J. Math. Tokushima Univ.
Vol. 52 (2018), 53 – 57

On Restricted Wythoff's Nim

By

Shin-ichi KATAYAMA and Tomoya KUBO

Shin-ichi KATAYAMA
Department of Mathematical Sciences, Graduate School of Science and Technology, Tokushima University, Minamijosanjima-cho 2-1, Tokushima 770-8506, JAPAN
e-mail address : *shinkatayama@tokushima-u.ac.jp*
and
Tomoya KUBO
Graduate School of Integrated Arts and Sciences, Tokushima University, Minamijosanjima-cho 1-1, Tokushima 770-8502, JAPAN
e-mail address : *itiji_banji@me.com*

Received October 12 2018

Abstract

We shall study the following restricted Wythoff's Nim. Let S_i $(1 \leq i \leq 3)$ be the set of positive integers. Each player can remove the number of tokens $s_1 \in S_1$ from the first pile and $s_2 \in S_2$ from the second pile and remove the same number of tokens $s_3 \in S_3$ from both piles. We shall identify (m, n) to a position of this nim, where m is the number of tokens in the first pile and n is the number of tokens in the second pile. In the case $|S_2| < \infty$, we will show the Sprague-Grundy sequence(or simply G-sequences) $g_S(m, n)$ is periodic for fixed m.

2010 Mathematics Subject Classification. Primary 91A46; Secondary 91A05

1 Introduction

In his paper [1], C. L. Bouton introduced the 2-player impartial combinatorial game, which is now called nim game. In [6], W. A. Wythoff modified the rule of this game as follows. The game is played by two players. There are two piles of tokens(or stones). Two players play alternately and either take from one of the piles an arbitrary number of tokens or from both piles of the same number of tokens. The player who takes up the last token is the winner. A position from which the player who made the last move, the previous player, can always win is called a P-position. The P-positions of original nim of Bouton are (k, k) with arbitrary $k \geq 0$ and the following P-positions of Wythoff's nim are related to the golden ratio.

Proposition 1.1 Wythoff (1905)
(m, n) is a P-position $\Longleftrightarrow (m, n) = (m_s, m_s + s)$ or $(m_s + s, m_s)$, where m_s is determined by $m_s = [s\phi]$. Here $\phi = \dfrac{1+\sqrt{5}}{2}$.

In this paper, we shall consider some restricted Wythoff's nim as follows. Let S_i $(1 \leq i \leq 3)$ be the set of positive integers. Each player can remove the number of tokens $s_1 \in S_1$ from the first pile and $s_2 \in S_2$ from the second pile and remove the same number of tokens $s_3 \in S_3$ from both piles. We shall identify (m, n) to a position of this Nim, where m is the number of tokens in the first pile and n is the number of tokens in the second pile. Assume the set of positive integers S_2 is finite. In the next section, we shall show the Sprague-Grundy sequence $g_S(m, n)$ is periodic for fixed m, i.e., there exist $a_m \geq 0$ and $p_m > 0$ such that $g_s(m, n + p_m) = g_s(m, n)$ for any $n \geq a_m$. Here p_m is called the period of this Sprague-Grundy sequence $g_s(m, n)$.

2 Proof of Main Theorem

Let $S_2 = \{s_1, s_2, \ldots, s_{s(2)} \mid 0 < s_1 < s_2 < \cdots < s_{s(2)}\}$ be the set of positive integers. The player is restricted to remove the number of tokens $s \in S_2$ from the second pile.

Theorem 2.1 *Under the above notations, $g_S(m, n)$ has a period p_m for any fixed m, that is,*

$$n \geq a_m \Longrightarrow g_S(m, n + p_m) = g_S(m, n), \quad \text{for any } n \geq a_m.$$

Proof. From the assumption on S_2, G-sequence satisfies $0 \leq g_S(m, n) \leq 2m + s(2)$. The case $m = 0$ is nothing but the case of restricted one pile nim and it is known that $g_S(0, n)$ is periodic. Thus, assume $g_S(m', n + p_{m'}) = g(m', n)$ for any $n \geq a_{m'}$ for the cases $m'(0 \leq m' \leq m - 1)$. $p_0, p_1, \ldots, p_{m-1}$ denote the periods for the cases $0 \leq m' \leq m - 1$. Put $a_* = max\{a_0, a_1, \ldots, a_{m-1}\}$, $p_* = LCM(p_0, p_1, \ldots, p_{m-1})$. Then the pigeonhole principle asserts that there exists a period $p = p_m$ $(p_*|p_m)$ as follows.

The number of patterns of consecutive $s(2)$ Grundy numbers $g_S(m,n)$ are at most $\ell_* = (2m+s(2)+1)^{s(2)}$. Consider $\ell_*+1 = (2m+s(2)+1)^{s(2)}+1$ pairs;

$$\begin{array}{l}(g_S(m,a_*), g_S(m,a_*+1), \ldots, g_S(m,a_*+s(2)-1)),\\ (g_S(m,a_*+p_*), g_S(m,a_*+p_*+1), \ldots, g_S(m,a_*+p_*+s(2)-1)),\\ \qquad\qquad\vdots\\ (g_S(m,a_*+\ell_*p_*), g_S(m,a_*+\ell_*p_*+1), \ldots, g_S(m,a_*+\ell_*p_*+s(2)-1)).\end{array}$$

Hence the pigeonhole principle asserts that there exists a pair ℓ_i, ℓ_j $(0 \le \ell_i < \ell_j \le \ell_*)$ which satisfies

$$\begin{array}{l}(g_S(m,a_*+\ell_i p_*), g_S(m,a_*+\ell_i p_*+1), \ldots, g_S(m,a_*+\ell_i p_*+s(2)-1))\\ = (g_S(m,a_*+\ell_j p_*), g_S(m,a_*+\ell_j p_*+1), \ldots, g_S(m,a_*+\ell_j p_*+s(2)-1))\end{array}$$

Put $a = a_* + p_*\ell_i$ and $p = p_*(\ell_j - \ell_i)$. From the above condition,

$$\left\{\begin{array}{ccc} g_S(m,a) & = & g_S(m,a+p) \\ \vdots & \vdots & \vdots \\ g_S(m,a+s(2)-1) & = & g_S(m,a+s(2)-1+p)\end{array}\right.$$

Thus $g_S(m,n)$ has period p for $a \le n \le a+s(2)-1$. Assume $g_S(m,n') = g_S(m,n'+p)$ for any n' $(a \le n' < n)$. Since $n - s_j \ge a, n - s_k \ge a$, $g_S(m,n+p) = mex\{g_S(m-s_i,n+p), g_S(m,n+p-s_j),\ g_S(m-s_k,n+p-s_k) \mid s_i,s_j,s_k \in S$ with $0 \le m-s_i$ and $0 \le m-s_k\} = mex\{g_S(m-s_i,n), g_S(m,n-s_j), g_S(m-s_k,\ n-s_k) \mid s_i,s_j,s_k \in S$ with $0 \le m-s_i$ and $0 \le m-s_k\} = g_S(m,n)$.
Hence, by induction, we have $n \ge a \Longrightarrow g_S(m,n+p) = g_S(m,n)$.

Now we shall consider the special case when $|S_1|, |S_2|$ and $|S_3|$ are finite. Put $S_1 = \{s_{1,1}, s_{1,2}, \ldots, s_{1,r(1)} \mid 0 < s_{1,1} < s_{1,2} < \cdots < s_{1,r(1)}\}$,
$S_2 = \{s_{2,1}, s_{2,2}, \ldots, s_{2,r(2)} \mid 0 < s_{2,1} < s_{2,2} < \cdots < s_{2,r(2)}\}$,
and $S_3 = \{s_{3,1}, s_{3,2}, \ldots, s_{3,r(3)} \mid 0 < s_{3,1} < s_{3,2} < \cdots < s_{3,r(3)}\}$. $s(0)$ denotes $max(s_{1,r(1)}, s_{2,r(2)}, s_{3,r(3)})$. Assume that there exist a positive integer p which satisfies

$$g_S(m,n+p) = g_S(m,n) \text{ for any } 0 \le m,n \le s(0)+p.$$

Then we have the following special case of the above theorem.

Corollary 2.2 *Under the above notation, $g_S(m,n)$ is purely periodic and satisfies $g_S(m,n+p) = g_S(m,n)$ for any $m,n \ge 0$.*

Using this corollary, we have the following example.

Table 1 (The case $S = (\{1,2\},\{1,2\},\{1,2\})$)

$m\backslash n$	0	1	2	3	4	5	6	7	8
0	0	1	2	0	1	2	0	1	2
1	1	2	0	1	2	0	1	2	0
2	2	0	1	2	0	1	2	0	1
3	0	1	2	0	1	2	0	1	2
4	1	2	0	1	2	0	1	2	0
5	2	0	1	2	0	1	2	0	1

Thus we have $g_S(m,n) \equiv g(m,n) \pmod 3$ and $g_S(m,n+3) = g_S(m,n)$ for any m and n.

3 Equivalent classes of $S \subset \mathbb{N}$

We call $S \subset \mathbb{N}$ and $S' \subset \mathbb{N}$ is equivanent if and only if

$$S \sim S' \iff g_S(k) = g_{S'}(k) \text{ (for any } k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}).$$

For n piles $(m_1, m_2, \ldots, m_n)$, the number of removable tokens from the pile m_k is restricted $s \in S_k$ for each S_k of $S = (S_1, S_2, \ldots, S_k, \ldots, S_n)$. Then we shall slightly generalizes the equivalent classes for $S = (S_1, S_2, \ldots, S_n)$, and $S' = (S'_1, S'_2, \ldots, S'_n)$,

$$S \sim S' \iff g_S(k_1, k_2, \ldots, k_n) = g_{S'}(k_1, k_2, \ldots, k_n) \text{ (for any } k_i \in \mathbb{N}_0 \ \ (1 \le i \le n)).$$

Then nim-sum implies

$$S \sim S' \iff S_i \sim S'_i \ (1 \le i \le n).$$

For any $S_i, S'_i \subset \mathbb{N}$ $(1 \le i \le 3)$, we shall write $S = (S_1, S_2, S_3)$, $S' = (S'_1, S'_2, S'_3)$ and consider the restricted Wythoff's nim. In the case S, the number of removable tokens from the first pile is restricted to $s \in S_1$, the number of removable tokens from the second pile is restricted to $s \in S_2$, and the number of removable tokens from both piles at the same time is restricted to $s \in S_3$. The case S' is same as the case S. Now we will consider two restricted Wythof's nim such as the number of tokens each player can remove from the piles are restricted $s \in S$ and $s \in S'$. Then for these restricted Wythoff's nim, there exist several S and S' with $S \sim S'$ but $g_S(m,n) \neq g_{S'}(m,n)$ for some (m,n).
Put $S = 2\mathbb{N}$ and $S' = \mathbb{N} - \{1\}$. Then it is known that $S \sim S'$. We have calculated $g_S(m,n)$ $g_{S'}(m,n)$ for small (m,n) as follows.

Table 2-1 (The case $S = 2\mathbb{N}$)

$m\backslash n$	0	1	2	3	4	5	6	7
0	0	0	1	1	2	2	3	3
1	0	0	1	1	2	2	3	3
2	1	1	2	2	0	0	4	4
3	1	1	2	2	0	0	4	4
4	2	2	0	0	1	1	5	5
5	2	2	0	0	1	1	5	5

Table 2-2 (The case $S' = \mathbb{N} - \{1\}$)

$m\backslash n$	0	1	2	3	4	5	6	7
0	0	0	1	1	2	2	3	3
1	0	0	1	1	2	2	3	3
2	1	1	2	2	0	0	4	4
3	1	1	2	2	3	0	0	4
4	2	2	0	3	1	4	5	5
5	2	2	0	0	4	1	5	5

Then one can verify that $S \sim S'$, but $g_S(3,4) = 0, g_{S'}(3,4) = 3$.

Put $(S = (\{1,2,3\},\{1,2,3\},\{1,2,3\})$ and $S' = (\{1,2,3\},\{1,2,3,5\},\{1,2,3\}))$. There are some examples $g_S(m,n)$ $g_{S'}(m,n)$ for small (m,n) as follows.

Table 3-1
$(S = (\{1,2,3\},\{1,2,3\},\{1,2,3\}))$

$m\backslash n$	0	1	2	3	4	5	6	7
0	0	1	2	3	0	1	2	3
1	1	2	0	4	1	2	0	4
2	2	0	1	5	3	0	1	2
3	3	4	5	6	2	7	4	5

Table 3-2
$(S' = (\{1,2,3\},\{1,2,3,5\},\{1,2,3\}))$

$m\backslash n$	0	1	2	3	4	5	6	7
0	0	1	2	3	0	1	2	3
1	1	2	0	4	1	2	0	4
2	2	0	1	5	3	0	1	2
3	3	4	5	6	2	7	4	6

One can also verify that $S \sim S'$, but $g_S(3,7) = 5, g_{S'}(3,7) = 6$ for this case.

References

[1] C. L. Bouton, Nim, A Game with a Complete Mathematical Theory, Ann. of Math., **3** (1901–1902), 35–39.

[2] A. Dress, A. Flammenkamp and N. Pink, Additive Periodicity of The Sprague-Grundy Function of Certain Nim Games, Adv. Appl. Math. **22** (1999), 249–270.

[3] S. Katayama and T. Kubo, Wythoff's Stone up Game and the Theorem of Rayleigh, Journal of the Tokushima Society for the History of Science **37** (2018), to appear.

[4] W. A. Liu, H. Li and B. Li, A Restricted Version of Wythoff's Nim, The Electric Journal of Combinatrics **18** (2011), # p 207.

[5] F. Sato, On the Mathematics of Extracting Games of Stones-Wonderful Relations between Games and Algebra, Sugaku-Shobou, 2014 (in Japanese)

[6] W. A. Wythoff, A Modification of the Game of Nim, Nieuw Arch, Wisk. **7** (1907), 199–202.