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On the l-part of the Class Groups of Imaginary Cyclic Fields of Conductor p and Degree $2l^n$

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Abstract

Let l be a fixed odd prime number, and $p = 2ql^e + 1$ an odd prime number with $(q, 2l) = 1$. For $0 \leq n \leq e$, let k_n be the subfield of the pth cyclotomic field $\mathbb{Q}(\zeta_p)$ of degree $2l^n$. It is an imaginary cyclic field of conductor p. We study the Galois module structure and Iwasawa type "class number formula" of the l-part of the class groups of k_n. Moreover, we give numerical examples for $p < 1,000,000$.

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1 Introduction

Let p be an odd prime number. In [7] (resp. [8]), the order (resp. the Galois module structure) of the minus ideal class groups of the pth cyclotomic fields are studied. In [3], the order and indivisibility by prime numbers of the minus class number of the pth cyclotomic fields are studied. In [4, 5, 6], the order and the Galois module structure of the 2-part of the ideal class groups of certain cyclic fields of conductor $8p$ or $8pq$, where $q \neq p$ is an odd prime number. In

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these papers, we can see Iwasawa type “class number formula” for the l-part of the ideal class groups, where l is a prime number which divides $p-1$.

Let l be a fixed odd prime number, and $p = 2ql^e + 1$ an odd prime number with $(q, 2l) = 1$. For $0 \le n \le e$, let k_n be the subfield of the pth cyclotomic field $\mathbb{Q}(\zeta_p)$ of degree $2l^n$. Let A_n be the l-part of the ideal class group of k_n. The formula claims that there exist non-negative integers n_0, μ, λ and an integer ν such that

$$|A_n| = l^{\mu l^n + \lambda n + \nu}$$

for $n_0 \le n \le e$ (cf. [3, Theorem 3]).

In this paper, we study some fundamental results on $\mathbb{Z}_l[[T]]$-modules in order to explain the formula and the Galois structure of A_n. We discuss conditions which determine $\mathbb{Z}_l[[T]]$-submodules by their indices, and satisfy

$$A_n \simeq (\mathbb{Z}/l^{s-1}\mathbb{Z})^{\oplus \lambda - b} \oplus (\mathbb{Z}/l^s\mathbb{Z})^{\oplus b}$$

as an abelian group. Moreover, we give numerical examples of the formula for imaginary cyclic fields of conductor $p < 1,000,000$ and degree $2l^n$.

2 Fundamental results on $\mathbb{Z}_l[[T]]$-modules

Let l be a fixed prime. Put $\Lambda = \mathbb{Z}_l[[T]]$. Let $f(T) \in \Lambda$ be a distinguished polynomial

$$f(T) = T^\lambda + a_{\lambda-1}T^{\lambda-1} + \cdots a_1 T + a_0,$$

where $p|a_i$ for all $0 \le i \le \lambda - 1$. Let M be a finitely generated torsion Λ-module with the characteristic polynomial $l^\mu f(T)$ for a non-negative integer μ. Then, we have a Λ-homomorphism φ:

$$\varphi : \quad M \to \bigoplus_{i=1}^{m} \Lambda/(l^{\mu_i}) \oplus \bigoplus_{j=1}^{r} \Lambda/(f_j(T))$$

with finite kernel and co-kernel such that

$$\mu = \sum_{i=1}^{m} \mu_i \quad \text{and} \quad f(T) = \prod_{j=1}^{r} f_j(T),$$

where $f_j(T)$ is distinguished (see [9, Theorem 13.12]). Put $\omega_n = (1+T)^{l^n} - 1$.

Proposition 2.1 (§13.3 in [9]). *Assume that $f(T)$ is relatively prime to ω_n for all n. Then $|M/\omega_n M|$ is finite for all n. Further, there exist a non-negative integer n_0 and an integer ν such that*

$$|M/\omega_n M| = l^{\mu l^n + \lambda n + \nu}$$

for all $n \ge n_0$.

The above assumption is not necessary for the following proposition.

Proposition 2.2 (§13.3 in [9]). *There exist a non-negative integer n_0 such that*

$$|\omega_{n-1}M/\omega_n M| = l^{\mu l^{n-1}(l-1)+\lambda}.$$

for all $n \geq n_0$.

From the above proposition, we easily obtain the following theorem, which explains Iwasawa type "class number formula" for a finite Λ-module $\tilde{M}$.

Theorem 2.3. *Let n_0 be a non-negative integer which satisfies the condition in Proposition 2.2. For $m \geq n_0$, put $\tilde{M} = M/\omega_m M$. Then,*

$$|\omega_{n-1}\tilde{M}/\omega_n \tilde{M}| = l^{\mu l^{n-1}(l-1)+\lambda}.$$

for all $n_0 \leq n \leq m$.

In order to study more precise structure, we show the following theorem.

Theorem 2.4. *Let $f(T)$ be a distinguished polynomial of degree λ in Λ. For $M = \Lambda/(f(T))$, every Λ-submodule M' of M is determined by the index $|M/M'|$ if and only if $\lambda = 1$ or $f(T)$ is an Eisenstein polynomial. Assume that $\lambda = 1$ or $f(T)$ is an Eisenstein polynomial. Then, if $|M/M'| = l^n$,*

$$M/M' \simeq (\mathbb{Z}/l^{s-1}\mathbb{Z})^{\oplus \lambda - b} \oplus (\mathbb{Z}/l^s\mathbb{Z})^{\oplus b},$$

where n and b are unique integers satisfying $n = \lambda(s-1) + b$ with $s \geq 1$ and $0 \leq b \leq \lambda - 1$.

Proof. If $\lambda = 1$, the Λ-submodule M' of M of index l^n is $l^n M$. Assume that $f(T)$ is an Eisenstein polynomial and $\lambda \geq 2$. Let $g(T) \in \Lambda \setminus (f(T))$. By l-adic Weierstrass preparation theorem, $g(T) \equiv l^{s'-1}u(T)h(T) \mod (f(T))$, where $s' \geq 1$, $u(T) \in \Lambda^\times$, and $h(T)$ a distinguished polynomial of degree $b' < \lambda$. Then, we have

$$\begin{aligned}(f(T), g(T)) &= (f(T), l^{s'-1}h(T)) \\ &= (f(T), l^{s'-1}h(T), l^{s'-1}f(T) - l^{s'-1}T^{\lambda - b'}h(T)) \\ &= (f(T), l^{s'-1}T^{b'}, l^{s'}).\end{aligned}$$

Let s be the minimum number of such s' contained in M' modulo $(f(T))$. Further let b be the minimum number of such b' with $s' = s$ contained in M' modulo $(f(T))$. M' is determined by s and b, i.e., $|M/M'| = l^{\lambda(s-1)+b}$.

If $f(T)$ is not of degree one nor an Eisenstein polynomial, consider $M_1 = (l, lT, f(T))/(f(T))$ and $M_2 = (l^2, T, f(T))/(f(T))$. Since $M_1 \neq M_2$ and $|M/M_1| = |M/M_2| = l^2$, this completes the proof. □

Remark 2.5. When $f(T)$ is an Eisenstein polynomial, there is another explanation (cf. [8, Proposition 3.4]). Let α be a root of of $f(T)$. Then $K = \mathbb{Q}_l(\alpha)$ is totally ramified extension of $\mathbb{Q}_l$. Let $\mathcal{O}_K$ be the maximal order of K. Since α is uniformizing element of $\mathcal{O}_K$, $\mathcal{O}_K = \mathbb{Z}_l[\alpha]$. Hence, the submodule M' of $M = \mathbb{Z}_l[\alpha]$ is determined by $|M/M'| = l^n$, where $M = (\alpha^n)$.

Remark 2.6. In order to explain the situation in Remark 2.5, we consider the following case. Let u be a non-square unit of $\mathbb{Z}_l$. Let $f(T) = T^2 - ul^2$. Then $\alpha = \pm\sqrt{u}l$ and $K = \mathbb{Q}_l(\sqrt{u})$. K is unramified extension of $\mathbb{Q}_l$ and $\mathcal{O}_K = \mathbb{Z}_l[\sqrt{u}]$. Though $\mathbb{Z}_l[\alpha]$ is an order of K, it is not the maximal order. $\mathbb{Z}_l[\alpha]$ has two submodules $M_1 = (l, l\alpha)$ and $M_2 = (l^2, \alpha)$ with the same index.

Remark 2.7. In [4, 5, 6], an Eisenstein polynomial

$$f(T) = (1+T)^{2^{e-1}} + 1 \in \mathbb{Z}_2[T]$$

plays a role. This element corresponds to $J + 1$, where J is the complex conjugate.

3 Numerical examples

Let l be a fixed odd prime number, and $p = 2ql^e + 1$ an odd prime number with $(q, 2l) = 1$. For $0 \leq n \leq e$, let k_n be the subfield of the pth cyclotomic field $\mathbb{Q}(\zeta_p)$ of degree $2l^n$. It is an imaginary cyclic field of conductor p. Let A_n be the l-part of the ideal class group of k_n. Then $A_n = A_n^- \oplus A_n^+$, where $A_n^- = \frac{1-J}{2}A_n$ and $A_n^+ = \frac{1+J}{2}A_n$. Let k_n^+ be the maximal totally real subfield of k_n. Since only one prime ramifies in $k_n^+/\mathbb{Q}$ and $[k_n^+ : \mathbb{Q}] = l^n$, the l-part of the ideal class group of k_n^+ is trivial. This implies $A_n = A_n^-$ and $A_n^+ = \{0\}$.

Let h_n^- be the relative class number of k_n. Put $h'_0 = h_0^-$ and

$$h'_n = \frac{h_n^-}{h_{n-1}^-}$$

for $1 \leq n \leq e$. We directly compute the value $\mathrm{ord}_l(h'_n)$ from $B_{1,\delta}$ in the following way. Let Q_k be the Hasse unit index of k (1 or 2, see [1, 2]) , and let w_k be the number of roots of unity in k. By the analytic class number formula [9, Theorem 4.17],

$$h_n^- = Q_{k_n} w_{k_n} \prod_{\delta} \left(-\frac{1}{2} B_{1,\delta} \right),$$

where δ runs over the odd Dirichlet characters of conductor p and order $2l^m$ with $m \leq n$. The generalized Bernoulli number for δ of order $2l^n$ is

$$B_{1,\delta} = \frac{1}{p} \sum_{a=1}^{p-1} a\delta(a) \in \mathbb{Q}(\zeta_{l^n}).$$

Put $\pi_n = \zeta_{l^n} - 1$. Then we have $\text{ord}_l(h'_n) = \text{ord}_{\pi_n}(B_{1,\delta})$ for any δ of order $2l^n$. Let g be a primitive root of $\mathbb{Z}/p\mathbb{Z}$, and let $F(T)$ be the polynomial in $\mathbb{Z}[T]$ such that

$$F(T) \equiv \sum_{a=1}^{p-1} a(1+T)^{i_a} \mod (\omega_e)$$

and that $\deg F(T) < l^e$, where i_a is an integer which satisfies $g^{i_a} \equiv a \mod p$. Then, by l-adic Weierstrass preparation theorem, we can write

$$F(T) = l^\mu f(T)u(T),$$

where μ is a non-negative integer μ, and $f(T)$ a distinguished polynomial of degree λ in $\mathbb{Z}_l[T]$, and $u(T) \in \Lambda^\times$. Note that $\text{ord}_l(h'_n) = \text{ord}_{\pi_n}(l^\mu f(\pi_n))$, and that $f(T)$ and ω_n are relatively prime for $0 \le n \le e$.

A $\mathbb{Z}_l[\text{Gal}(k_e/k_0)]$-module A_e is considered as a $\Lambda/(\omega_e)$-module by identifying a generator γ of $\text{Gal}(k_e/k_0)$ with $1+T$. Since only one prime totally ramifies in $k_n/\mathbb{Q}$, $A_n \simeq A_e/\omega_n A_e$. We obtain Iwasawa type "class number formula": $\text{ord}_l(h_n) = \mu l^n + \lambda n + \nu$ or $\text{ord}_l(h'_n) = \mu l^{n-1}(l-1) + \lambda$ by applying Theorem 2.3 to $\tilde{M} = A_e$ with $m = e$.

Further, the norm map $N_{n,n-1} : A_n \to A_{n-1}$ is surjective. Since the minus part of the unit group of k_n is trivial, the natural map $i_{n-1,n} : A_{n-1} \to A_n$ is injective. This implies that the kernel of $N_{n,n-1}$ is not trivial if A_{n-1} is not trivial. Therefore, if $\text{ord}_l(h_0) > 0$, then $\text{ord}_l(h'_n) > 0$ for $1 \le n \le e$. On the other hand, if $\text{ord}_l(h_0) = 0$, then $\text{ord}_l(h'_n) = 0$ for $1 \le n \le e$.

In Tables 1–5, we give $\text{ord}_l(h'_n)$ for $0 \le n \le e$ when $p < 1{,}000{,}000$, $e \ge e_l$ and $\text{ord}_l(h'_i) \ge 2$ for some i, where $e_3 = 5$, $e_5 = 3$, $e_7 = 2$ and $e_l = 1$ for $l \ge 11$.

Remark 3.1. On Iwasawa type "formula" in Tables 1–5, we have

$$\mu = 0, \quad 1 \le \lambda \le 6 \quad \text{and} \quad -2 \le \nu \le 8.$$

For $(l,p) = (3, 363043)$, $\lambda = 6$ and $\nu = 2$. For $(l,p) = (3, 131707)$, $\lambda = 5$ and $\nu = -2$. For $(l,p) = (3, 837379)$, $\lambda = 2$ and $\nu = 8$. In [4, 5], there are a lot of cases with $\mu > 0$ when $l = 2$ and $k_n \subseteq \mathbb{Q}(\zeta_{8pq})$.

Remark 3.2. In Tables 1–5, there are four pairs (l,p) whose A_n is not a cyclic $\mathbb{Z}_l[\text{Gal}(k_n/k_0)]$-module. Stars mark these primes p (97687, 967627 843211 in Table 1, 857099 in Table 4). Since A_0 is bicyclic for these pairs, A_n is generated by 2 elements over $\mathbb{Z}_l[\text{Gal}(k_n/k_0)]$. Except for these pairs, A_n is a cyclic $\mathbb{Z}_l[\text{Gal}(k_n/k_0)]$-module and

$$A_n \simeq \Lambda/(f(T), \omega_n),$$

where $f(T)$ is the distinguished polynomial defined above. Furthermore, if $\lambda = 1$ or $\text{ord}_l(h'_0) = 1$, we can apply Theorem 2.4 to A_n.

Table 1: $l = 3$, $\mathrm{ord}_3(h'_n)$ when $e \geq 5$ and $\mathrm{ord}(h'_i) \geq 2$ for some i.

p	e	0	1	2	3	4	5	6	7	8	9	λ	ν
196831	9	3	1	1	1	1	1	1	1	1	1	1	3
363043	7	4	4	6	6	6	6	6	6			6	2
345547	7	4	1	1	1	1	1	1	1			1	4
505927	6	5	2	2	2	2	2	2				2	5
578827	6	4	1	1	1	1	1	1				1	4
243487	6	2	5	4	4	4	4	4				4	3
94771	6	2	4	3	3	3	3	3				3	3
*97687	6	2	3	4	4	4	4	4				4	1
129763	6	2	1	1	1	1	1	1				1	2
304723	6	1	2	4	4	4	4	4				4	-1
260983	6	1	2	3	3	3	3	3				3	0
605071	6	1	2	3	3	3	3	3				3	0
858763	6	1	2	3	3	3	3	3				3	0
925831	6	1	2	2	2	2	2	2				2	1
104491	5	4	2	2	2	2	2					2	4
219187	5	4	1	1	1	1	1					1	4
*967627	5	3	3	3	3	3	3					3	3
910279	5	3	2	2	2	2	2					2	3
25759	5	3	1	1	1	1	1					1	3
175447	5	3	1	1	1	1	1					1	3
557443	5	3	1	1	1	1	1					1	3
769339	5	3	1	1	1	1	1					1	3
241543	5	2	3	4	4	4	4					4	1
327079	5	2	3	4	4	4	4					4	1
190027	5	2	2	2	2	2	2					2	2
502039	5	2	2	2	2	2	2					2	2
770311	5	2	2	2	2	2	2					2	2
787807	5	2	2	2	2	2	2					2	2
*843211	5	2	2	2	2	2	2					2	2
73387	5	2	1	1	1	1	1					1	2
264871	5	2	1	1	1	1	1					1	2
274591	5	2	1	1	1	1	1					1	2
321247	5	2	1	1	1	1	1					1	2
576883	5	2	1	1	1	1	1					1	2
656587	5	2	1	1	1	1	1					1	2
690607	5	2	1	1	1	1	1					1	2
821827	5	2	1	1	1	1	1					1	2
833491	5	2	1	1	1	1	1					1	2
907363	5	2	1	1	1	1	1					1	2
926803	5	2	1	1	1	1	1					1	2
837379	5	1	9	2	2	2	2					2	8
802387	5	1	4	2	2	2	2					2	3
37423	5	1	3	2	2	2	2					2	2
397063	5	1	3	2	2	2	2					2	2
425251	5	1	3	2	2	2	2					2	2
536059	5	1	3	2	2	2	2					2	2
955963	5	1	3	2	2	2	2					2	2
973459	5	1	3	2	2	2	2					2	2
131707	5	1	2	5	5	5	5					5	-2
933607	5	1	2	4	4	4	4					4	-1
156979	5	1	2	3	3	3	3					3	0
989011	5	1	2	3	3	3	3					3	0
612847	5	1	2	2	2	2	2					2	1
714907	5	1	2	2	2	2	2					2	1
920971	5	1	2	2	2	2	2					2	1
948187	5	1	2	2	2	2	2					2	1

Table 2: $l = 5$, $\mathrm{ord}_5(h'_n)$ when $e \geq 3$ and $\mathrm{ord}(h'_i) \geq 2$ for some i.

p	e	0	1	2	3	4	5	p	e	0	1	2	3
118751	5	2	1	1	1	1	1	683251	3	1	3	3	3
418751	5	2	1	1	1	1	1	137251	3	1	2	2	2
276251	4	2	1	1	1	1		239251	3	1	2	2	2
763751	4	2	1	1	1	1		408251	3	1	2	2	2
941251	4	1	2	2	2	2		417751	3	1	2	2	2
227251	3	3	1	1	1			474751	3	1	2	2	2
144751	3	2	2	2	2			475751	3	1	2	2	2
819251	3	2	2	2	2			494251	3	1	2	2	2
92251	3	2	1	1	1			540251	3	1	2	2	2
117251	3	2	1	1	1			569251	3	1	2	2	2
126751	3	2	1	1	1			591751	3	1	2	2	2
136751	3	2	1	1	1			635251	3	1	2	2	2
233251	3	2	1	1	1			714751	3	1	2	2	2
295751	3	2	1	1	1			754751	3	1	2	2	2
364751	3	2	1	1	1			873251	3	1	2	2	2
705751	3	2	1	1	1			910751	3	1	2	2	2
831751	3	2	1	1	1			994751	3	1	2	2	2
365251	3	1	4	4	4								

Table 3: $l = 7$, $\mathrm{ord}_7(h'_n)$ when $e \geq 2$ and $\mathrm{ord}(h'_i) \geq 2$ for some i.

p	e	0	1	2	3	p	e	0	1	2
258623	3	2	1	1	1	612011	2	2	1	1
365639	3	2	1	1	1	760187	2	2	1	1
409543	3	2	1	1	1	833099	2	2	1	1
421891	3	2	1	1	1	892291	2	2	1	1
332711	3	1	2	2	2	945211	2	2	1	1
640039	3	1	2	2	2	951091	2	2	1	1
796447	3	1	2	2	2	966379	2	2	1	1
962459	3	1	2	2	2	906403	2	1	3	3
886019	2	3	1	1		408563	2	1	2	2
20287	2	2	1	1		422087	2	1	2	2
160231	2	2	1	1		630827	2	1	2	2
205507	2	2	1	1		717851	2	1	2	2
229027	2	2	1	1		930119	2	1	2	2
498527	2	2	1	1		959911	2	1	2	2
507347	2	2	1	1						

Table 4: $l = 11$, $\mathrm{ord}_{11}(h'_n)$ when $e \geq 1$ and $\mathrm{ord}(h'_i) \geq 2$ for some i.

p	e	0	1	2	3	p	e	0	1	p	e	0	1
130439	3	2	1	1	1	466951	1	2	1	19559	1	1	2
113983	2	2	1	1		500083	1	2	1	28447	1	1	2
234499	2	2	1	1		503207	1	2	1	35267	1	1	2
435359	2	2	1	1		517243	1	2	1	59159	1	1	2
518123	2	2	1	1		548791	1	2	1	89431	1	1	2
31219	2	1	2	2		580691	1	2	1	194899	1	1	2
314359	2	1	2	2		582451	1	2	1	210739	1	1	2
947431	2	1	2	2		601943	1	2	1	211927	1	1	2
105491	1	2	3			607003	1	2	1	215183	1	1	2
662443	1	2	3			642907	1	2	1	249107	1	1	2
385771	1	2	2			644227	1	2	1	257687	1	1	2
639563	1	2	2			695047	1	2	1	301643	1	1	2
*857099	1	2	2			704287	1	2	1	341771	1	1	2
957331	1	2	2			722459	1	2	1	376399	1	1	2
35311	1	2	1			722899	1	2	1	511787	1	1	2
46399	1	2	1			736363	1	2	1	558251	1	1	2
71699	1	2	1			743711	1	2	1	616243	1	1	2
82567	1	2	1			745999	1	2	1	671903	1	1	2
127931	1	2	1			763423	1	2	1	700811	1	1	2
166783	1	2	1			824099	1	2	1	740191	1	1	2
214787	1	2	1			904399	1	2	1	758231	1	1	2
234323	1	2	1			914827	1	2	1	835847	1	1	2
287387	1	2	1			991387	1	2	1	901891	1	1	2
322631	1	2	1			875183	1	1	4	925607	1	1	2
340583	1	2	1			608851	1	1	3	934979	1	1	2
431267	1	2	1			666139	1	1	3	946727	1	1	2
432499	1	2	1			928643	1	1	3	974359	1	1	2
449131	1	2	1			2971	1	1	2	985007	1	1	2
464311	1	2	1			13619	1	1	2				

Table 5: $l \geq 13$, $\mathrm{ord}_l(h'_n)$ when $e \geq 1$ and $\mathrm{ord}(h'_i) \geq 2$ for some i.

l	p	e	0	1	2	3	l	p	e	0	1
13	672283	3	1	2	2	2	17	45119	1	2	1
	712843	2	2	1	1			157999	1	2	1
	506663	2	1	2	2			284819	1	2	1
	821003	2	1	2	2			354791	1	2	1
	104287	1	2	2				372539	1	2	1
	639263	1	2	2				447611	1	2	1
	96487	1	2	1				536147	1	2	1
	109903	1	2	1				66947	1	1	3
	136319	1	2	1				944963	1	1	3
	266059	1	2	1				154871	1	1	2
	288731	1	2	1				208591	1	1	2
	317071	1	2	1				217499	1	1	2
	335999	1	2	1				236471	1	1	2
	351391	1	2	1				317663	1	1	2
	375467	1	2	1				444007	1	1	2
	539839	1	2	1				477259	1	1	2
	545663	1	2	1				714443	1	1	2
	548003	1	2	1			19	84551	1	2	1
	635363	1	2	1				552103	1	2	1
	783719	1	2	1				683887	1	2	1
	883247	1	2	1				424271	1	1	2
	890683	1	2	1				838927	1	1	2
	912523	1	2	1				882247	1	1	2
	986519	1	2	1				886427	1	1	2
	239539	1	1	3			23	345599	1	2	1
	337039	1	1	3				713599	1	2	1
	724751	1	1	3				197203	1	1	2
	898171	1	1	3				926671	1	1	2
	33931	1	1	2			29	239831	1	1	2
	52963	1	1	2				433087	1	1	2
	64091	1	1	2				691651	1	1	2
	97579	1	1	2				828067	1	1	2
	125243	1	1	2			31	740591	1	2	1
	210523	1	1	2				257611	1	1	2
	223367	1	1	2				517267	1	1	2
	223939	1	1	2				878851	1	1	2
	257791	1	1	2			41	441079	1	1	2
	322999	1	1	2			43	771163	1	1	2
	323987	1	1	2			47	214603	1	1	2
	407707	1	1	2			101	749219	1	1	2
	570467	1	1	2							
	628811	1	1	2							
	635051	1	1	2							
	680707	1	1	2							
	865307	1	1	2							
	882571	1	1	2							
	891827	1	1	2							
	922351	1	1	2							

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