq \alpha_0\}$ is a compact set in X.*

Corollary 2.4. *Let X be a compact Hausdorff topological space. Then every directed family of points $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ in X converges to a certain nonempty compact set.*

Proposition 2.5. *Assume that X is a Hausdorff topological space. Then a directed family of points $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ in X converges to a point x if, for any neighborhood U of x, there exists some $\alpha_0 \in \mathcal{A}$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.*

Theorem 2.6. *Let X be a topological space. Let $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ be a directed family of points in X. We use the above notation. Then we have the following:*

(D1) *If $x_\alpha = x$ for all α, $x_\alpha \to \overline{\{x\}}$ holds.*

(D2) *If $x_\alpha \to A$ for a nonempty set A of accumulation points of $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{y_\beta\}$ is a cofinal directed subfamily of $\{x_\alpha\}$, $y_\beta \to A$ holds.*

(D3) *If a directed subfamily $\{y_\beta\}$ of $\{x_\alpha\}$ has always a converginy directed subfamily $\{z_\gamma\}$ and $z_\gamma \to A$ for a nonempty set A of accumulation points of $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ and A is determined independently of the choice of $\{y_\beta\}$, then $x_\alpha \to A$ holds.*

(D4) *Assume that $x_\alpha \to A(\alpha \in \mathcal{A})$ holds for a nonempty set A of accumulation points of $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ and, for each $\alpha \in \mathcal{A}$, $y_{\alpha\beta} \to x_\alpha(\beta \in \mathcal{B}_\alpha)$ holds. Then we define a directed set of direct product $\mathcal{C} = \mathcal{A} \times \prod_\alpha \mathcal{B}_\alpha$ and define the projections $p : \mathcal{C} \to \mathcal{A}$ and $p_\alpha : \mathcal{C} \to \mathcal{B}_\alpha$. If we define $z_\gamma = y_{\alpha\beta}$ where $\gamma \in \mathcal{C}, \alpha = p(\gamma), \beta = p_\alpha(\gamma)$ hold, then $z_\gamma \to A$ holds.*

We denote the limit set of x_α as $\lim x_\alpha$ or $\lim_{\alpha\in\mathcal{A}} x_\alpha$. We say that a point in $\lim x_\alpha$ is a limit point of x_α. $x_\alpha \to A$ is equivalent to say that A is the intersection of the closures of sets, each of which is composed of any cofinal directed subfamily of $\{x_\alpha\}$.

If a directed subfamily $\{y_\beta\}$ of $\{x_\alpha\}$ is not cofinal with $\{x_\alpha\}$, then some accumulation points of $\{y_\beta\}$ does not belong to the limit set of $\{x_\alpha\}_{\alpha\in\mathcal{A}}$.

3. Convergence of filters

Definition 3.1. Let X be a topological space and Φ a filter in X and A a nonempty set of accumulation points of every $F \in \Phi$. Then we denote a system of all neighborhoods of A by $\mathfrak{U}(A)$. We say that the filter Φ converges to the set A if A satisfies the following conditions :

(1) $\Phi \supset \mathfrak{U}(A)$ holds.

(2) A is the maximum one which satisfies the condition (1).

Then we denote this $\Phi \to A$ and we say that the set A is the limit set of Φ or simply the limit of Φ. If the filter generated by a filter basis $\mathcal{B}$ converges to A, then we say that the filter basis $\mathcal{B}$ converges to A.

Then we have the following.

Corollary 3.2. *We use the notation of Definition 3.1. Then the set A is calculated by the relation*

$$\bigcap_{F \in \Phi} \overline{F} = A.$$

Namely, the limit or the limit set can be obtained by the set operation.

Then we have the following.

Theorem 3.3. *Let X be a Hausdorff topological space. Then a filter Φ converges to a certain nonempty compact set A if there exists some element F of Φ such that $\overline{F}$ is compact.*

Corollary 3.4. *Let X be a compact Hausdorff topological space. Then every filter Φ converges to a certain nonempty compact set A.*

Proposition 3.5. *Assume that X is a Hausdorff topological space. Then a filter Φ converges to a point x if $\Phi \supset \mathfrak{U}(x)$ holds.*

Theorem 3.6. *Let X be a topological space. We use the above notation. Then we have the following:*

(L1) *For a point a in X, a filter $\Phi_a = \{B; a \in B \subset X\}$ converges to $\overline{\{a\}}$.*

(L2) *If two filters Φ and Ψ satisfy the conditions $\Phi \to A$ and $\Psi \supset \Phi$, then $\Psi \to A$ holds.*

(L3) *If, for a family of filters $\{\Phi_\lambda\}$, every $\Phi_\lambda \to A$ holds, then $\cap_\lambda \Phi_\lambda = \Phi \to A$ holds.*

(L4) *Assume the following* (i) $\sim$ (iii). (i) $X \supset Y$, (ii) *For every nonempty closed set B in Y, there exists a filter Φ_B in X such that $\Phi_B \to B$ and* (iii) *a filter Ψ in X generated by a filter basis $\mathcal{B}$ in Y converges to A. Then $\cup_{B \in \mathcal{B}} \left(\cap_{\emptyset \neq C = \overline{C} \subset B} \Phi_C\right)$ converges also to A.*

4. Relations between various convergences

In this section, we mention relations of various convergences.

We have the following theorems.

Theorem 4.1. *Let X be a topological space.*

(1) *For a directed family of points $\{x_\alpha\}_{\alpha \in \mathcal{A}}$, the family of subsets in X $\{\{x_\alpha; \alpha \in \mathcal{A}, \alpha \geq \alpha_0\}; \alpha_0 \in \mathcal{A}\}$ is a filter basis in X.*

(2) *For the filter Φ generated by the filter basis defined in* (1), *$x_\alpha \to A$ if and only if $\Phi \to A$.*

Then Theorem 3.6 induces Theorem 2.6 and Corollary 3.5 induces Corollary 2.5.

Let X and Y be two topological spaces and $f : X \to Y$ a function (or mapping) and D the domain of f. Let A be a certain set of accumulation points of D. Then $\{f(U \cap D \setminus A); U \in \mathfrak{U}(A)\}$ becomes a filter basis in Y, where $\mathfrak{U}(A)$ means a system of neighborhoods of A. Further assume $U \cap D \setminus A \neq \emptyset$. Let Φ be the filter generated by the above filter basis. Then $f(x) \to B$ as $x \to A$ if and only if $\Phi \to B$.

By the above, the various convergences mentioned until now can be represented by the convergence of filters.

5. Convergence and topology

In a topological space, we could define the notions of convergence of directed families of points and of filters. Conversely, we can introduce a topology using the notion of convergence. We have the following.

Theorem 5.1. *Assume that, in a topological space X, all filters are determined to converge or not and that the properties* (L1) $\sim$ (L4) *are satisfied in X. Then, if we define the convergence of directed families of points in X as above, then the properties* (D1) $\sim$ (D4) *are satisfied.*

Thereby, if we define the union of the limit sets of all directed families $\{x_\alpha\}$ such as $x_\alpha \in A$ to be $\overline{A}$, then the axiom of closures is satisfied for $\overline{A}$. Thereby we can define a topology of X. With respect to this topology, we have the following properties:

(i) $B = \overline{B} \subset \overline{A}$ *if and only if there exists $\{x_\alpha\}(x_\alpha \in A)$ such that $x_\alpha \to B$.*

(ii) U *is a neighborhood of $A = \overline{A}$ if and only if, for any $\{x_\alpha\}$ such as $x_\alpha \to A$, there exists to α_0 such that $\{x_\alpha; \alpha \geq \alpha_0\} \subset U$ holds.*

When we define the convergence of directed families of points by way of the topology, these properties (i) *and* (ii) *as above hold also. Then if we define the new topology by the processes: the topology $\to$ the convergence of filters $\to$ the convergence of directed families of points $\to$ the new topology, the new topology coincides with the first given topology. Further, if we define the new notion of convergence of filters (or directed families of points) starting from the given notion of convergence of filters (or directed families of points), this also coincides with the first given one. By the above, to give the notion of topology and to give the notion of convergence of filters or directed families of points are entirely identical.*

When we mention the notions in a topological space using the terminology of convergence, we have the following. The fact that a topological space X is compact is identical with the fact that all perfect directed families of points converge. This is also identical with the fact that all ultrafilters converge. Further this is also identical with the fact that, for every directed family of points, there exists a converging directed subfamilies of points.

Theorem 5.2. *Let X and Y be two topological space and $f : X \to Y$ a mapping and D the domain of f. In order that f is continuous at a certain nonempty closed set $A(\subset D)$ of accumulation points of D, it is necessary and sufficient that each one of the following conditions* (i) $\sim$ (iii) *is satisfied:*

(i) *For every directed family of points $\{x_\alpha\}$ such as $x_\alpha \to A$, we have $f(x_\alpha) \to f(A)$.*

(ii) *For every filter Φ such as $\Phi \to A$, we have $f(\Phi) = \{f(M); M \in \Phi\} \to f(A)$.*

(iii) *In the sense of the limit of values of the function f, if $x \to A$ holds, we have $f(x) \to f(A)$.*

The definition of the topology on the basis of the notion of convergence was originated by M. Fréchet[5]. By using the notion of convergence of filters or directed families of points, the correspondence of the convergence and the topology in a topological space becomes perfect. For that purpose, E. H. Moore and H. L. Smith introduced the notion of convergence of directed families of points[8]. Nevertheless, the definition of Moore-Smith is ambiguous in the case other than the case of Hausdorff topological space. Therefore, in this article, we improved these notions as metioned before.

References

[1] G. Birkhoff, Moore-Smith convergence in general topology, Ann. of Math., **38**(1937), 39-56.

[2] N. Bourbaki, Topologie générale, chap. 1, Actualités Sci. Indust., Hermann, 1940, 4th ed. 1965.

[3] H. Cartan, Théorie des filtres, C. R. Acad. Sci. Paris, **205**(1937), 595-598.

[4] H. Cartan, Filtres et ultrafiltres, C. R. Acad. Sci. Paris, **205**(1937), 777-779.

[5] M. Fréchet, Sur quelques points du calcul fonctionnel, Rend. Circ. Mat. Palermo, **22**(1906), 1-74.

[6] K. Ito, Iwanami sūgaku jiten, 3rd ed, Iwanami Shoten, 1985. (in Japanese).

[7] Y. Kawada and Y. Mimura, Introduction to modern mathematies, II, Iwanami Shoten, 1965. (in Japanese).

[8] E. H. Moore and H. L. Smith, A general theory of limits, Amer. J. Math., **44**(1922), 102-121.