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Decay Rates of Solutions for Non-Degenerate Kirchhoff Type Dissipative Wave Equations

By

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Abstract

Consider the Cauchy problem for the non-degenerate Kirchhoff type dissipative wave equations with the initial data belonging to $(H^2(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))\times(H^1(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))$. Using the Fourier transform method in the $L^2\cap L^1$-frame, we can improve the decay rates of the energies given by the energy method of the L^2-frame.

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1 Introduction

In this paper we study on decay estimates of the solution $u(t)$ for the non-degenerate Kirchhoff type dissipative wave equations :

$$\begin{cases} \rho u'' + a\left(\|A^{1/2}u(t)\|^2\right) Au + u' = 0 & \text{in} \quad \mathbb{R}^N \times [0,\infty)\,, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) & \text{in} \quad \mathbb{R}^N\,, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is an unknown real value function, $' = \partial/\partial t$, $A = -\Delta = -\sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator with domain $\mathcal{D}(A) = H^2(\mathbb{R}^N)$, ρ is positive constant, and $\|\cdot\|$ is the usual norm of $L^2(\mathbb{R}^N)$, that is,

$$\|f\| = \left(\int_{\mathbb{R}^N} |f(x)|^2 dx\right)^{\frac{1}{2}} \quad \text{for} \quad f \in L^2(\mathbb{R}^N)\,.$$

For the non-local nonlinear term $a(M) \in C^0([0,\infty)) \cap C^2((0,\infty))$, we assume that as follows :

<u>Hyp.1</u> $\quad K_1 \le a(M) \le K_2 + K_3 M^\gamma \quad$ for $M \ge 0$

<u>Hyp.2</u> $\quad 0 \le a'(M)M \le K_4 a(M) \quad$ for $M > 0$

Hyp.3 $\quad a'(M)M + |a''(M)|M^2 \le K_5 M^\gamma \quad$ for $M > 0$

with $\gamma > 0$ and $K_j > 0$ $(j = 1, 2, 3, 4, 5)$.

For a typical example, we have

$$a(M) = 1 + M^\gamma \quad \text{with} \quad \gamma > 0\,.$$

Equation (1.1) describes small amplitude vibrations of an elastic string when the dimension is one (see Kirchhoff [5] for the original equation, and [1], [2], [8]).

In the previous papers [12] and [13], we have derived the following decay estimates of the solution $u(t)$ of (1.1) (see [3], [9], [11] for degenerate equations).

Theorem 1.1 *Suppose that the initial data* (u_0, u_1) *belong to* $H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^1)$ *and* $u_0 \neq 0$, *and Hyp.1 and Hyp.2 are fulfilled. Then, there exists* $\rho_0 > 0$ *such that for any* $\rho \in (0, \rho_0)$ *the problem* (1.1) *admits a unique global solution* $u(t)$ *in the class* $C^0([0,\infty); H^2(\mathbb{R}^N)) \cap C^1([0,\infty); H^1(\mathbb{R}^N)) \cap C^2([0,\infty); L^2(\mathbb{R}^N))$ *satisfying* $\|A^{1/2}u(t)\|^2 \ge Ce^{-\alpha t}$ *with some* $\alpha > 0$.

Moreover, if Hyp.3 is fulfilled, then the solution $u(t)$ *satisfies*

$$\|A^{1/2}u(t)\|^2 \le C(1+t)^{-1}\,, \tag{1.2}$$

$$\|u'(t)\|^2 + \|Au(t)\|^2 \le C(1+t)^{-\theta}\,, \tag{1.3}$$

$$\|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \le C(1+t)^{-\sigma} \quad \textit{for} \quad t \ge 0\,, \tag{1.4}$$

where

$$\theta = \min\{2\,,\, 1 + 2\gamma\}\,,$$
$$\sigma = \min\{3\,,\, (1+\gamma)(1+2\gamma)\}\,.$$

In order to derive (1.2)–(1.4) in the previous paper [12], we use the functions $E(t)$, $F(t)$, $L(t)$ defined by

$$E(t) = \rho\|u'(t)\|^2 + \int_0^{M(t)} a(\mu)\, d\mu\,, \quad M(t) = \|A^{1/2}u(t)\|^2\,, \tag{1.5}$$

$$F(t) = \rho\|A^{1/2}u'(t)\|^2 + a(M(t))\|Au(t)\|^2\,, \tag{1.6}$$

$$L(t) = \rho\|u''(t)\|^2 + a(M(t))\|A^{1/2}u'(t)\|^2 + \frac{a'(M(t))}{2}|M'(t)|^2\,, \tag{1.7}$$

respectively.

Moreover, we have derived that the functions $E(t)$, $F(t)$, $L(t)$ satisfy the following inequalities :

$$\sup_{t\le s\le t+1} E(s)^2 \le C\left(E(t) + \sup_{t\le s\le t+1}\|u(s)\|^2\right)(E(t)-E(t+1))\,, \tag{1.8}$$

$$\begin{aligned}\sup_{t\le s\le t+1} F(s)^2 \le{}& C\left(F(t) + \sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} M(s)\right)(F(t)-F(t+1))\\ &+ C\left(\sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} M(s)\right)\sup_{t\le s\le t+1} f(s)^2\,,\end{aligned} \tag{1.9}$$

where

$$f(t)^2 = \min\{M(t)^{2\gamma+1}\,,\ M(t)^{2\gamma}\|Au(t)\|^2\}\,, \tag{1.10}$$

$$\begin{aligned}\sup_{t\le s\le t+1} L(s)^2 \le{}& C\left(L(t) + \sup_{t\le s\le t+1} h(s)^2 + \sup_{t\le s\le t+1}\|u'(s)\|^2\right)(L(t)-L(t+1))\\ &+ C\left(\sup_{t\le s\le t+1} h(s)^2 + \sup_{t\le s\le t+1}\|u'(s)\|^2\right)\sup_{t\le s\le t+1} h(s)^2\,,\end{aligned} \tag{1.11}$$

where

$$h(t)^2 = \begin{cases}\|u'(t)\|^{2\gamma}\|A^{1/2}u'(t)\|^2 & \text{if}\quad 0<\gamma<\dfrac{1}{2}\\ M(t)^{\gamma-\frac{1}{2}}\|u'(t)\|\|A^{1/2}u'(t)\|^2 & \text{if}\quad \gamma\ge\dfrac{1}{2}\,.\end{cases} \tag{1.12}$$

When the initial data (u_0, u_1) belong to $L^1(\mathbb{R}^N)\times L^1(\mathbb{R}^N)$, we can derive the decay rate of L^2-norm of the solution $u(t)$, and then, we will improve the decay rates of (1.2)–(1.4),

Our main result is as follows.

Theorem 1.2 *If the initial data (u_0, u_1) belong to $L^1(\mathbb{R}^N)\times L^1(\mathbb{R}^N)$ in addition to the assumption of Theorem 1.1, then the solution $u(t)$ of* (1.1) *satisfies*

$$\|u(t)\|^2 \le C(1+t)^{-\eta}\,,\quad \|A^{1/2}u(t)\|^2 \le C(1+t)^{-1-\eta}\,, \tag{1.13}$$

$$\|u'(t)\|^2 + \|Au(t)\|^2 \le C(1+t)^{-\omega}\,, \tag{1.14}$$

$$\|u'(t)\|^2 + \|Au(t)\|^2 \le C(1+t)^{-\mu}\quad \textit{for}\quad t\ge 0\,, \tag{1.15}$$

where

$$\begin{aligned}&\eta = \min\{N/2\,,\ 2\}\,,\\ &\omega = \min\{2+\eta\,,\ (1+2\gamma)(1+\eta)\}\,,\\ &\mu = \min\{3+\eta\,,\ (1+\gamma)(2+\eta)\,,\ (1+\gamma)(1+2\gamma)(1+\eta)\}\,.\end{aligned}$$

The notations we use in the paper are standard. Positive constants will be denoted by C and will change from line to line.

2 Integral Forms

We denote the Fourier transform of $g(x)$ by

$$\mathcal{F}(g(x))(\xi) \equiv \widehat{g}(\xi) \equiv (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-i\xi\cdot x} g(x)\, dx\,,$$

where $\xi \cdot x = \sum_{j=1}^{N} \xi_j x_j$.

Through the Fourier transform, we can rewrite (1.1) to the following equation :

$$\begin{cases} \rho\widehat{u}'' + \widehat{u}' + a(0)|\xi|^2\widehat{u} = f(M(t))\widehat{Au} & \text{in} \quad \mathbb{R}^N_\xi \times [0,\infty)\,, \\ \widehat{u}(\xi,0) = \widehat{u_0}(\xi) \quad \text{and} \quad \widehat{u}'(\xi,0) = \widehat{u_1}(\xi) & \text{in} \quad \mathbb{R}^N_\xi\,, \end{cases} \tag{2.1}$$

where $f(M) = a(0) - a(M)$. Then, we obtain the integral form for (2.1) :

$$\widehat{u}(\xi,t) = \widehat{u_L}(\xi,t) + \widehat{u_N}(\xi,t) \tag{2.2}$$

where we set

$$\widehat{u_L}(\xi,t) = \frac{1}{2}\left(\phi_1(\xi,t) + \phi_2(\xi,t)\right)\widehat{u_0}(\xi) + \phi_2(\xi,t)\widehat{u_1}(\xi)\,, \tag{2.3}$$

$$\widehat{u_N}(\xi,t) = \int_0^t \phi_2(\xi,t-s) f(M(s))\widehat{Au}(\xi,s)\, ds\,, \tag{2.4}$$

and

$$\phi_1(\xi,t) = e^{\frac{-1+\lambda}{2\rho}t} + e^{\frac{-1-\lambda}{2\rho}t} = \begin{cases} 2e^{-\frac{t}{2\rho}} \cosh \dfrac{\lambda t}{2\rho} & \text{if} \quad \xi \in X_1 \cup X_2\,, \\ 2e^{-\frac{t}{2\rho}} \cos \dfrac{\sigma t}{2\rho} & \text{if} \quad \xi \in X_3 \cup X_4\,, \end{cases}$$

$$\phi_2(\xi,t) = \frac{1}{\lambda}\left(e^{\frac{-1+\lambda}{2\rho}t} - e^{\frac{-1-\lambda}{2\rho}t}\right) = \begin{cases} \dfrac{t}{\rho} e^{-\frac{t}{2\rho}} \dfrac{2\rho}{\lambda t} \sinh \dfrac{\lambda t}{2\rho} & \text{if} \quad \xi \in X_1 \cup X_2\,, \\ \dfrac{t}{\rho} e^{-\frac{t}{2\rho}} \dfrac{2\rho}{\lambda t} \sin \dfrac{\sigma t}{2\rho} & \text{if} \quad \xi \in X_3 \cup X_4\,, \end{cases}$$

with $\lambda = \sqrt{1 - 4\rho a(0)|\xi|^2}$ and $\sigma = \sqrt{4\rho a(0)|\xi|^2 - 1}$,

$$X_1 = \{\xi \mid |\xi| < (2^3\rho a(0))^{-\frac{1}{2}}\}\,,$$
$$X_2 = \{\xi \mid (2^3\rho a(0))^{-\frac{1}{2}} \le |\xi| < (2^2\rho a(0))^{-\frac{1}{2}}\}\,,$$
$$X_3 = \{\xi \mid (2^2\rho a(0))^{-\frac{1}{2}} \le |\xi| < (2\rho a(0))^{-\frac{1}{2}}\}\,,$$
$$X_4 = \{\xi \mid (2\rho a(0))^{-\frac{1}{2}} \le |\xi|\}\,.$$

We denote the cut-off functions $\chi_j(\cdot)$ for $j = 1,2,3,4$ by

$$\chi_j(\xi) = \begin{cases} 1 & \text{if} \quad \xi \in X_j\,, \\ 0 & \text{if} \quad \xi \notin X_j\,. \end{cases}$$

Proposition 2.1 *The solution* $u(t)$ *of* (1.1) *satisfies*

$$\|u(t)\| \le C(1+t)^{-\frac{N}{4}} + C\int_0^t (1+t-s)^{-1}|f(M(s))|\|u(s)\|\,ds + \int_0^t e^{-\frac{t-s}{2\rho}}|f(M(s))|\|Au(s)\|\,ds \quad \textit{for} \quad t \ge 0\,. \tag{2.5}$$

Proof. (1) First, we estimate the linear part (2.3). From the Parseval identity, we observe

$$\begin{aligned}\|u_L(t)\| &\le C\sum_{j=1}^{4}\|\chi_j(\xi)(|\phi_1(\xi,t)|+|\phi_2(\xi,t)|)(|\widehat{u_0}(\xi)|+|\widehat{u_1}(\xi)|)\| \\ &\le CI_1(t)(\|\widehat{u_0}(\xi)\|_{L^\infty}+\|\widehat{u_1}(\xi)\|_{L^\infty}) + C\sum_{j=2}^{4}I_j(t)(\|\widehat{u_0}(\xi)\|+\|\widehat{u_1}(\xi)\|) \\ &\le CI_1(t)(\|u_0\|_{L^1}+\|u_1\|_{L^1}) + C\sum_{j=2}^{4}I_j(t)(\|u_0\|+\|u_1\|)\end{aligned}$$

where we set

$$I_j(t) = \begin{cases} \|\chi_1(\xi)(|\phi_1(\xi,t)|+|\phi_2(\xi,t)|)\| & \text{if} \quad j=1\,, \\ \sup_{\xi\in X_j}(|\phi_1(\xi,t)|+|\phi_2(\xi,t)|) & \text{if} \quad j=2,3,4\,. \end{cases}$$

(i) When $\xi \in X_1$, we observe that $1/\sqrt{2} < \lambda \le 1$ and $-1+\lambda \le -2\rho a(0)|\xi|^2$ and

$$\sup_{\xi\in X_1}(|\phi_1(\xi,t)|+|\phi_2(\xi,t)|) \le Ce^{-a(0)|\xi|^2 t}\,,$$

and hence,

$$\begin{aligned} I_1(t) &\le C\|\chi_1(\xi)e^{-a(0)|\xi|^2 t}\| \le C\left(\int_{X_1} e^{-2a(0)|\xi|^2 t}d\xi\right)^{\frac{1}{2}} \\ &\le C\left(\int_0^\delta |\xi|^{N-1}e^{-2a(0)|\xi|^2 t}d|\xi|\right)^{\frac{1}{2}} \le C(1+t)^{-\frac{N}{4}}\end{aligned}$$

with $\delta = (2^3\rho a(0))^{-\frac{1}{2}} > 0$.

(ii) When $\xi \in X_2$, we observe that $0 < \lambda \le 1/\sqrt{2}$ and

$$\sup_{\xi\in X_2} |\phi_1(\xi,t)| \le Ce^{-(1-\frac{1}{\sqrt{2}})\frac{t}{2\rho}} ,$$

$$\begin{aligned}\sup_{\xi\in X_2} |\phi_2(\xi,t)| &\le Cte^{-\frac{t}{2\rho}} \sup_{\xi\in X_2} \left|\frac{2\rho}{\lambda t}\sinh\frac{\lambda t}{2\rho}\right| \\ &= Cte^{-\frac{t}{2\rho}} \sup_{\xi\in X_2} \left|\frac{2\rho}{\lambda t}\int_0^1 \frac{d}{d\theta}\left(\sinh\frac{\lambda t}{2\rho}\theta\right) d\theta\right| \\ &\le Cte^{-\frac{t}{2\rho}} \sup_{\xi\in X_2} \left|\cosh\frac{\lambda t}{2\rho}\right| \le Cte^{-(1-\frac{1}{\sqrt{2}})\frac{t}{2\rho}} ,\end{aligned}$$

and hence,

$$I_2(t) \le Ce^{-\nu t}$$

with some $0 < \nu \le (1-1/\sqrt{2})/(2\rho)$.

(iii) When $\xi \in X_3$, we observe that $0 \le \sigma < \sqrt{2}$ and

$$\sup_{\xi\in X_3} |\phi_1(\xi,t)| \le Ce^{-\frac{t}{2\rho}} ,$$

$$\begin{aligned}\sup_{\xi\in X_3} |\phi_2(\xi,t)| &\le Cte^{-\frac{t}{2\rho}} \sup_{\xi\in X_3} \left|\frac{2\rho}{\sigma t}\sin\frac{\sigma t}{2\rho}\right| \\ &= Cte^{-\frac{t}{2\rho}} \sup_{\xi\in X_3} \left|\frac{2\rho}{\sigma t}\int_0^1 \frac{d}{d\theta}\left(\sin\frac{\sigma t}{2\rho}\theta\right) d\theta\right| \le Cte^{-\frac{t}{2\rho}} ,\end{aligned}$$

and hence,

$$I_3(t) \le Ce^{-\nu t}$$

with some $0 < \nu \le 1/(2\rho)$.

(iv) When $\xi \in X_4$, we observe that $\sigma \ge \sqrt{2}$ and

$$\sup_{\xi\in X_4} (|\phi_1(\xi,t)| + |\phi_2(\xi,t)|) \le Ce^{-\frac{t}{2\rho}} \quad \text{and} \quad I_4(t) \le Ce^{-\frac{t}{2\rho}} .$$

Therefore, we obtain

$$\|u_L(t)\| \le C(1+t)^{-\frac{N}{4}}(\|u_0\|_{L^1} + \|u_1\|_{L^1}) + Ce^{-\nu t}(\|u_0\| + \|u_1\|) \tag{2.6}$$

with some $\nu > 0$.

(2) Next, we estimate the nonlinear part (2.4). From the Parseval identity, we observe

$$\|u_N(t)\| \le \int_0^t \|\phi_2(\xi,t-s)\widehat{Au}(\xi,s)\| |f(M(s))|\, ds$$

and

$$\|\phi_2(\xi,t-s)\widehat{Au}(\xi,s)\| \le \sum_{j=1}^{4} \|\chi_j(\xi)\phi_2(\xi,t-s)\widehat{Au}(\xi,s)\|$$

$$\le \sum_{j=1}^{3} \|\chi_j(\xi)|\xi|^2\phi_2(\xi,t-s)\widehat{u}(\xi,s)\| + \|\chi_4(\xi)\phi_2(\xi,t-s)\widehat{Au}(\xi,s)\|$$

$$\le C\sum_{j=1}^{3} J_j(t-s)\|u(s)\| + CJ_4(t-s)\|Au(s)\|$$

where

$$J_j(t) = \begin{cases} \sup\limits_{\xi\in X_1} |\xi|^2|\phi_2(\xi,t)| & \text{if} \quad j=1\,, \\ \sup\limits_{\xi\in X_j} |\phi_2(\xi,t)| & \text{if} \quad j=2,3,4\,. \end{cases}$$

(i) When $\xi \in X_1$, we observe $1/\sqrt{2} < \lambda \le 1$ and $-1+\lambda \le -2\rho a(0)|\xi|^2$ and

$$\sup_{\xi\in X_1} |\phi_2(\xi,t)| \le Ce^{-a(0)|\xi|^2 t}\,,$$

and hence,

$$J_1(t) \le C\sup_{\xi\in X_1} |\xi|^2 e^{-a(0)|\xi|^2 t} \le C(1+t)^{-1}\,.$$

(ii) When $\xi \in X_2$, we observe that $0 < \lambda \le 1/\sqrt{2}$ and

$$\sup_{\xi\in X_2} |\phi_2(\xi,t)| \le Cte^{-(1-\frac{1}{\sqrt{2}})\frac{t}{2\rho}} \quad \text{and} \quad J_2(t) \le Ce^{-\nu t}$$

with some $0 < \nu \le (1-1/\sqrt{2})/(2\rho)$.

(iii) When $\xi \in X_3$, we observe $0 \le \sigma < \sqrt{2}$ and

$$\sup_{\xi\in X_3} |\phi_2(\xi,t)| \le Cte^{-\frac{t}{2\rho}} \quad \text{and} \quad J_3(t) \le Ce^{-\nu t}$$

with some $0 < \nu \le 1/(2\rho)$.

(iv) When $\xi \in X_4$, we observe that $\sigma \ge \sqrt{2}$ and

$$\sup_{\xi\in X_4} |\phi_2(\xi,t)| \le Ce^{-\frac{t}{2\rho}} \quad \text{and} \quad J_4(t) \le Ce^{-\frac{t}{2\rho}}\,.$$

Therefore, we obtain

$$\|u_N(t)\| \le C\int_0^t (1+t-s)^{-1}|f(M(s))|\|u(s)\|\,ds$$
$$+ C\int_0^t e^{-\frac{t-2}{2\rho}}|f(M(s))|\|Au(s)\|\,ds\,. \tag{2.7}$$

Thus, from (2.6) and (2.7) we conclude (2.5). □

3 Energy Decay

Proposition 3.1 *Suppose that the initial data* (u_0, u_1) *belong to* $(H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \times (H^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$. *Then, the solution* $u(t)$ *of* (1.1) *satisfies*

$$\|u(t)\|^2 \leq C(1+t)^{-\eta}\,, \tag{3.1}$$

$$E(t) = \rho\|u'(t)\|^2 + \int_0^{M(t)} a(\mu)\,d\mu \leq C(1+t)^{-1-\eta}\,, \tag{3.2}$$

$$M(t) = \|A^{1/2}u(t)\|^2 \leq C(1+t)^{-1-\eta}\,, \tag{3.3}$$

where $\eta = \min\{N/2\,,\,2\}$.

Proof. We observe from Hyp.2 that

$$\begin{aligned} F(M) = a(0) - a(M) &= -\int_0^1 \frac{d}{dt}a(\theta M)\,d\theta \\ &\leq \int_0^1 K_4 \frac{(\theta M)^\gamma}{\theta M} M\,d\theta \leq CM^\gamma\,, \end{aligned}$$

and from (2.5) that

$$\begin{aligned} \|u(t)\| \leq C(1+t)^{-\frac{N}{4}} &+ C\int_0^t (1+t-s)^{-1} M(s)^\gamma \|u(s)\|\,ds \\ &+ C\int_0^t e^{-\frac{t-s}{2\rho}} M(s)^\gamma \|Au(s)\|\,ds\,, \end{aligned} \tag{3.4}$$

and from (1.2) and (1.3) that

$$\begin{aligned} \|u(t)\| &\leq C(1+t)^{-\frac{N}{4}} + C\int_0^t (1+t-s)^{-1}(1+s)^{-\gamma}\|u(s)\|\,ds \\ &\quad + C\int_0^t e^{-\frac{t-s}{2\rho}}(1+s)^{-(\frac{3}{2}+\gamma)}ds \\ &\leq C(1+t)^{-\frac{\eta}{2}} + C\int_0^t (1+t-s)^{-1}(1+s)^{-\gamma}\|u(s)\|\,ds \end{aligned}$$

with $\eta = \min\{N/2\,,\,2\}$.

Setting

$$\mu(t) = \sup_{0\leq s\leq t}(1+s)^{\frac{\eta}{2}}\|u(s)\|\,,$$

since $\|u(t)\|$ is bounded, we have that for $0 \leq \tau \leq t$ and any $\varepsilon > 0$,

$$\begin{aligned} &(1+\tau)^{\frac{\eta}{2}}\|u(\tau)\| \\ &\leq C + C(1+\tau)^{\frac{\eta}{2}}\left(\int_0^{\frac{\tau}{2}} + \int_{\frac{\tau}{2}}^{\tau}\right)(1+\tau-s)^{-1}(1+s)^{-\gamma-\frac{\eta}{2}(1-\varepsilon)}ds\mu(\tau)^{1-\varepsilon} \\ &\leq C + C\mu(t)^{1-\varepsilon}\,, \end{aligned}$$

and hence,

$$\mu(t) \le C \quad \text{or} \quad \|u(t)\|^2 \le C(1+t)^{-\eta} . \tag{3.5}$$

Using (1.8) together with (3.5) again, we have

$$\sup_{t\le s\le t+1} E(s)^2 \le C\left(E(t) + (1+t)^{-\eta}\right)(E(t) - E(t+1)) . \tag{3.6}$$

Applying Lemma 3.2 to (3.6), we obtain

$$E(t) \le C(1+t)^{-1-\eta}$$

which implies (3.2) and (3.3). □

We used the following Lemma for the energy estimate (see [4], [6], [7], [10] for the proof).

Lemma 3.2 *Let $\phi(t)$ be a non-negative function on $[0,\infty)$ and satisfy*

$$\sup_{t\le s\le t+1} \phi(s)^{1+\alpha} \le (k_0\phi(t)^\alpha + k_1(1+t)^{-\beta})(\phi(t) - \phi(t+1)) + k_2(1+t)^{-\gamma}$$

with certain constants $k_0, k_1, k_2 \ge 0$, $\alpha > 0$, $\beta > -1$, and $\gamma > 0$. Then, the function $\phi(t)$ satisfies

$$\phi(t) \le C_0(1+t)^{-\theta} , \qquad \theta = \min\{\frac{1+\beta}{\alpha} , \frac{\gamma}{1+\alpha}\}$$

for $t \ge 0$ with some constant C_0 depending on $\phi(0)$.

4 Improved Decay Estimates

Proposition 4.1 *Under the assumptions of Theorem 1.1 and Proposition 3.1, the solution $u(t)$ of (1.1) satisfies*

$$F(t) = \rho\|A^{1/2}u'(t)\|^2 + a(M(t))\|Au(t)\|^2 \le C(1+t)^{-\omega} , \tag{4.1}$$

$$\|u'(t)\|^2 + \|u''(t)\|^2 \le C(1+t)^{-\omega} , \tag{4.2}$$

where $\omega = \min\{2+\eta\, , \, (1+2\gamma)(1+\eta)\}$.

Proof. From (1.6), (1.10), (3.3) we observe

$$f(t) \le C(1+t)^{-(1+2\gamma)(1+\eta)} \quad \text{and} \quad f(t) \le C(1+t)^{-2\gamma(1+\eta)}F(t) . \tag{4.3}$$

Using (1.9) together with (3.3) and (4.3), we have

$$\sup_{t\le s\le t+1} F(s)^2 \le C\left(F(t) + (1+t)^{-(1+\eta)}\right)(F(t) - F(t+1)) + C(1+t)^{-(1+2\gamma)(1+\eta)} \sup_{t\le s\le t+1} F(s)$$

and from the Young inequality, we have

$$\sup_{t\le s\le t+1} F(s)^2 \le C\left(F(t)+(1+t)^{-(1+\eta)}\right)(F(t)-F(t+1)) \\ + C(1+t)^{-2(1+2\gamma)(1+\eta)}\,. \tag{4.4}$$

Thus, applying Lemma 3.2 to (4.4), we have

$$F(t) \le C(1+t)^{-\omega} \quad \text{with} \quad \omega = \min\{2+\eta\,,\,(1+2\gamma)(1+\eta)\}\,.$$

Multiplying (1.1) by $2u'(t)$ and integrating it over $\mathbb{R}^N$, we have

$$\rho\frac{d}{dt}\|u'(t)\|^2 + 2\|u'(t)\|^2 = -2a(M(t))(Au(t),u(t))\,,$$

and from the Young inequality, we have

$$\rho\frac{d}{dt}\|u'(t)\|^2 + \|u'(t)\|^2 \le a(M(t))^2\|Au(t)\|^2 \le C(1+t)^{-\omega}\,,$$

and hence, we derive the desired estimate (4.2). □

Proposition 4.2 *Under the assumptions of Theorem 1.1 and Proposition 3.1, the solution $u(t)$ of (1.1) satisfies*

$$L(t) = \rho\|u''(t)\|^2 + +a(M(t))\|Au'(t)\|^2 + \frac{a'(M(t))}{2}|M'(t)|^2 \\ \le C(1+t)^{-\mu}\,, \tag{4.5}$$

where $\mu = \{3+\eta\,,\,(1+\gamma)(2+\eta)\,,\,(1+\gamma)(1+2\gamma)(1+\eta)\}$.

Proof. (i) When $0<\gamma<\frac{1}{2}$, we observe from (1.12), (4.1), (4.2) that

$$h(t)^2 \le C(1+t)^{-(1+\gamma)\omega} \quad \text{and} \quad h(t)^2 \le C(1+t)^{-\gamma\omega}L(t)\,. \tag{4.6}$$

Using (1.11) together with (4.2) and (4.6), we have

$$\sup_{t\le s\le t+1} L(s)^2 \le C\left(L(t)+(1+t)^{-\omega}\right)(L(t)-L(t+1)) \\ + C(1+t)^{-(1+\gamma)\omega}\sup_{t\le s\le t+1} L(s)$$

and from the Young inequality, we have

$$\sup_{t\le s\le t+1} L(s)^2 \le C\left(L(t)+(1+t)^{-\omega}\right)(L(t)-L(t+1)) \\ + C(1+t)^{-2(1+\gamma)\omega} \tag{4.7}$$

Thus, applying Lemma 3.2 to (4.7), we have

$$L(t) \le C(1+t)^{-\mu} \quad \text{for} \quad \mu = \min\{1+\omega\,,\,(1+\gamma)\omega\}\,,$$

which implies (4.5) for $0 < \gamma < \frac{1}{2}$.

(ii) When $\gamma \ge \frac{1}{2}$, we observe from (1.12), (3.3), (4.1), (4.2) that

$$h(t)^2 \le C(1+t)^{-\frac{3}{2}\omega} \quad \text{and} \quad h(t)^2 \le C(1+t)^{-\frac{1}{2}\omega} L(t)\,. \tag{4.8}$$

Using (1.11) together with (4.2) and (4.8), we have

$$\sup_{t\le s\le t+1} L(s)^2 \le C\left(L(t) + (1+t)^{-\omega}\right)(L(t) - L(t+1)) + C(1+t)^{-\frac{3}{2}\omega} \sup_{t\le s\le t+1} L(s)$$

and from the Young inequality, we have

$$\sup_{t\le s\le t+1} L(s)^2 \le C\left(L(t) + (1+t)^{-\omega}\right)(L(t) - L(t+1)) + C(1+t)^{-3\omega}\,. \tag{4.9}$$

Thus, applying Lemma 3.2 to (4.9), we have

$$L(t) \le C(1+t)^{-\mu} \quad \text{with} \quad \mu = \{1+\omega\,,\,\frac{3}{2}\omega\} = 1+\omega\,,$$

which implies (4.5) for $\gamma \ge \frac{1}{2}$. □

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