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Melnikov Integral Formula for Beam Sea Roll Motion Utilizing a Non-Hamiltonian Exact Heteroclinic Orbit

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Abstract

Chaos appearing in a ship roll equation in beam seas, known as the escape equation, has been intensively investigated so far because it is closely related to capsizing accident. In particular, many applications of Melnikov integral formula have been reported in the existing literature. However, in all the analytical works concerning with the escape equation, Melnikov integral is formulated utilizing a separatrix for Hamiltonian part or a numerically obtained heteroclinic orbit for non-Hamiltonian part, of the original escape equation. To overcome such limitations, this paper attempts to utilise an analytical expression of the non-Hamiltonian part. As a result, an analytical procedure making use of a heteroclinic orbit of non-Hamiltonian part within the framework of Melnikov integral formula is provided.

1. Introduction

In the research field of nonlinear dynamical system theory, it is well known that the Feigenbaum cascade of period doubling bifurcation could lead to chaos [1)], and so far, considerable research with regard to this phenomenon has been reported. The chaotic behaviour of ship roll motion in beam seas was studied by Virgin [2)], Thompson [3), 4)], Kan et al [5), 6)], and others. Thompson used the escape equation with a second-order polynomial fitting the restoring term and discussed about capsize (or escape) and chaos. Kan et al. observed in their model experiment the capsizing phenomenon caused by period-doubling bifurcation [5)]. Further they investigated the escape equation with nonlinear cubic restoring term using the numerical time simulation, and confirmed close the relationship between capsize and chaos [6)]. On the other hand, Murashige et al. calculated the Lyapunov exponents from the measured time history of flooded ship model [7)], and they confirmed that the ship rolling motion could tend to a chaotic attractor. Moreover they did detailed numerical stidies with a theoretical model [8)].

Melnikov integral formula enables us to test for the existence of transverse homoclinic connection of invariant manifold of a saddle [9), 10)]. It implies the beginning of the fractal metamorphoses, and one of prerequisites for the chaotic behaviour. As an example of direct application of this method to the ship roll problem using the escape equation with cubic restoring term, Kan et al. [6)] analytically estimated the condition of chaotic behaviour. Although it is required to analytically or numerically obtain the heteroclinic orbit in time domain to calculate the Melnikov integral of highly dissipative system, an analytical expression of non-Hamiltonian part is not easily obtained in general. Therefore Wu et al. calculate the Melnikov integral using a numerically obtained heteroclinic orbit [11)]. Though numerical integration requires verification for its accuracy of infinite integral, it seems to be an extremely powerful technique even for high-dimensional systems.

A general solution for nonlinear differential equation is not always available. In case of the escape equation with cubic restoring term it is not. Thereby Kan et al [6)] calculated the Melnikov integral based upon the separatrix of Hamiltonian part of escape equation as alternative to solving its non-Hamiltonian part. However, although we cannot find a general solution of the escape equation, heteroclinic orbits themselves can be obtained using a solution technique, which is used for analysing nonlinear waves.

A heteroclinic orbit is an orbit connecting two saddles, and such an orbit can be realized for certain parameters i.e.

the set of heteroclinic bifurcation points [12]. For instance, it is well known that surf-riding threshold in following and quartering seas can be represented as a heteroclinic bifurcation point [13]. Considerable effort for the research on the surf-riding threshold has been carried out, and one of the authors also proposed an analytical technique for estimating the surf-riding threshold. In this technique, a sinusoidal periodic surge force induced by waves was approximated using a 3rd order polynomial, and then an analytical formula to estimate the surf-riding threshold is obtained [14]. This approximated surge equation is identical to the non-Hamiltonian part of the escape equation. Therefore, in present paper, we attempt to apply the same procedure to the escape equation for providing an analytical formula for the threshold of chaos.

2. Escape Equation

A uncoupled roll motion with wave excitation is modelled as [6];

$$I\frac{d^2\Phi}{dt^2}+N\frac{d\Phi}{dt}+W\cdot GM\cdot\Phi\left(1-\Phi/\Phi_V\right)\left(1+\Phi/\Phi_V\right)=M_{rs}+M_r\cos\left(\omega t+\delta\right) \quad (1),$$

where Φ is roll angle. In forcing term i.e. right side of equation (1), M_{rs} denotes the 2nd order steady wave-induced roll moment whereas M_r does the amplitude of the 1st order wave-induced roll moment. It is assumed that M_{rs} and M_r have relatively small values against terms in the left side of equation (1). Obviously $\Phi=\pm\Phi_V$ are saddles. Here we divide M_{rs} into two:

$$M_{rs}=M_1+M_2 \quad (2).$$

This separation is to create the heteroclinic orbit. In practice value M_1 is determined by condition of heteroclinic bifurcation described in next section. The factorisation yields:

$$W\cdot GM\cdot\Phi\left(1-\Phi/\Phi_V\right)\left(1+\Phi/\Phi_V\right)-M_1=W\cdot GM\left(\Phi-\Phi_1\right)\left(\Phi_2-\Phi\right)\left(\Phi-\Phi_3\right)/\Phi_V^2 \quad (3),$$

where the relation $\Phi_1<\Phi_2<\Phi_3$ is held. Considering the following transformation:

$$\varphi=\frac{\Phi-\Phi_1}{\Phi_3-\Phi_1} \quad (4),$$

then restoring terms becomes:

$$\frac{W\cdot GM\left(\Phi-\Phi_1\right)\left(\Phi_2-\Phi\right)\left(\Phi-\Phi_3\right)}{\Phi_V^2}=\frac{W\cdot GM\left(\Phi_3-\Phi_1\right)^3}{\Phi_V^2}\varphi\left(\varphi-\frac{\Phi_2-\Phi_1}{\Phi_3-\Phi_1}\right)\left(1-\varphi\right) \quad (5).$$

Therefore we have:

$$\frac{d^2\varphi}{dt^2}+\frac{N}{I}\frac{d\varphi}{dt}+\frac{W\cdot GM\left(\Phi_3-\Phi_1\right)^2}{I\Phi_V^2}\varphi\left(\varphi-\frac{\Phi_2-\Phi_1}{\Phi_3-\Phi_1}\right)\left(1-\varphi\right)=\frac{M_2}{I\left(\Phi_3-\Phi_1\right)}+\frac{M_r}{I\left(\Phi_3-\Phi_1\right)}\cos\left(\omega t+\delta\right) \quad (6).$$

The definitions of new variables as :

$$\begin{cases}\tilde{\beta}\equiv N/I,\ \tilde{\mu}\equiv W\cdot GM\left(\Phi_3-\Phi_1\right)^2/I\Phi_V^2,\ \tilde{a}\equiv\left(\Phi_2-\Phi_1\right)/\left(\Phi_3-\Phi_1\right)\\ b_0\equiv M_2/I\left(\Phi_3-\Phi_1\right), b\equiv M_r/I\left(\Phi_3-\Phi_1\right)\end{cases} \quad (7),$$

yield the following equation:

$$\frac{d^2\varphi}{dt^2}+\tilde{\beta}\frac{d\varphi}{dt}+\tilde{\mu}\cdot\varphi\left(1-\varphi\right)\left(\varphi-\tilde{a}\right)=b_0+b\cos\left(\omega t+\delta\right) \quad (8).$$

Note that if $\tilde{a}=0.5$, then left side of this equation becomes symmetrical. This equation is utilised for all the considerations in this paper.

3. Solution of Non-Hamiltonian Heteroclinic Orbit

Considering the case of $b_0 = 0$ and $b = 0$ in equation (8):

$$\frac{d^2\varphi}{dt^2} + \tilde{\beta}\frac{d\varphi}{dt} + \tilde{\mu}\cdot\varphi(1-\varphi)(\varphi-\tilde{a}) = 0 \tag{9}$$

This equation is identical with FHN (FitzHugh-Nagumo) equation [15), 16)] except for some coefficients (see appendix 1), so that the solution method for nonlinear waves [17)] to find a travelling wave is applicable. Here assuming that equation (9) has a heteroclinic orbit. Then let us postulate a non-Hamiltonian heteroclinic orbit:

$$\dot{\varphi} = \tilde{c}\varphi(1-\varphi) \tag{10}$$

Differentiation of equation (10) with regard to time yields:

$$\ddot{\varphi} = \tilde{c}\frac{d\left(\varphi-\varphi^2\right)}{dt} = \tilde{c}\left(\dot{\varphi}-2\varphi\dot{\varphi}\right) = \tilde{c}\dot{\varphi}(1-2\varphi) = \tilde{c}^2\varphi(1-\varphi)(1-2\varphi) \tag{11}$$

If we substitute above equation into equation (9), then we can obtain:

$$\tilde{c}^2\varphi(1-\varphi)(1-2\varphi) + \tilde{\beta}\tilde{c}\varphi(1-\varphi) + \tilde{\mu}\cdot\varphi(1-\varphi)(\varphi-\tilde{a}) = 0 \tag{12}$$

Here taking a monomial order of ${}^{\forall}\varphi : \varphi \in (0,1)$:

$$\varphi\left(\tilde{\mu}-2\tilde{c}^2\right) + \left(\tilde{c}^2 + \tilde{\beta}\tilde{c} - \tilde{\mu}\tilde{a}\right) = 0 \tag{13}$$

In order to satisfy the above equation for ${}^{\forall}\varphi : \varphi \in (0,1)$, the following relations are required.

$$\begin{cases} \tilde{\mu} - 2\tilde{c}^2 = 0 \\ \tilde{c}^2 + \tilde{\beta}\tilde{c} - \tilde{\mu}\tilde{a} = 0 \end{cases} \tag{14a,b}$$

From equation (14a) we have:

$$\tilde{c} = \pm\sqrt{\tilde{\mu}/2} \tag{15}$$

The positive sign corresponds to a heteroclinic orbit on upper side of phase plane whereas the negative sign does to oneon lower side of phase plane. Substituting this condition into equation (14b), we have:

$$\frac{\tilde{\mu}}{2} \pm \tilde{\beta}\sqrt{\frac{\tilde{\mu}}{2}} - \tilde{\mu}\tilde{a} = 0 \Rightarrow \tilde{\mu}\left(\frac{1}{2}-\tilde{a}\right) \pm \tilde{\beta}\sqrt{\frac{\tilde{\mu}}{2}} = 0 \tag{16}$$

Here the conditions of $\tilde{a} > 0.5$ corresponds to the heteroclinic orbit on upper plane while $\tilde{a} < 0.5$ does to that on lower plane since positive roll damping, i.e. $\tilde{\beta} > 0$ and positive metacentric height, i.e. $\tilde{\mu} > 0$ should be satisfied for an normal intact ship in general. Equation (16) can be solved using a simple iteration procedure with respect to a single variable, such as M_1, when the bifurcation point is required as a function of M_1. Table 1 indicates the comparison of critical value σ_C providing the heteroclinic orbit. Here σ_C denotes the non-dimensionalized value M_1 as $\sigma = M_1 / W \cdot GM \cdot \Phi_V$. Further calculation condition is set to be the same as those of Table 3 provided by Wu and McCue [11)] and the values σ_C obtained by Wu and McCue are noted. Since the results obtained by the present procedure well agree with the numerical ones by Wu and McCue [11)], it is concluded that the analytical method proposed here is verified and numerical results of Wu and McCue [11)] has sufficiently high accuracy. If we solve Equation (10), we can easily obtain as a solution in time domain:

$$\varphi^0(t) = \frac{1}{1+\exp\left(-\tilde{c}t+\tilde{d}\right)} = \frac{1}{2}+\frac{1}{2}\tanh\left(\frac{\tilde{c}t-\tilde{d}}{2}\right) \qquad (17).$$

Here $\tilde{d} \in (-\infty, \infty)$ denoted the arbitrary integral constant determined by an initial condition. Taking $\varphi = 0.5$ at $t = 0$ yields $\tilde{d} = 0$, then equation (17) becomes:

$$\varphi^0(t) = \frac{1}{2}+\frac{1}{2}\tanh\frac{\tilde{c}t}{2} \qquad (18).$$

This equation is utilized for calculating the Melnikov integral in next section. Moreover for the rolling equation with 4th-order polynomial, we can similarly obtain an analytical solution of heteroclinic orbit for limited condition. It is planned to be presented in separate publication in the future.

If we consider the case of $\tilde{a} = 0.5$, the solution of equation (16) is $\tilde{\beta} = 0$ or $\tilde{\mu} = 0$. $\tilde{\mu} = 0$ implies the non-existence of restoring term, so that this solution is not relevant to the current problem. Therefore $\tilde{\beta} = 0$ should be regarded as a solution. If $\tilde{\beta} = 0$ i.e. the Hamiltonian system, the separatrix connecting $\varphi = 0$ and $\varphi = 1$ is realised only when $\tilde{a} = 0.5$. It can be easily proved as follows. If we consider the case of $\tilde{\beta} = 0$ in equation (9), simple manipulation yields:

$$\frac{d^2\varphi}{dt^2} = \tilde{\mu}\left[\varphi^3 - (\tilde{a}+1)\varphi^2 + \tilde{a}\varphi\right] \qquad (19).$$

Multiplying each side of equation (19) by $d\varphi/dt$ and integrating with regard to time *t*, then we can obtain:

$$\frac{d\varphi}{dt} = \pm\varphi\sqrt{2\tilde{\mu}\left(\frac{1}{4}\varphi^2 - \frac{\tilde{a}+1}{3}\varphi + \frac{\tilde{a}}{2}\right)} \qquad (20).$$

Here $\varphi = 0$ at $d\varphi/dt = 0$ is assumed. Then at $\varphi = 1$, $d\varphi/dt$ takes the value of:

$$\frac{d\varphi}{dt} = \pm\sqrt{\tilde{\mu}\left(\frac{2\tilde{a}-1}{6}\right)} \qquad (21).$$

If we require $d\varphi/dt = 0$, the condition of $\tilde{a} = 0.5$ is necessary. Therefore symmetrical equation has a separatrix connecting two saddles only for the case of $\tilde{\beta}=0$, otherwise a separatrix becomes a homoclinic orbit.

Finally we briefly consider whether expression (10) with $\tilde{a}\neq 0.5$ can represent all the heteroclinic orbits. Equation (16) denotes the set consisting of heteroclinic bifurcation points in a parameter plane spanned by $\tilde{\beta}$ and $\tilde{\mu}$ as two solution one-dimensional manifolds. The system has only one heteroclinic orbit for certain parameter combination of $\tilde{\beta}_0$ and $\tilde{\mu}_0$ by the uniqueness of the solution. It cannot deny that there could exist other heteroclinic orbits for other parameter combinations. The heteroclinic orbit introduced here, however, becomes identical to equation (A.5) in Kan[6)] when $\tilde{\beta}$ has zero (see appendix 2), so that it is supposed that the heteroclinic orbit in which we have interest, is realised for the parameter combination of $\tilde{\beta}$ and $\tilde{\mu}$ obtained from equation (16). Therefore it is concluded that the equation (16) is the required solution for our analysis within the framework of the present research.

4. Calculation of Melnikov Integral

Using the heteroclinic orbit obtained the above and following the methodology introduced by Salam [10], the Melnikov integral can be calculated. State equation (8) can be rewritten as vectorial representation:

$$\frac{d}{dt}\begin{pmatrix}\varphi\\ \dot{\varphi}\end{pmatrix}=\begin{pmatrix}\dot{\varphi}\\ -\tilde{\beta}\dot{\varphi}-\tilde{\mu}\varphi(1-\varphi)(\varphi-\tilde{a})\end{pmatrix}+\begin{pmatrix}0\\ b_0+b\sin\omega t\end{pmatrix}\equiv\mathbf{F}(\mathbf{x})+\mathbf{G}(t) \qquad (22).$$

As shown above, the solution on $b_0=0$ and $b=0$ can be obtained as equation (21). Here apply Melnikov integral method.

$$D\mathbf{F}(\mathbf{x})=\begin{pmatrix}0 & 1\\ \tilde{\mu}\left[3\varphi^2-2(\tilde{a}+1)\varphi+\tilde{a}\right] & -\tilde{\beta}\end{pmatrix} \qquad (23)$$

yields

$$\mathrm{tr}D\mathbf{F}(\mathbf{x})=-\tilde{\beta} \qquad (24).$$

Note that wedge product is defined as $\mathbf{a}\wedge\mathbf{b}=a_1b_2-a_2b_1$. Then $\mathbf{F}\left(q^0(t)\right)\wedge\mathbf{G}(t+t_0)$ can be calculated as:

$$\mathbf{F}\left(q^0(t)\right)\wedge\mathbf{G}(t+t_0)=\dot{\varphi}(b_0+b\sin\omega(t+t_0)) \qquad (25).$$

Therefore, Melnikov function $M(t_0)$ is determined as:

$$\begin{aligned}M(t_0)&=\int_{-\infty}^{\infty}\mathbf{F}\left(\varphi^0(t)\right)\wedge\mathbf{G}(t+t_0)\cdot\exp\left(-\int_0^t\mathrm{tr}D\mathbf{F}(\mathbf{x})ds\right)dt\\ &=\int_{-\infty}^{\infty}b\sin\omega(t+t_0)\tilde{c}\varphi^0(t)\left(1-\varphi^0(t)\right)\cdot\exp\left(\int_0^t\tilde{\beta}ds\right)dt+\int_{-\infty}^{\infty}b_0\tilde{c}\varphi^0(t)\left(1-\varphi^0(t)\right)\cdot\exp\left(\int_0^t\tilde{\beta}ds\right)dt\\ &=b\tilde{c}(I_i\cos\omega t_0+I_r\sin\omega t_0)+b_0\tilde{c}I(0)\\ &=b\tilde{c}\sqrt{I_r^2+I_i^2}\sin\left(\omega t_0+\tan^{-1}(I_i/I_r)\right)+b_0\tilde{c}I(0)\end{aligned} \qquad (26).$$

Here I is defined as the following Fourier transformation:

$$I(\omega)\equiv\int_{-\infty}^{\infty}\varphi^0(t)\left(1-\varphi^0(t)\right)e^{\tilde{\beta}t}e^{i\omega t}dt \qquad (27),$$

where we put $I_r=\mathrm{Re}[I]$ and $I_i=\mathrm{Im}[I]$. The condition having simple zero of equation (26) can be represented as:

$$\frac{I(0)}{\sqrt{I_r^2+I_i^2}}=b/b_0 \qquad (28).$$

Each component can be calculated by using Cauchy's integral theorem (see appendix 3). Equation (18) with (A6) provides the condition of onset of chaos in the escape equation discussed here.

5. Concluding Remarks

A fully analytical solution of heteroclinic orbit is used for calculating the Melnikov integral to estimate the onset of chaotic behaviour of escape equation, as an alternative to the technique using a separatrix of the Hamiltonian part of the escape equation or a numerically obtained heteroclinic orbit of its non-Hamiltonian part. Verification of the proposed technique is shown by comparison with existing numerical work. Uniqueness of the heteroclinic orbit having the form of equation (10) should be mathematically examined in future.

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Appendix 1

Hodgkin and Huxley have shown that the shape and speed of pulses in the nerve of a squid are well-approximated by numerical solution of Hodgkin and Huxley equation [18]. Other closely related models were discussed by Fitzhugh or Nagumo [16]. Nagumo [15] simplified the Hodgkin and Huxley equation as follows:

$$\frac{\partial e}{\partial t} = \frac{\partial^2 e}{\partial x^2} + e(1-e)(e-a) - b\int e dt \qquad \text{(A1).}$$

Where e is a function of x and t, and $0 < a < 1$. Assuming $b = 0$ and $e(x,t) = e(x+ct)$ yields:

$$\frac{d^2 e}{dt^2} - c\frac{de}{dt} + e(1-e)(e-a) = 0 \qquad \text{(A2).}$$

This equation is identical to Equation (9) except for some coefficients.

Appendix 2

We explain that $\tilde{\beta}=0$ at equation (21) leads to equation (A.5) in Kan [6]. As mentioned above, $\tilde{d}$ is arbitrary constant. Putting $\tilde{d}=0$ and taking $\tilde{c}=\sqrt{\tilde{\mu}/2}$ in (15), then the following equation can be obtained:

$$\varphi=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{t\sqrt{\tilde{\mu}}}{2\sqrt{2}}\right) \qquad \text{(A3).}$$

This orbit is defined within the open set $\varphi\in(0,1)$, so that utilizing the change of variable $\varphi=(\psi+1)/2$ yields:

$$\psi=\tanh\left(\sqrt{\frac{\mu}{2}}t\right) \qquad \text{(A4).}$$

This result is identical with equation (A.5) in Kan [6].

Appendix 3

Here we briefly state about the integral having the form of equation (27). This equation can be rewritten as follows:

$$I(\omega) = \frac{1}{4}\int_{-\infty}^{\infty} \frac{\exp\left(\tilde{\beta}t + i\omega t\right)}{\cosh^2(\tilde{c}t/2)} dt \qquad \text{(A5)}.$$

Taking the integral rout as shown in Figure 1, Cauchy's integral theorem easily leads to the following result:

$$I(\omega) = \frac{\pi\left(\tilde{\beta} + i\omega\right)\csc\left[\pi\left(\tilde{\beta} + i\omega\right)/\tilde{c}\right]}{\tilde{c}^2 \operatorname{sgn}\tilde{c}} \qquad \text{(A6)}.$$

Note that a singular point of equation (A5), i.e. $t = \pi i(2n+1)/\tilde{c}$, is a pole of order 2. Here n denotes the arbitrary integer.

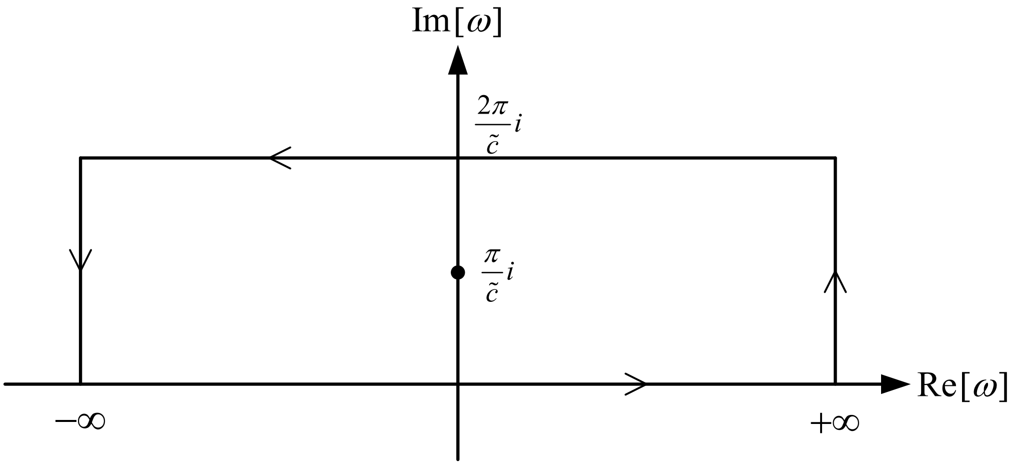

Figure 1 An integral route for positive $\tilde{c}$.

Table 1 Comparison between numerical and present analytical results

$\tilde{\beta}$	σ_C (Nemericalre results)[11]	σ_C (Analytical results)
0.05	0.023577	0.023557
0.1	0.047036	0.047036

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Nomenclature

GM	Metacentric height
H	Wave height
I	Moment of inertia in roll
N	Damping coefficient in roll
W	Ship mass
Φ	Roll angle
Φ_V	Vanishing angle of roll restoring moment