

Upper Decay Estimates for Non-Degenerate Kirchhoff Type Dissipative Wave Equations

By

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Abstract

We study on the Cauchy problem for non-degenerate Kirchhoff type dissipative wave equations $\rho u'' + a(\|A^{1/2}u(t)\|^2) Au + u' = 0$ and $(u(0), u'(0)) = (u_0, u_1)$, where $u_0 \neq 0$ and the nonlocal nonlinear term $a(M) = 1 + M^\gamma$ with $\gamma > 0$. Under the suitably smallness condition, we derive the upper decay estimates of the solution $u(t)$ for the case of $0 < \gamma < 1$ in addition to $\gamma \geq 1$.

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1 Introduction

Let H be a real Hilbert space with inner product (\cdot, \cdot) and norm $\|\cdot\|$.

In this paper we investigate on the upper decay estimates of the solution $u(t)$ for the non-degenerate Kirchhoff type dissipative wave equations :

$$\begin{cases} \rho u'' + a(\|A^{1/2}u(t)\|^2) Au + u' = 0, & t \geq 0 \\ (u(0), u'(0)) = (u_0, u_1) \in \mathcal{D}(A) \times \mathcal{D}(A^{1/2}), \end{cases} \quad (1.1)$$

where $u = u(t)$ is an unknown real value function, ρ is a positive constant, $' = d/dt$, A is a linear operator on H with dense domain $\mathcal{D}(A)$.

We assume that the operator A is self-adjoint and nonnegative such that $(Av, v) \geq 0$ for $v \in \mathcal{D}(A)$. The α -th power of A with dense domain $\mathcal{D}(A^\alpha)$ is denoted by A^α for $\alpha > 0$, and the graph-norm of A^α is denoted by $\|v\|_\alpha = (\|v\|^2 + \|A^\alpha v\|^2)^{\frac{1}{2}}$ for $v \in \mathcal{D}(A^\alpha)$. We use that $\|A^{1/2}v\|^2 = (Av, v)$ for $v \in \mathcal{D}(A^{1/2})$.

For the non-local nonlinear term $a(M) \in C^0([0, \infty)) \cap C^2((0, \infty))$, we assume that as follows :

Hyp.1 $K_1 \leq a(M) \leq K_2 + K_3 M^\gamma$ for $M \geq 0$

Hyp.2 $0 \leq a'(M)M \leq K_4 a(M)$ for $M > 0$

Hyp.3 $a'(M)M + |a''(M)|M^2 \leq K_5 M^\gamma$ for $M > 0$

with $\gamma > 0$ and $K_j > 0$ ($j = 1, 2, 3, 4, 5$).

From Hyp.1, we see that

$$K_1 M \leq \int_0^M a(\mu) d\mu \leq \left(K_2 + \frac{K_3}{\gamma + 1} M^\gamma \right) M. \quad (1.2)$$

For typical examples, we have that

$$a(M) = 1 + M^\gamma \quad \text{with} \quad \gamma > 0.$$

When the dimension is one, (1.1) describes small amplitude vibrations of an elastic string (see [3], [6]).

We denote the energy $E(t)$ for (1.1) by

$$E(t) = \rho \|u'(t)\|^2 + \int_0^{M(t)} a(\mu) d\mu \quad \text{with} \quad M(t) = \|A^{1/2}u(t)\|^2. \quad (1.3)$$

By fundamental calculation, we have the energy identity

$$\frac{d}{dt} E(t) + 2\|u'(t)\|^2 = 0 \quad (1.4)$$

and

$$E(t) + 2 \int_0^t \|u'(s)\|^2 ds = E(0) \quad (1.5)$$

with

$$E(0) = \rho \|u_1\|^2 + 2 \int_0^{\|A^{1/2}u_0\|^2} a(\mu) d\mu.$$

Moreover, we introduce the quantities $G(0)$ and $B(0)$ on the initial data (u_0, u_1) :

$$G(0) = \frac{\|Au_0\|^2}{\|A^{1/2}u_0\|^2} + \rho \frac{\|A^{1/2}u_0\|^2 \|A^{1/2}u_1\|^2 - |(A^{1/2}u_0, A^{1/2}u_1)|}{a(\|A^{1/2}u_0\|^2) \|A^{1/2}u_0\|^4}$$

and

$$B(0) = \max\left\{ \frac{\|u_1\|^2}{\|A^{1/2}u_0\|^2}, \frac{1 + K_4}{K_4} (K_2 + K_3 (K_1^{-1} E(0))^\gamma)^2 G(0) \right\}.$$

In the previous paper [12], we have proved the following the global existence theorem (see [1], [2], [9], [13] for local solutions).

Theorem 1.1 *Suppose that Hyp.1 and Hyp.2 are fulfilled. If the initial data (u_0, u_1) belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ and $u_0 \neq 0$, and moreover, the coefficient ρ and the initial data (u_0, u_1) satisfy*

$$2\rho G(0)^{\frac{1}{2}} B(0)^{\frac{1}{2}} < \frac{1}{K_4 + 1},$$

then the problem (1.1) admits a unique global solution $u(t)$ in the class

$$C^0([0, \infty); \mathcal{D}(A)) \cap C^1([0, \infty); \mathcal{D}(A^{1/2})) \cap C^2([0, \infty); H)$$

and the solution $u(t)$ satisfies

$$\|u(t)\|^2 \leq C(\|u_0\|^2 + E(0)), \quad (1.6)$$

$$K_1 M(t) \leq E(t) \leq E(0), \quad (1.7)$$

$$\rho \frac{|M'(t)|}{M(t)} \leq \frac{1}{K_4 + 1}, \quad (1.8)$$

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(0), \quad \frac{\|u'(t)\|^2}{M(t)} \leq B(0), \quad (1.9)$$

and $M(t) \geq Ce^{-\alpha t}$ with some $\alpha > 0$ for $t \geq 0$.

We do not need the assumption that $\gamma \geq 1$ in our argument (see [4] for $\gamma \geq 1$ that is, $a(\cdot) \in C^1([0, \infty))$, and $a'(M) \geq K_0 > 0$ for $\gamma > 0$ (see [11] for $a(M) = (1 + M)^\gamma$ with $\gamma > 0$).

The purpose of this paper to derive upper decay estimates of the solution $u(t)$ of (1.1) for the case of $0 < \gamma < 1$ in addition to $\gamma \geq 1$, under Hyp.1, Hyp.2, Hyp.3.

Our main result is as follows.

Theorem 1.2 *Suppose that the assumption of Theorem 1.1 and Hyp.3 are fulfilled. Then, the solution $u(t)$ of (1.1) satisfies*

$$\|A^{1/2}u(t)\|^2 \leq C(1+t)^{-1},$$

$$\|u'(t)\|^2 + \|Au(t)\|^2 \leq \begin{cases} C(1+t)^{-(1+2\gamma)} & \text{if } 0 < \gamma < \frac{1}{2}, \\ C(1+t)^{-2} & \text{if } \gamma \geq \frac{1}{2}, \end{cases}$$

$$\|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \leq \begin{cases} C(1+t)^{-(1+\gamma)(1+2\gamma)} & \text{if } 0 < \gamma < \frac{1}{2}, \\ C(1+t)^{-3} & \text{if } \gamma \geq \frac{1}{2} \end{cases}$$

for $t \geq 0$.

The proof of Theorem 1.2 will be given by Propositions 2.2–2.5 in the next section.

The notations we use in the paper are standard. Positive constants will be denoted by C and will change from line to line.

2 Decay Rates

The following generalized Nakao type inequality is useful to derive decay estimates of energies (see [5], [7], [8], [10] for the proof).

Lemma 2.1 *Let $\phi(t)$ be a non-negative function on $[0, \infty)$ and satisfy*

$$\sup_{t \leq s \leq t+1} \phi(s)^{1+\alpha} \leq (k_0 \phi(t)^\alpha + k_1(1+t)^{-\beta})(\phi(t) - \phi(t+1)) + k_2(1+t)^{-\gamma}$$

with certain constants $k_0, k_1, k_2 \geq 0$, $\alpha > 0$, $\beta > -1$, and $\gamma > 0$. Then, the function $\phi(t)$ satisfies

$$\phi(t) \leq C_0(1+t)^{-\theta}, \quad \theta = \min\left\{\frac{1+\beta}{\alpha}, \frac{\gamma}{1+\alpha}\right\}$$

for $t \geq 0$ with some constant C_0 depending on $\phi(0)$.

Using Lemma 2.1, we obtain the following energy decay for the energy $E(t)$.

Proposition 2.2 *Under the assumption of Theorem 1.1, the energy $E(t)$ satisfies*

$$E(t) = \rho \|u'(t)\|^2 + \int_0^{M(t)} a(\mu) d\mu \leq C(1+t)^{-1}, \quad (2.1)$$

and the solution $u(t)$ satisfies

$$\|A^{1/2}u(t)\|^2 + \|Au(t)\|^2 + \|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \leq C(1+t)^{-1} \quad (2.2)$$

for $t \geq 0$.

Proof. Integrating (1.4) over $[t, t+1]$, we have

$$2 \int_t^{t+1} \|u'(s)\|^2 ds = E(t) - E(t+1) \quad (\equiv 2D(t)^2). \quad (2.3)$$

Then there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u'(t_j)\|^2 \leq 4D(t)^2 \quad \text{for } j = 1, 2. \quad (2.4)$$

On the other hand, taking the inner product of (1.1) with $u(t)$, we have

$$a(M(t))M(t) = \rho \left(\|u'(t)\|^2 - \frac{d}{dt}(u'(t), u(t)) \right) - (u'(t), u(t)). \quad (2.5)$$

Integrating (2.5) over $[t_1, t_2]$, we have that

$$\begin{aligned} & \int_{t_1}^{t_2} a(M(s))M(s) ds \\ & \leq \rho \int_t^{t+1} \|u'(s)\|^2 ds + \rho \sum_{j=1}^2 \|u'(t_j)\| \|u(t_j)\| + \int_t^{t+1} \|u'(s)\| \|u(s)\| ds \end{aligned}$$

and from (2.3), (2.4), and Hyp.1 that

$$K_1 \int_{t_1}^{t_2} M(s) ds \leq \rho D(t)^2 + CD(t) \sup_{t \leq s \leq t+1} g(s) \quad \text{with} \quad g(t)^2 = \|u(t)\|^2, \quad (2.6)$$

and from (1.2), (1.3), (1.7), (2.3), (2.6) that

$$\begin{aligned} \int_{t_1}^{t_2} E(s) ds & \leq \rho \int_t^{t+1} \|u'(s)\|^2 ds + \int_{t_1}^{t_2} \left(K_2 + \frac{K_3}{\gamma+1} M(s)^\gamma \right) M(s) ds \\ & \leq CD(t)^2 + CD(t) \sup_{t \leq s \leq t+1} g(s). \end{aligned} \quad (2.7)$$

Integrating (2.3) over $[t, t_2]$, we have (2.3) and (2.7) that

$$\begin{aligned} E(t) & = E(t_2) + 2 \int_t^{t_2} \|u'(s)\|^2 ds \\ & \leq 2 \int_{t_1}^{t_2} E(s) ds + \int_t^{t+1} \|u'(s)\|^2 ds \\ & \leq CD(t)^2 + CD(t) \sup_{t \leq s \leq t+1} g(s). \end{aligned}$$

Since it holds that $2D(t)^2 = E(t) - E(t+1) \leq E(t)$ by (2.3), we observe

$$\begin{aligned} E(t)^2 & \leq C \left(D(t)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) D(t)^2 \\ & \leq C \left(E(t) + \sup_{t \leq s \leq t+1} g(s)^2 \right) (E(t) - E(t+1)). \end{aligned} \quad (2.8)$$

Thus, using $E(t) \leq E(0)$ and $g(t) = \|u(t)\|^2 \leq C$ by (1.6) and (1.7), we have

$$E(t)^2 \leq C(E(t) - E(t+1)), \quad (2.9)$$

and hence, applying Lemma 2.1 to (2.9), we obtain (2.1).

Moreover, we obtain that $M(t) \leq K_1^{-1} E(t) \leq C(1+t)^{-1}$ by (1.7), $\|Au(t)\|^2 + \|u'(t)\|^2 \leq CM(t) \leq C(1+t)^{-1}$ by (2.4), and furthermore, $\|u''(t)\|^2 \leq C(1+t)^{-1}$ by (1.1), that is, the desired estimate (2.2) holds true. \square

Proposition 2.3 *Under the assumption of Theorem 1.2, it holds that*

$$F(t) \equiv \rho \|A^{1/2}u'(t)\|^2 + a(M(t))\|Au(t)\|^2 \leq C(1+t)^{-\omega} \quad \text{for } t \geq 0 \quad (2.10)$$

with $\omega = \min\{2, 1 + 2\gamma\}$.

Proof. Taking the inner product of (1.1) with $2Au'(t)$, we have that

$$\begin{aligned} \frac{d}{dt}F(t) + 2\|A^{1/2}u'(t)\|^2 &= a'(M(t))M'(t)\|Au(t)\|^2 \\ &\leq CM(t)^{\gamma+\frac{1}{2}} \frac{\|Au(t)\|^2}{M(t)} \|A^{1/2}u'(t)\| \end{aligned} \quad (2.11)$$

and from the Young inequality that

$$\frac{d}{dt}F(t) + \|A^{1/2}u'(t)\|^2 \leq Cf(t)^2 \quad \text{with } f(t)^2 = M(t)^{2\gamma+1} \frac{\|Au(t)\|^4}{M(t)^2}. \quad (2.12)$$

Integrating (2.12) over $[t, t+1]$, we have

$$\int_t^{t+1} \|A^{1/2}u'(s)\|^2 ds = F(t) - F(t+1) + C \sup_{t \leq s \leq t+1} f(s)^2 \quad (\equiv D(t)^2). \quad (2.13)$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|A^{1/2}u'(t_j)\|^2 \leq 4D(t)^2 \quad \text{for } j = 1, 2. \quad (2.14)$$

On the other hand, taking the inner product of (1.1) with $Au(t)$, we have

$$a(M(t))\|Au(t)\|^2 = \rho \left(\|A^{1/2}u'(t)\|^2 - \frac{d}{dt}(A^{1/2}u', A^{1/2}u) \right) - (A^{1/2}u', A^{1/2}u)$$

and hence

$$F(t) = 2\rho \|A^{1/2}u'(t)\|^2 - \rho \frac{d}{dt}(A^{1/2}u', A^{1/2}u) - (A^{1/2}u', A^{1/2}u). \quad (2.15)$$

Integrating (2.15) over $[t_1, t_2]$, we have from (2.13) and (2.14) that

$$\begin{aligned} &\int_{t_1}^{t_2} F(s) ds \\ &\leq 2\rho \int_t^{t+1} \|A^{1/2}u'(s)\|^2 ds + \rho \sum_{j=1}^2 \|A^{1/2}u'(t_j)\| \|A^{1/2}u(t_j)\| \\ &\quad + \int_t^{t+1} \|A^{1/2}u'(s)\| \|A^{1/2}u(s)\| ds \\ &\leq CD(t)^2 + CD(t) \sup_{t \leq s \leq t+1} g(s) \quad \text{with } g(t)^2 = M(t). \end{aligned} \quad (2.16)$$

Moreover, there exists $t_* \in [t_1, t_2]$ such that

$$F(t_*) \leq 2 \int_{t_1}^{t_2} F(s) ds. \quad (2.17)$$

For $\tau \in [t, t+1]$, integrating (2.11) over $[\tau, t_*]$ (or $[t_*, \tau]$), we have from (2.12) and (2.17) that

$$\begin{aligned} F(\tau) &= F(t_*) + \int_{\tau}^{t_*} \left(2\|A^{1/2}u'(s)\|^2 - a'(M(s))M'(s)\|Au(s)\|^2 \right) ds \\ &\leq 2 \int_{t_1}^{t_2} F(s) ds + C \int_t^{t+1} \|A^{1/2}u'(s)\|^2 ds + C \int_t^{t+1} f(s)^2 ds \\ &\leq CD(t)^2 + CD(t) \sup_{t \leq s \leq t+1} g(s) + C \sup_{t \leq s \leq t+1} f(s)^2. \end{aligned}$$

Since it holds that

$$D(t)^2 = F(t) - F(t+1) + C \sup_{t \leq s \leq t+1} f(s)^2 \leq F(t) + \sup_{t \leq s \leq t+1} f(s)^2$$

by (2.13), we observe

$$\begin{aligned} &\sup_{t \leq s \leq t+1} F(s)^2 \\ &\leq C \left(D(t)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) D(t)^2 + C \sup_{t \leq s \leq t+1} f(s)^4 \\ &\leq C \left(F(t) + \sup_{t \leq s \leq t+1} f(s)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) (F(t) - F(t+1)) \\ &\quad + CF(t) \sup_{t \leq s \leq t+1} f(s)^2 + C \left(\sup_{t \leq s \leq t+1} f(s)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) \sup_{t \leq s \leq t+1} f(s)^2 \end{aligned}$$

and hence

$$\begin{aligned} &\sup_{t \leq s \leq t+1} F(s)^2 \\ &\leq C \left(F(t) + \sup_{t \leq s \leq t+1} f(s)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) (F(t) - F(t+1)) \\ &\quad + C \left(\sup_{t \leq s \leq t+1} f(s)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) \sup_{t \leq s \leq t+1} f(s)^2. \quad (2.18) \end{aligned}$$

Since it holds that

$$f(t)^2 = \begin{cases} M(t)^{2\gamma+1} \frac{\|Au(t)\|^4}{M(t)^2} \leq CM(t)^{2\gamma+1} \leq C(1+t)^{-(1+2\gamma)} \\ M(t)^{2\gamma} \frac{\|Au(t)\|^2}{M(t)} \|Au(t)\|^2 \leq CM(t)^{2\gamma} \|Au(t)\|^2 \leq C(1+t)^{-2\gamma} F(t) \end{cases}$$

and $g(t)^2 = M(t) \leq C(1+t)^{-1}$, we have

$$\begin{aligned} \sup_{t \leq s \leq t+1} F(s)^2 &\leq C(F(t) + (1+t)^{-1})(F(t) - F(t+1)) \\ &\quad + C(1+t)^{-(1+2\gamma)} \sup_{t \leq s \leq t+1} F(s) \end{aligned}$$

and hence

$$\begin{aligned} \sup_{t \leq s \leq t+1} F(s)^2 &\leq C(F(t) + (1+t)^{-1})(F(t) - F(t+1)) \\ &\quad + C(1+t)^{-2(1+2\gamma)}. \end{aligned} \quad (2.19)$$

Thus, applying Lemma 2.1 to (2.19), we obtain

$$F(t) \leq C(1+t)^{-\omega} \quad \text{with} \quad \omega = \min\{2, 1+2\gamma\}$$

which implies the desired estimate (2.10). \square

Proposition 2.4 *Under the assumption of Theorem 1.2, it holds that*

$$\|u'(t)\| \leq C(1+t)^{-\omega} \quad \text{for} \quad t \geq 0 \quad (2.20)$$

with $\omega = \min\{2, 1+2\gamma\}$.

Proof. Taking the inner product of (1.1) with $2u'(t)$, we have

$$\rho \frac{d}{dt} \|u'(t)\|^2 + 2\|u'(t)\|^2 = -2a(M(t))(Au(t), u'(t)),$$

and by the Young inequality we observe

$$\rho \frac{d}{dt} \|u'(t)\|^2 + \|u'(t)\|^2 \leq a(M(t))^2 \|Au(t)\|^2.$$

Thus, from (1.7) and (2.10) we drive the desired estimate (2.20). \square

Proposition 2.5 *Under the assumption of Theorem 1.2, it holds that*

$$\begin{aligned} L(t) &\equiv \rho \|u''(t)\|^2 + a(M(t)) \|A^{1/2}u'(t)\|^2 + \frac{a'(M(t))}{2} |M'(t)|^2 \\ &\leq C(1+t)^{-\sigma} \quad \text{for} \quad t \geq 0 \end{aligned} \quad (2.21)$$

with $\sigma = \min\{3, (1+\gamma)(1+2\gamma)\}$.

Proof. Taking the inner product of (1.1) differentiated with respect to t with $2u''(t)$, we have

$$\begin{aligned} \frac{d}{dt} L(t) + 2\|u''(t)\|^2 &= 3a'(M(t))M'(t) \|A^{1/2}u'(t)\|^2 + \frac{a''(M(t))}{2} (M'(t))^3 \end{aligned} \quad (2.22)$$

$$\leq Cf(t)^2 \quad \text{with} \quad f(t)^2 = M(t)^\gamma \frac{|M'(t)|}{M(t)} \|A^{1/2}u'(t)\|^2. \quad (2.23)$$

Integrating (2.23) over $[t, t + 1]$, we have

$$2 \int_t^{t+1} \|u''(s)\|^2 ds \leq L(t) - L(t+1) + C \sup_{t \leq s \leq t+1} f(s)^2 \quad (\equiv 2D(t)^2). \quad (2.24)$$

Then, there exist two numbers $t_1 \in [t, t + 1/4]$ and $t_2 \in [t + 3/4, t + 1]$ such that

$$\|u''(t_j)\|^2 \leq 4D(t)^2 \quad \text{for } j = 1, 2. \quad (2.25)$$

On the other hand, taking the inner product of (1.1) differentiated with respect to t with $u'(t)$, we have

$$\begin{aligned} & a(M(t)) \|A^{1/2} u'(t)\|^2 + \frac{a'(M(t))}{2} |M'(t)|^2 \\ &= \rho \left(\|u''(t)\|^2 - \frac{d}{dt} (u''(t), u'(t)) \right) - (u''(t), u'(t)) \end{aligned}$$

and hence

$$L(t) = 2\rho \|u''(t)\|^2 - \rho \frac{d}{dt} (u''(t), u'(t)) - (u''(t), u'(t)). \quad (2.26)$$

Integrating (2.26) over $[t_1, t_2]$, we observe from (2.24) and (2.25) that

$$\begin{aligned} & \int_{t_1}^{t_2} L(s) ds \\ & \leq 2\rho \int_t^{t+1} \|u''(s)\|^2 ds + \rho \sum_{j=1}^2 \|u''(t_j)\| \|u'(t_j)\| + \int_t^{t+1} \|u''(s)\| \|u'(s)\| ds \\ & \leq CD(t)^2 + CD(t) \sup_{t \leq s \leq t+1} g(s) \quad \text{with } g(t)^2 = \|u'(t)\|^2. \end{aligned} \quad (2.27)$$

Moreover, there exists $t_* \in [t_1, t_2]$ such that

$$L(t_*) \leq 2 \int_{t_1}^{t_2} L(s) ds. \quad (2.28)$$

For $\tau \in [t, t + 1]$, integrating (2.22) over $[\tau, t_*]$ (or $[t_*, \tau]$), we have from (2.23) and (2.28) that

$$\begin{aligned} & L(\tau) = L(t_*) \\ & + \int_{\tau}^{t_*} \left(2\rho \|u''(s)\|^2 - 3a'(M(s)) M'(s) \|A^{1/2} u'(s)\|^2 + \frac{a(M(s))}{2} (M'(s))^3 \right) ds \\ & \leq 2 \int_{t_1}^{t_2} L(s) ds + C \int_t^{t+1} \|u''(s)\|^2 ds + C \int_t^{t+1} f(s)^2 ds \\ & \leq CD(t)^2 + CD(t) \sup_{t \leq s \leq t+1} g(s) + C \sup_{t \leq s \leq t+1} f(s)^2. \end{aligned}$$

Since it holds that

$$D(t)^2 = L(t) - L(t+1) + C \sup_{t \leq s \leq t+1} f(s)^2 \leq L(t) + \sup_{t \leq s \leq t+1} f(s)^2$$

by (2.24), we observe

$$\begin{aligned} & \sup_{t \leq s \leq t+1} L(s)^2 \\ & \leq C \left(D(t)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) D(t)^2 + C \sup_{t \leq s \leq t+1} f(s)^4 \\ & \leq C \left(L(t) + \sup_{t \leq s \leq t+1} f(s)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) (L(t) - L(t+1)) \\ & \quad + CL(t) \sup_{t \leq s \leq t+1} f(s)^2 + C \left(\sup_{t \leq s \leq t+1} f(s)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) \sup_{t \leq s \leq t+1} f(s)^2 \end{aligned}$$

and hence

$$\begin{aligned} & \sup_{t \leq s \leq t+1} L(s)^2 \\ & \leq C \left(L(t) + \sup_{t \leq s \leq t+1} f(s)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) (L(t) - L(t+1)) \\ & \quad + C \left(\sup_{t \leq s \leq t+1} f(s)^2 + \sup_{t \leq s \leq t+1} g(s)^2 \right) \sup_{t \leq s \leq t+1} f(s)^2. \end{aligned} \quad (2.29)$$

(i) When $0 < \gamma < \frac{1}{2}$, we put $\omega = 1 + 2\gamma$. Since it holds that

$$\begin{aligned} f(t)^2 & \leq 2 \frac{\|Au(t)\|}{M(t)^{\frac{1}{2}}} \frac{\|u'(t)\|^{1-2\gamma}}{M(t)^{\frac{1}{2}}} \|u'(t)\|^{2\gamma} \|A^{1/2}u'(t)\|^2 \\ & \leq C \|u'(t)\|^{2\gamma} \|A^{1/2}u'(t)\|^2 \leq \begin{cases} C(1+t)^{-(1+\gamma)\omega} \\ C(1+t)^{-\gamma\omega} L(t) \end{cases} \end{aligned}$$

and $g(t)^2 = \|u'(t)\|^2 \leq C(1+t)^{-\omega}$, we have

$$\begin{aligned} \sup_{t \leq s \leq t+1} L(t)^2 & \leq C (L(t) + (1+t)^{-\omega}) (L(t) - L(t+1)) \\ & \quad + C(1+t)^{-(1+\gamma)\omega} \sup_{t \leq s \leq t+1} L(s) \end{aligned}$$

and hence

$$\begin{aligned} \sup_{t \leq s \leq t+1} L(t)^2 & \leq C (L(t) + (1+t)^{-\omega}) (L(t) - L(t+1)) \\ & \quad + C(1+t)^{-2(1+\gamma)\omega}. \end{aligned} \quad (2.30)$$

Thus, applying Lemma 2.1 to (2.30), we obtain

$$L(t) \leq C(1+t)^{-\sigma} \quad \text{with} \quad \sigma = \{\omega + 1, (1 + \gamma)\omega\} = (1 + \gamma)(1 + 2\gamma)$$

which implies the desired estimate (2.21) for $0 < \gamma < \frac{1}{2}$.

(ii) When $\gamma \geq \frac{1}{2}$, we put $\omega = 2$. Since it holds that

$$\begin{aligned} f(t)^2 &\leq 2M(t)^{\gamma-\frac{1}{2}} \frac{\|Au(t)\|}{M(t)^{\frac{1}{2}}} \|u'(t)\| \|A^{1/2}u'(t)\| \\ &\leq CM(t)^{\gamma-\frac{1}{2}} \|u'(t)\| \|A^{1/2}u'(t)\| \leq \begin{cases} C(1+t)^{-(\gamma+\frac{3\omega-1}{2})} \\ C(1+t)^{-(\gamma+\frac{\omega-1}{2})}L(t) \end{cases} \end{aligned}$$

and $g(t)^2 = \|u'(t)\|^2 \leq C(1+t)^{-\omega}$, we have

$$\begin{aligned} \sup_{t \leq s \leq t+1} L(t)^2 &\leq C(L(t) + (1+t)^{-\omega})(L(t) - L(t+1)) \\ &\quad + C(1+t)^{-(\gamma+\frac{3\gamma-1}{2})} \sup_{t \leq s \leq t+1} L(s) \end{aligned}$$

and hence

$$\begin{aligned} \sup_{t \leq s \leq t+1} L(t)^2 &\leq C(L(t) + (1+t)^{-\omega})(L(t) - L(t+1)) \\ &\quad + C(1+t)^{-2(\gamma+\frac{3\gamma-1}{2})}. \end{aligned} \tag{2.31}$$

Thus, applying Lemma 2.1 to (2.31), we obtain

$$L(t) \leq C(1+t)^{-\sigma} \quad \text{with} \quad \sigma = \{\omega + 1, \gamma + \frac{3\gamma-1}{2}\} = 3$$

which implies the desired estimate (2.21) for $\gamma \geq \frac{1}{2}$. \square

Proof of Theorem 1.2. Gathering Propositions 2.2–2.5, we conclude Theorem 1.2. \square

References

- [1] A. Arosio and S. Garavaldi, On the mildly degenerate Kirchhoff string, *Math. Methods Appl. Sci.* **14** (1991) 177–195.
- [2] A. Arosio and S. Panizzi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.* **348** (1996) 305–330.
- [3] G.F. Carrier, On the non-linear vibration problem of the elastic string, *Quart. Appl. Math.* **3** (1945) 157–165.
- [4] M. Ghisi and M. Gobino, Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: time-decay estimates, *J. Differential Equations* **245** (2008) 2979–3007.

- [5] S. Kawashima, M. Nakao, and K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, *J. Math. Soc. Japan* **47** (1995) 617–653.
- [6] G. Kirchhoff, *Vorlesungen über Mechanik*, Teubner, Leipzig, 1883.
- [7] M. Nakao, Decay of solutions of some nonlinear evolution equations, *J. Math. Anal. Appl.* **60** (1977) 542–549.
- [8] M. Nakao and K. Ono, Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations, *Math. Z.* **214** (1993) 325–342.
- [9] K. Ono, Global existence and decay properties of solutions for some mildly degenerate nonlinear dissipative Kirchhoff strings, *Funkcial. Ekvac.* **40** (1997) 255–270.
- [10] K. Ono, On sharp decay estimates of solutions for mildly degenerate dissipative wave equations of Kirchhoff type, *Math. Methods Appl. Sci.* **34** (2011) 1339–1352.
- [11] K. Ono, Asymptotic behavior of solutions for Kirchhoff type dissipative wave equations in unbounded domains *J. Math. Tokushima Univ.*, **61** (2017) 37–54.
- [12] K. Ono, Lower decay estimates for non-degenerate Kirchhoff type dissipative wave equations, *J. Math. Tokushima Univ.*, **52** (2018) 39–52.
- [13] W.A. Strauss, *Nonlinear wave equations*, CBMS Regional Conference Series in Mathematics, Vol.73, Amer. Math. Soc., Providence, RI, 1989.